Fixed Point Results and Weak Orbital Continuity

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Abstract

In this paper we show that the continuity conditions assumed in the main results of [Meir-Keeler Type And Caristi Type Fixed Point Theorem, *Appl. Anal. Discrete Math.*, **13(2019)**, 849-858] and [A characterization of completeness of Menger PM-spaces, *J. Fixed Point Theory Appl.*, **21:90(2019)**] can be relaxed further. We also obtain local fixed point results for Meir-Keeler and Caristi type mappings. Our results generalize some previous results and provide new answers to Rhoades' problem on the existence of contractive definitions which admit discontinuity at fixed point.

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1 Introduction

In 2004, Agarwal et al [1] have obtained some interesting global and local fixed point theorems for Meir-Keeler and Caristi type mappings. Further, Pant et al [7] obtain the analogues of the main results of Agarwal et al [1] under weaker continuity conditions. The results in [7] hold for continuous as well as discontinuous mappings and provide new answers to Rhoades' problem ([12], p.242) on the existence of contractive definitions which admit discontinuity at the fixed point. In 1999, Pant [10] resolved this problem and obtained the first result for contractive mappings which are discontinuous at the fixed point. Further, fixed point theorems for discontinuous mappings have been extensively studied by various researchers (see [2], [3], [11], and references therein).
Now, we recall some weaker forms of continuity.

Definition 1 ([4]). *If T is a self-mapping of a metric space (X, d), then the set $O(T, x) = \{T^n x : n = 0, 1, 2, \ldots\}$ is called the orbit of T at x and T is called orbitally continuous if $u = \lim_i T^{m_i} x$ implies $Tu = \lim_i TT^{m_i} x$.*

Definition 2 ([6]). *A self-mapping T of a metric space (X,d) is called k-continuous, $k = 1,2,3,\dots$ if $T^k x_n \to Tt$ whenever $\{x_n\}$ is a sequence in X such that $T^{k-1}x_n \to t$.*

Remark 1. *Continuity of T implies orbital continuity but converse is not true [4]. 1-continuity is equivalent to continuity, and obviously, k-continuity implies $(k+1)$ continuity for $k \in \mathbb{N}$ [6].*

Let (X,d) be a complete metric space, $x_0 \in X$ and $r > 0$. Define the followings:

$$B(x_0,r) = \{x \in X : d(x,x_0) < r\}, \qquad \overline{B(x_0,r)} = \textit{the closure of } B(x_0,r).$$

If $f : \overline{B(x_0,r)} \to X$ is a map and $x, y \in \overline{B(x_0,r)}$, let us denote

$$M(x,y) = \max\{d(x,fx), d(y,fy)\}.$$

In 2019 Pant et al [5] obtained the local version of Meir-Keeler and Caristi type fixed point theorem as follows:

Theorem 3 (Theorem 2.7 of [5]). *Let (X,d) be a complete metric space, $x_0 \in X$ and $r > 0$. Suppose $f : \overline{B(x_0,r)} \to X$ is a map such that*

(i) $d(fx,fy) < M(x,y)$ *whenever* $M(x,y) > 0$,

(ii) *given* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that* $M(x,y) < \epsilon + \delta \Rightarrow d(fx,fy) \le \epsilon$,

(iii) $d(x_0, f^n x_0) < r, n = 1,2,\dots$.

If f^k is continuous or if f is k-continuous for some $k \geq 1$ or if f is orbitally continuous then f possesses a unique fixed point, say, $t \in \overline{B(x_0,r)}$. Moreover, f is continuous at the fixed point t if and only if $\lim_{x\to t} M(x,t) = 0$.

Theorem 4 (Theorem 2.14 of [5]). *Let (X,d) be a complete metric space, $x_0 \in X, r > 0$ and $f : \overline{B(x_0,r)} \to X$. Suppose there exists a function $\phi : X \to [0,\infty)$ such that for each $x \in \overline{B(x_0,r)}$ we have*

(i) $d(x,fx) \le \phi(x) - \phi(fx)$

(ii) $\phi(x_0) < r$.

If f is orbitally continuous or if f^k is continuous or if f is k-continuous for some $k \ge 1$ then f possesses a fixed point, say $t \in \overline{B(x_0,r)}$.

Recently Pant et. al [7] introduced a weaker form of orbital continuity.

Definition 5 ([7]). *A self-mapping T of a metric space (X,d) is called weak orbitally continuous if the set $\{y \in X : \lim_i T^{m_i}y = u \Rightarrow \lim_i TT^{m_i}y = fu\}$ is nonempty whenever the set $\{x \in X : \lim_i T^{m_i}x = u\}$ is nonempty.*

Example 6 ([7]). *Let* $X = [0, 2]$ *equipped with the Euclidean metric. Define* $T : X \to X$ *by*

$$T(x) = \frac{1+x}{2} \; if \; x < 1, \qquad T(x) = 0 \; if \; 1 \leq x < 2, \qquad T(2) = 2.$$

Then $T^n 0 \to 1$ *and* $T(T^n 0) \to 1 \neq T1$. *Therefore* T *is not orbitally continuous. However* T *is weakly orbitally continuous. If we take* $x = 2$ *then* $T^n 2 \to 2$ *and* $T(T^n 2) \to 2 = T2$ *and, hence* T *is weakly orbitally continuous. Further,* T *is also not k-continuous. To see this, consider the sequence* $\{T^n 0\}$ *then for any integer* $k \geq 1$, *we have* $T^{k-1}(T^n 0) \to 1$ *and* $T^k(T^n 0) \to 1 \neq T1$.

Example 6 shows that orbital continuity implies weak orbital continuity but the converse need not be true.

Our aim of this paper is to show the results of [5] and [9] hold for weaker continuity condition and provide new solutions to the Rhoades' problem of the existence of such contractive mappings which admit discontinuity at the fixed point. We also obtain local fixed point results for Meir-Keeler and Caristi type mappings. Several examples are also given to illustrate our results.

2 Main Results

Theorem 7. *Let* (X, d) *be a complete metric space,* $x_0 \in X$ *and* $r > 0$. *Suppose* $f : \overline{B(x_0, r)} \to X$ *is a map such that*

(i) $d(fx, fy) < M(x, y)$ *whenever* $M(x, y) > 0$,

(ii) *given* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that* $M(x, y) < \epsilon + \delta \Rightarrow d(fx, fy) \leq \epsilon$,

(iii) $d(x_0, f^n x_0) < r, n = 1, 2, \ldots$.

If f *is weakly orbitally continuous then* f *possesses a unique fixed point, say,* $t \in \overline{B(x_0, r)}$. *Moreover,* f *is continuous at the fixed point* t *if and only if* $\lim_{x \to t} M(x, t) = 0$.

Proof: We observe that, under condition (i), condition (ii) is equivalent to

$$\epsilon < M(x, y) < \epsilon + \delta \Rightarrow d(fx, fy) \leq \epsilon. \tag{2.1}$$

If (ii) is satisfied then condition (1) is obviously satisfied. On the other hand suppose (i) and (1) are satisfied. If $M(x, y) \leq \epsilon$ then by (i) we get $d(fx, fy) < M(x, y) \leq \epsilon$; and if $\epsilon < M(x, y) < \epsilon + \delta$ then (1) implies $d(fx, fy) \leq \epsilon$. Thus $M(x, y) < \epsilon + \delta \Rightarrow d(fx, fy) \leq \epsilon$ and (ii) is satisfied. Define a sequence $\{x_n\}$ by $x_1 = fx_0, x_2 = fx_1, \ldots, x_n = fx_{n-1}, \ldots$.Then following the proof of Theorem 2.7 in [5], we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete and $x_n \in B(x_0, r)$, there exists t in $\overline{B(x_0, r)}$ such that $x_n \to t$ and $f^k x_n \to t$ for each $k \geq 1$.

Suppose that f is weakly orbital continuous. Since $f^n x_0 \to t$ for each x_0, by virtue of weak orbital continuity of f we get $f^n y_0 \to t$ and $f^{n+1} y_0 \to ft$ for some y_0 in X. This implies $t = ft$ since $f^{n+1} y_0 \to t$. Thus t is a fixed point of f. Uniqueness of the fixed point follows from (i).

Example 8. *Let* $X = [0, 2]$, $x_0 = \frac{3}{4}, r = \frac{1}{2}$. *Define* $f : \overline{B(x_0, r)} \to X$ *by*

$$fx = 1 \ \ if \ \ \frac{1}{4} \le x \le 1, \quad fx = 0 \ \ if \ \ 1 < x \le \frac{5}{4}$$

Then f *satisfies all conditions of Theorem 7 and has a unique fixed point* $x = 1$ *at which* f *is discontinuous. It is clear to see that* f *satisfies condition (ii) with* $\delta(\epsilon) = 1$, *when* $\epsilon \ge 1$ *and* $\delta(\epsilon) = 1 - \epsilon$ *when* $\epsilon < 1$.

Remark 2. *Theorem 7 generalizes Theorem 2.7 of [5] and also provides a new solution to Rhoades' problem of existence of contractive mappings that admit discontinuity at fixed point.*

Theorem 9. *Let* (X, d) *be a complete metric space,* $x_0 \in X, r > 0$ *and* $f : \overline{B(x_0, r)} \to X$. *Suppose there exists a function* $\phi : X \to [0, \infty)$ *such that for each* $x \in \overline{B(x_0, r)}$ *we have*

(i) $d(x, fx) \le \phi(x) - \phi(fx)$

(ii) $\phi(x_0) < r$.

If f *is weakly orbitally continuous then* f *possesses a fixed point, say* $t \in \overline{B(x_0, r)}$.

Proof: Define a sequence $\{x_n\}$ by $x_1 = fx_0, x_2 = fx_1, \ldots, x_n = fx_{n-1}, \ldots$. Then followings the proof of Theorem 2.14 of [5], we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete and $x_n \in \overline{B(x_0, r)}$ for each n, there exists t in $\overline{B(x_0, r)}$ such that $x_n \to t$. Moreover, for each $k \ge 1$ we get $fx_n \to t$ and $f^k x_n \to t$.

Suppose that f is weakly orbitally continuous. Since $\{f^n x_0\}$ converges for each x_0 in X, weak orbital continuity implies that there exists $y_0 \in X$ such that $f^n y_0 \to t$ and $f^{n+1} y_0 \to ft$ for some t in X. This implies that $t = ft$. Thus t is a fixed point of f.

Example 10. *Let* $X = [0, 3], x_0 = \frac{3}{4}$ *and* $r = \frac{5}{4}$ *equipped with the Euclidean metric. Define* $T : X \to X$ *by*

$$T(x) = \begin{cases} \frac{1+x}{2} & if \quad x < 1 \\ 0 & if \quad 1 \le x < 2; x = 3 \\ 2 & if \quad x = 2 \\ 3 & if \quad x > 2 \end{cases}.$$

Let us define $\phi : X \to [0, \infty)$ *by*

$$\phi(x) = \begin{cases} 1 - x & if \quad x < 1 \\ 1 + x & if \quad x \ge 1 \end{cases}.$$

Then $\phi(\frac{3}{4}) < \frac{5}{4}$ *and* $d(x, fx) \le \phi(x) - \phi(fx)$ *for each* $x \in \overline{B(x_0, r)} = [0, 2]$, *so* f *satisfies conditions (i) and (ii) of Theorem 9. Clearly,* f *is weakly orbitally continuous but not k-continuous or orbitally continuous. Hence* f *satisfies all the conditions of Theorem 9 and* f *has a fixed point* $x = 2 \in \overline{B(x_0, r)}$. *However,* f *does not satisfy condition (i) for all* $x \in X$; *for* $x \in (2, 3)$ *we have* $d(x, fx) \ge \phi(x) - \phi(fx)$. *Obviously, Theorems 2.14 and 2.15 of [5] and Theorem 2.10 of Pant et al. [7] do not hold in this case.*

Remark 3. *Theorem 9 generalizes Theorem 2.14 of [5] and also provides a local version of Theorem 2.10 due to Pant et al [7]. In the Example 10, f has a fixed point $x = 2$ at which f is discontinuous. Hence, Theorem 9 is Caristi type fixed point theorem to provide an answer to the question of existence of contractive mappings which admit discontinuity at fixed point.*

We now show that the main result of Pant et al [9] in the probablistic metric space also holds for weakly orbitally continuous mappings.

Following the notation of [9], let D^+ be the set of all distribution functions $F : \mathbb{R} \to [0,1]$, such that F is a nondecreasing, left-continuous mapping that satisfies $F(0) = 0$ and $\sup_{x\in\mathbb{R}} F(x) = 1$. The space D^+ is partially ordered by the usual pointwise ordering of functions, i.e., $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for D^+ in this order is the distribution function given by

$$\epsilon_0(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0 \end{cases} .$$

Definition 11. *A binary operation $T : [0,1]\times[0,1] \to [0,1]$ is continuous t-norm if T satisfies the following conditions:*

- (a) *T is commutative and associative;*
- (b) *T is continuous;*
- (c) *$T(a,1) = a$ for all $a \in [0,1]$;*
- (d) *$T(a,b) \leq T(c,d)$ whenever $a \leq c$ and $b \leq d$, and $a,b,c,d \in [0,1]$.*

Examples of t-norm are $T(a,b) = \min\{a,b\}$ and $T(a,b) = ab$.

Definition 12. *A Menger probabilistic metric space (briefly, Menger PM-space) is a triple (X,F,T), where X is a nonempty set, T is a continuous t-norm, and F is a mapping from $X \times X$ into D^+ such that, if $F_{x,y}$ denotes the value of F at the pair (x,y), the following conditions hold:*

- $(PM1)$ *$F_{x,y}(t) = \epsilon_0(t)$ if and only if $x = y$;*
- $(PM2)$ *$F_{x,y}(t) = F_{y,x}(t)$;*
- $(PM3)$ *$F_{x,z}(t+s) \geq T(F_{x,y}(t), F_{y,z}(s))$ for all $x,y,z \in X$ and $s,t \geq 0$.*

Recently, Pant et al [9] obtained the following theorem:

Theorem 13 (Theorem 3.2 of [9])**.** *Let (X,F,T) be a complete Menger PM-space, and let f be self-mapping of X satisfying the conditions:*

- (i) *For every $\epsilon \in (0,1)$, there exists $\delta \in (0,\epsilon]$, such that*

$$\epsilon - \delta < \min\{F_{x,fx}(t), F_{y,fy}(t)\} < \epsilon \Rightarrow F_{fx,fy}(t) \geq \epsilon.$$

- (ii) $F_{fx,fy}(t) > \min\{F_{x,fx}(t), F_{y,fy}(t)\}$, *for all $x, y \in X$. If f is k-continuous or f^k is continuous or f is orbitally continuous, then f has a unique fixed point.*

Now, we obtain our result for weaker continuity condition.

Theorem 14. *Let (X, F, T) be a complete Menger PM-space, and let f be self-mapping of X satisfying the conditions (i) and (ii) of Theorem 13. If f is weakly orbitally continuous, then f has a unique fixed point.*

Proof: Let x_0 be any point in X. Define a sequence $\{x_n\}_{n\in N}$ in X recursively by $x_n = fx_{n-1}$. Following the proof of Theorem 3.2 in [9], it is easy to see that $\{x_n\}_{n\in N}$ is a Cauchy sequence. Since (X, F, T) is complete, there exists a point z in X, such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} fx_n = z$.

Suppose that f is weakly orbitally continuous. Since $\{f^n x_0\}$ converges for each x_0 in X, weak orbital continuity implies that there exists $y_0 \in X$ such that $f^n y_0 \to t$ and $f^{n+1} y_0 \to ft$ for some t in X. This implies that $t = ft$. Thus t is a fixed point of f.

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