Combinatorial Problem in North Indian Music System with 16- Musical Tone

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Abstract

Since ancient times mathematics and music have been closely associated. It is true that problems arise in the theory of music, are of a mathematical nature. The objective of the present paper is to use enumerative combinatorics. It deals with the area of combinatorics, related to the number of ways that certain pattern can formed. In this paper by taking 16-musical tones we created the raga in north Indian music system and create a generating function by dividing 12- musical notes in 96 equal temperament. It includes the problem finding number of various combinations to play the 16-tone piano by Julian.

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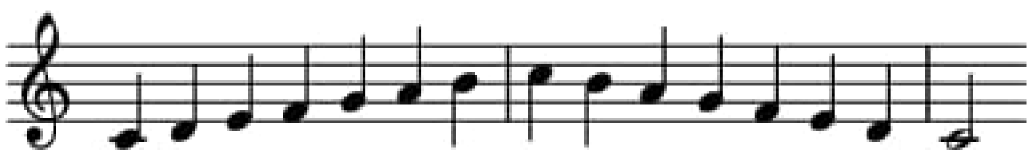
1 Introduction

In Indian music system there are two major traditions; the north Indian classical music or Hindustani music and South Indian music or carnatic music. Indian classical music has two foundational element, raga and tala. The raga based on swara (notes including microtones), forms the fabric of a melodic structure, while the tala measures the times cycle. The sequence of notes in all music system, is known as a chord, can be mapped into integral sequence and the ascent sequence in the Indian music system is called an aarohan while the sequence is descent is called the avarohan. In Indian music system each raga to be a part of a unique sequence of form notes in the ascent and descent that determines the characteristic of the raga and the musical forms and composition that originate from the raga. In general composition of north Indian music and other forms of musical improvisations must contain the notes, which are included in the scales of raga except that from ornamentation and grace, other notes and microtones could be added as in “Gamakas”[12] (distantly analogous to vibrato of western music) of north Indian music system. The overall link between north Indian music and combinatorics includes vast area of several studies. The construction of raga there are some grammatical rules are govern, notes are the important part of raga a set of notes are called scales. Also “Shruti”[12] is an important concept of in music, where it means the smallest interval of pitch, that the singer or musical instrument can produce. The scale may have four, five, six or seven notes (swara). The aarohan or avrohan of a raga that have four notes are called "surtara"(tetra tonic)raga. Tetrachord ragas are not common, they do exist, as exemplified by the raga "Mahathi "[2], those have five notes are called "audava"(pentatonic) raga, those have six notes are called "shadava"(hex-tonic) raga and with seven notes is called "sampurna"(heptatonic) raga or the complete scale (octave completed $\hat{S}$ included). The aarohan and avarohan of a raga's can have a number of combination of scales chosen from the 12-note system enumerated in table (I) forming an integer sequence. In the scale is uniform in both aarohan and avarohan without any repetition of note, that raga is considered as "non kinky"(or non vakra) raga. Here considered the enumeration of only non "vakra"raga. Now create the scale of non "vakra"raga in north Indian music system are constructed by selecting the 11 notes (excluding the upper Sa or C (in western), noted as $\hat{S}$) in table (I). When 11 notes are mapped into integers then in combinatorial terms this corresponds to the enumeration of integer sequence under constraints and equivalence as adherent by the theory of north Indian music. The aim of this paper is to construct the mathematical foundation for such an entire and yet non repetitive enumeration and generating function provides a base for the formulation of new ragas.

2 Preliminaries

Definition 1. *(Scale): In music theory a scale is any set of musical notes or pitch. An octave can be completed by 12 notes. Scale can be defined as a subset of {0,1,2,3,4,5,6,7,8,9,10,11} arranged in ascending order. Transposition of a scale is a subset obtained by a mapping of the form $x \to x + \alpha (mod\ 12)$. Where α is constant.*



"a C major scale ascending and descendig"

Definition 2. *(Derangement) :In combinatorial mathematics, a derangement is a permutation of the set σ of n elements with no fixed point i.e. $\sigma(i) \neq i$ for all $i \in \{1, 2, ..., n\}$.*

Definition 3. *(Enumerative combinatorics): Enumerative combinatorics is an area of combinatorics that deals with the number of ways that certain patterns can be formed. This type of problem are counting combination and counting permutation. k- Permutations are the ordered arrangements of k distinct elements selected from a set. When k is equal to the size of the set, these are the permutation of the set. And a k- combination of a set S is a subset of k distinct elements of S. if the set has n element, the number of k- combination is equal to the binomial coefficient.*

$$\binom{n}{k} = \frac{n(n-1)...(n-k+1)}{k(k-1)...1}$$

Where $k \leq n$, and which is zero when $k > n$.

Definition 4. *(Inclusion exclusion or sieve formula): Let A_1, A_2, ... An be subset of a finite set S. Let $N_0 = |S|$ and N_j be the sum of size of all j-way intersections* $N_j = \sum_{1 \le i_1 < i_2 < ... < i_j \le n} |A_{i_1} \cap A_{i_2} \cap ... \cap A_{i_j}|$ *for* $j = 1, 2, ..., n$. *Then* $|A_1 \cup A_2 \cup ... \cup A_n| = N_1 - N_2 + N_3 - N_4 ... \pm N_n = \sum (-1)^{j-1} N_j ; j = 1, 2, ...n$

Definition 5. *(Generating function):*
1. "A generating function is a device somewhat similar to a bag. Instead of carrying many little objects detachedly, which could be embarrassing, we put them all in a bag, and then we have only one object to carry, the bag"(George Pólya)[10].

2. "A generating function is a clothesline on which we have up a sequence of numbers for display"(Herbert Wilf) [5]

3. Generating function is used to describe families of combinatorial objects. Let $\mathcal{F}$ denoted the family of objects and let $F(x)$ be its generating function. Then $F(x) = \sum f_n x^n$, $n \geq 0$. Where f_n is the number of combinatorial objects of size n. The number of combinarorial objects of size n is therefore given by the coefficient of x_n. For two families $(\mathcal{F} \times \mathcal{G})$ has generating function $F(x) \times G(x)$.

3 North Indian music

Pandit Vishnu Nārāyan Bhātkhande [7] was an Indian musicologist who wrote the first modern treatise on north Indian classic music, an art which had been propagated earlier for a few centuries mostly through oral tradition. During that earlier time, the art had undergone several changes, rendering the raga grammar document in scant old outdated texts. Raga used to classify into raga (male), ragini (female) and putra (children). Bhātkhande reclassified them into currently used that system. In Indian classical music there are 32 thaat but only 10 thaat are used in construction of raga. In north Indian or Hindustani music thaat is a "parent scale". Thaat may consist of up to seven scale degrees or swara. In Indian music there are seaven notes $Sa - Re - Ga - Ma - Pa - Dha - Ni$. These seven notes are the foundation of music. As in $table(I)$ $Sa - Re - Ga - Ma - Pa - Dha - Ni$ the last being $\hat{S}$ (Sa or C high in Western) and has an octavial relation to S. Octave means "saptak"there are three octave or 3 "saptak"in Indian classical music. First is lower octave is called "Mandra saptak"(the dot below the swara).

$\underset{\cdot}{Ni}, \quad \underset{\cdot}{Dha}, \quad \underset{\cdot}{Pa}, \quad \underset{\cdot}{Ma}, \quad \underset{\cdot}{Ga}, \quad \underset{\cdot}{Re}$

Second is middle octave is called "Maddhya saptak"(there is no symbol)

$Re, \quad Ga, \quad Ma, \quad Pa, \quad Dha, \quad Ni$

And third is upper octave is called "Taar saptak"(the dot above the swara)

$\dot{Re}, \quad \dot{Ga}, \quad \dot{Ma}, \quad \dot{Pa}, \quad \dot{Dha}, \quad \dot{Ni}$

Swara Re, Ga, Ma, Dha, and Ni can be either natural (shudha) or flat (komal) or sharp (tivra), Sa and Pa are achal swara. It starts from "middle octave"after that in "lower octave"and then "upper octave"with its ascending notes (aarohan) and descending notes (avrohna).

4 Julian Carrillo Piano

In north Indian music system there is 12-notes. A system of tuning is or an equal temperament is a musical temperament, in which the frequency interval between every pair of adjacent notes has the same ratio. In classical music and western music generally considered the 12-tone equal temperament also known as 12- equal temperament, which divide octave into 12 equal parts. Some music divide octave differently. In 1940 ,Carrillo[8] patented fifteen metamorphose piano for producing whole tones, third tones, quarter tones, fifth tones, sixth tones, seven, eight, nine,..., sixteenth tones. The 16th tone piano is an upright piano with 97 keys. It is tuned to 96 - equal temperament. The interval between adjacent keys is a 16^{th} –tone (12.5 cents), there are 8 steps within each equal tempered semitones.

5 Generating function of raga using Combinatorics

In north Indian music system there are 12 notes of the octave S, $R1$, $R2$, $G1$, $G2$, $M1$, $M2$, P, $D1$, $D2$, $N1$, $N2$ the raga must have S and P, one of the M's ,one each of the R's and G's and one each of the D's and N's. Also R must necessarily precede G and D must necessarily proceed N. For constructing generating function (or generating sequence) of raga, consider the enumeration of all possible "non vakra"ragas for the scale is pentatonic, hex tonic and so on. In north Indian music system there are different types of scale such as audava (ascent)-audava (descent), audava (ascent)- shaadava (descent), audava (ascent)- sampurna (descent) etc., can be accomplished utilizing powerful enumerative combinatorics function. We shall construct a "Pattern inventory"in Pólya's term of all such non –vakra or non kinky ragas of north Indian music system. We construct generating function for the ascent and multiple corresponding generating functions for the descent to get the complete pattern inventory of ragas. The maximal chord length is allowed is 7 in a sampurana non- vakra type. The aarohan and avarohn must contain the combination of notes. This gives the $2 \times 4 \times 4 = 32$ thaat in north Indian music. Same as in carnatic music gives $2 \times 6 \times 6 = 72$ Melakrata raga (Thaat). If all notes occur in a heptatonic sequence that is S, R, G, M, P, D and N. For the enumeration of hex tonic aarohana , first the patterns are enumerated for the hex tonic scale as shown in *table* (II) and then the numbers for each pattern. A hex tonic pattern such as $S\ G\ M\ P\ D\ N\ \hat{S}$ canbe mathematically characterized as $\bar{R}$, since it is missing R (known as rishabh) from a complete heptatonic scale. Thus there are six pattern characterized by $\bar{R}$, $\bar{G}$, $\bar{M}$, $\bar{P}$, $\bar{D}$, and $\bar{N}$. Note that S cannot be missing from a raga also we have construct generating functions for 96-equal temperament. First construct hex tonic ascent generating function as: (here $ascH$

denote the ascent generating function and $dscH$ denote the descent generating function for hex tonic)

$$ascH = 16\bar{R} + 48\bar{G} + 48\bar{M} + 96\bar{P} + 16\bar{D} + 48\bar{N}$$
$$dscH = 48\bar{N} + 16\bar{D} + 96\bar{P} + 48\bar{M} + 48\bar{G} + 16\bar{R}.$$

In this enumeration the total number of hex tonic aarohans is obtained by substituting $\bar{R} = 1$, $\bar{G} = 1$,...$\bar{N} = 1$ in the above expressions which yields 272 hex tonic aarohan and 272 hex tonic avaorhan.

The pentatonic scale are those that have two missing notes relative to the heptatonic scale and are thus denoted in mathematically terms such as $\bar{R}\bar{G}$, $\bar{R}\bar{M}$,...etc. as enumerated in *table* (III). The patterns are shown in *table* (III), and the generating function for the pentatonic ascent is given by (here $ascP$ denoted the pentatonic ascent generating function. Similarly we can generate $dscP$)
$ascP = 16\bar{R}\bar{G} + 8\bar{R}\bar{M} + 16\bar{R}\bar{P} + 2\bar{R}\bar{D} + 8\bar{R}\bar{N} + 24\bar{G}\bar{M} + 48\bar{G}\bar{P} + 8\bar{G}\bar{D} + 24\bar{G}\bar{N} + 48\bar{M}\bar{P} + 8\bar{M}\bar{D} + 24\bar{M}\bar{N} + 16\bar{P}\bar{D} + 48\bar{P}\bar{N} + 16\bar{D}\bar{N}.$

In the above enumeration scheme replacing all binomial terms by 1 or equivalently summing the coefficient gives the total number of symmetric pentatonic ragas or pentatonic ascents as 314.
Although ragas with tetra tonic (tetra chord) scale are rare, they do occur as illustrated before, and thus they are enumerated here for completeness. Such enumerations can also be useful in computer synthesis of musical tetra tonic generating function is given by

$ascT = 8\bar{R}\bar{G}\bar{M} + 16\bar{R}\bar{G}\bar{P} + 2\bar{R}\bar{G}\bar{D} + 8\bar{R}\bar{G}\bar{N} + 8\bar{R}\bar{M}\bar{P} + \bar{R}\bar{M}\bar{D} + 4\bar{R}\bar{M}\bar{N} + 2\bar{R}\bar{P}\bar{D} + 8\bar{R}\bar{P}\bar{N} + 2\bar{R}\bar{D}\bar{N} + 24\bar{G}\bar{M}\bar{P} + 4\bar{G}\bar{M}\bar{D} + 12\bar{G}\bar{M}\bar{N} + 8\bar{G}\bar{P}\bar{D} + 24\bar{G}\bar{P}\bar{N} + 8\bar{G}\bar{D}\bar{N} + 8\bar{M}\bar{P}\bar{D} + 24\bar{M}\bar{P}\bar{N} + 8\bar{M}\bar{D}\bar{N} + 8\bar{P}\bar{D}\bar{N}.$

Thus the total number of tetra- tonic scales, also referred to in music theory as tetra chords (sequence of 4 notes), is obtained by substituting all trinomial ascent is 187.
The tri-chords (triplets) or sequence of three notes, one of which is S, are enumerated by the expression for $ascTr$.

$ascTr = 8RG + 8RM + 4RP + 12RD + 4RN + 2GM + GP + 4GD + GN + 2MP + 8MD + 2MN + 4PD + PN + 8DN.$

In the above expression instead of complementary notation, the notes themselves are used for the binomials, for example RG to denote the sequence SRG in the tetra chord. Thus the total number of tetra chord is obtained by adding the coefficient in $ascTr$ which equal 69. The number of di-chords or a sequence of 2 notes, one of which has to be S, is simply 11 since that is the number of distinct notes in table I (note $\hat{S}$ is related to S by an octave).

The equivalence classes of ragas in each such pattern are enumerated using the principle of inclusion and exclusion or the sieve formula [3] [11] [13] such as derangements or the problem of m*é*nage or Euler function, generates the number of primes to any integer n and less than n.

Let $p_1, p_2, p_3, ..., p_n$ be a set of n constraint stipulated by the north Indian music theory. Then the generating function F for the enumeration such that none of the constraints $p_1, p_2, p_3, ..., p_n$ is satisfied is given by the Sieve formula.

$F = f(0) - f(1) + f(2) - f(3) + ... + (-1)^i f(i) + ...(-1)^n f(n)$, where $f(i)$ denotes the generating function for the enumeration that satisfies exactly i of the properties $p_1, p_2, p_3, ..., p_n$.
To illustrate, the number of symmetrical "sampurna- sampurna"raga in the north Indian music system, the numbers $f(0), f(1), f(2), f(3), ..., f(6)$ are obtained as:
$F = f(0) - f(1) + f(2) - f(3) + f(4) - f(5) + f(6)$

The above enumeration is a straight forward application since all notes occur in a heptatonic sequence, that is S, R, G, M, P, D, N occur.

Theorem 1. *Let $\mathcal{F}$ and $\mathcal{G}$ be the families of objects let $F(x)$ and $G(x)$ be their generating functions. Then $C = \mathcal{F} \times \mathcal{G}$ has generating function $C(x) = F(x) \times G(x)$.*

Proof. Let c_n be the number of objects of size n in the Cartesian product $C = \mathcal{F} \times \mathcal{G}$. These objects $c = (f, g)$ are obtained by picking an object $f \in \mathcal{F}$ of size $k \leq n$ and an object $g \in \mathcal{G}$ of size $n - k$. thus $c_n = f_k g_{n-k}$,$0 \leq k \leq n$
now consider the product of generating functions;

$$F(x) \times G(x) = (\sum f_k x^k) \times (\sum g_k x^k); k \geq 0.$$

In order to get a monomial x^n in this product, one must multiply a monomial $f_k x^k$ for $k \leq n$ from the first sum with a monomial $g_{n-k} x^{n-k}$ from the second sum. Thus one has

$$F(x) \times G(x) = \sum_{n \geq 0} (\sum f_k g_{n-k}) x^n; 0 \leq k \leq n.$$

□

All of the above expression can be combined into a pattern inventory of aarohans of raga that we refer to as a raga ascent inventory, $ascI$, gives as a polynomial in x, where x^n denote the term for n-tonic ascent.

$$ascI = 1 + x + 11x^2 + 69x^3 + 187x^4 + 314x^5 + 272x^6 + 32x^7$$

Where term first is a trivial null set, the second term corresponds to a single note or just S, the term x^2, x^3, x^4, x^5, x^6, x^7 are the number of di-chords, tri-chords, tetra chords, pentatonic chords, hex -tonic chords and heptatonic chords respectively. For a raga to be stable its scale must have at least a tetra chord, and thus terms with powers more than or equal to 4 are relevant for the scales of ragas.
For descent, raga inventory enumerated by the generating function dscI given by

$$dscI = 1 + y + 11y^2 + 69y^3 + 187y^4 + 314y^5 + 272y^6 + 32y^7$$

The total generating function for all of the ragas is given by the product of the ascent and descent inventories or

$$\begin{aligned} I \quad = ascI \times dscI = \quad & (1 + x + 11x^2 + 69x^3 + 187x^4 + 314x^5 + 272x^6 + 32x^7) \times \\ & (1 + y + 11y^2 + 69y^3 + 187y^4 + 314y^5 + 272y^6 + 32y^7) \end{aligned}$$

In the above generating function the coefficient of $x^n y^m$ enumerates the number of ragas with n- tonic notes in ascent omitting higher octave $\hat{S}$ and m- tonic in

the descent. For example in the above expression coefficient of x^6y^7 is the number of shadava- sampurna ragas, which is 8704. The coefficient of x^4 is the number of symmetrical tetra chord which is 187 and the coefficient of x^4y^4 is the total number of all tetra chord ragas, which are 34969. For the audava, shaadava and sampurna scales all of the symmetrical ragas are enumerated by the terms x^5, x^6 and x^7, respectively. The number of ragas with at least pentatonic or higher scales in the ascent and descent is enumerated in table (IV). It should be mentioned that Pattamal [9] has proposed a scientific naming scheme for some of the ragas, and the numbers obtained before are not rigorously correct as these empirical methods either missed some of not combination or those methods do not fully consider equivalence restrictions. In the present scheme, we have carefully provided a mathematical arrangement within combinatorial principles that provides equivalence, symmetry and other restrictions and it is yet exhaustive as the polynomial inventory rigidly considers all of the combinations.
More detailed combinatorial generating functions can be constructed by considering the generating functions, $ascH$, $ascP$, $ascT$, $ascTr$. for example detailed enumerated for the pentatonic- hex tonic ragas is given by

$$ascP \times dscH = (16\bar{R}\bar{G} + 8\bar{R}\bar{M} + 16\bar{R}\bar{P} + 2\bar{R}\bar{D} + 8\bar{R}\bar{N} + 24\bar{G}\bar{M} + 48\bar{G}\bar{P} + 8\bar{G}\bar{D} + 24\bar{G}\bar{N} + 48\bar{M}\bar{P} + 8\bar{M}\bar{D} + 24\bar{M}\bar{N} + 16\bar{P}\bar{D} + 48\bar{P}\bar{N} + 16\bar{D}\bar{N}) \times (16\bar{R}' + 48\bar{G}' + 48\bar{M}' + 96\bar{P}' + 16\bar{D}' + 48\bar{N}').$$

The above expression enumerated all combination of pentatonic- hex tonic ragas. For example, the number of raga missing G and $\bar{N}$ in ascent and raga M missing in descent is given by the coefficient of G $\bar{N}$ $\bar{M}$, which is 1152 consequently, combining different ascent generating functions with different descent generating functions, all ragas, both symmetrical and unsymmetrical are enumerated.
Table (*II*) and *Table* (*III*) contain the detailed enumerations for the most common ragas of different types. Science the number of heptachord is 32, the number of heptatonic with any combination is obtained by multiplying the corresponding generating function by 32.

Conclusion

Thus construction of combinatorial generating function of pentatonic, hex- tonic raga are formulated and detailed analysis is given in this paper with the help of such expression one can find number of ragas missing some swars in ascent and in descent order, here generating function contains all symmetrical and unsymmetrical tetra chords, tri- chords, bi- chords, etc.

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Table – I Notation of Notes in Hindustani (North Indian), mathematical, western and South Indian (Carnatic) music systems.

North Indian	Math	Western	South Indian
Sa	S (static)	C	Sa
Komal Re	R1	D flat	Ra (shudha)
Shudha Re	R2	D natural	Ri (chatusruthi)
Komal Ga	G1	D sharp	Ru (shatsruthi)
Shudh Re	R2	E double flat	Ga (shudha)
Komal Ga	G1	E flat	Gi (sadharana)
Shudha Ga	G2	E natural	Gu (anthara)
Shudha Ma	M1	F natural	Ma (shudha)
Tivra Ma	M2	F sharp	Mi (prathi)
Pa	Pa	G	Pa (panchamam)
Komal Dha	D1	A flat	Dha (shudha)
Shudha Dha	D2	A natural	Dhi (chatusrithi)
Komal Ni	N1	A sharp	Dhu (shatsruthi)
Shudha Dha	D2	B double flat	Na (shudha)
Komal Ni	N1	B flat	Ni(kaisiki)
Shudha Ni	N2	B natural	Nu (kakali)
Sa (high)	Sa	C (Higher)	Sa (high)

Table – II shadva (hexatonic) ragas (scales) of North Indian music system number in parentheses are symmetrical (i.e. same descent and ascent) ragas[a]

Arohana(ascent)	Avrohana (descent)	Polynomial	Number
S R G M P D $\hat{S}$	$\hat{S}$ D P M G R S	$\bar{N}^2$	2304 (48)
S R G M P N $\hat{S}$	$\hat{S}$ D P M G R S	$\bar{D}\bar{N}$	768
S R G M D N $\hat{S}$	$\hat{S}$ D P M G R S	$\bar{P}\bar{N}$	4608
S R G P D N $\hat{S}$	$\hat{S}$ D P M G R S	$\bar{M}\bar{N}$	2304
S R M P D N $\hat{S}$	$\hat{S}$D P M G R S	$\bar{G}\bar{N}$	2304
S G M P D N $\hat{S}$	$\hat{S}$D P M G R S	$\bar{R}\bar{N}$	768
S R G M P D $\hat{S}$	$\hat{S}$N P M G R S	$\bar{N}\bar{D}$	768
S R G M P N $\hat{S}$	$\hat{S}$ N P M G R S	$\bar{D}^2$	256 (16)
S R G M D N $\hat{S}$	$\hat{S}$ N P M G R S	$\bar{P}\bar{D}$	1536
S R G P D N $\hat{S}$	$\hat{S}$ N P M G R S	$\bar{M}\bar{D}$	768
S R M P D N $\hat{S}$	$\hat{S}$ N P M G R S	$\bar{G}\bar{D}$	768
S G M P D N $\hat{S}$	$\hat{S}$ N P M G R S	$\bar{R}\bar{D}$	256
S R G M P D $\hat{S}$	$\hat{S}$ N D M G R S	$\bar{N}\bar{P}$	4608
S R G M P N $\hat{S}$	$\hat{S}$ N D M G R S	$\bar{D}\bar{P}$	1536
S R G M D N $\hat{S}$	$\hat{S}$ N D M G R S	$\bar{P}^2$	9216 (96)
S R G P D N $\hat{S}$	$\hat{S}$ N D M G R S	$\bar{M}\bar{P}$	4608
S R M P D N $\hat{S}$	$\hat{S}$ N D M G R S	$\bar{G}\bar{P}$	4608
S G M P D N $\hat{S}$	$\hat{S}$ N D M G R S	$\bar{R}\bar{P}$	1536
S R G M P D $\hat{S}$	$\hat{S}$ N D P G R S	$\bar{N}\bar{P}$	2304
S R G M P N $\hat{S}$	$\hat{S}$ N D P G R S	$\bar{D}\bar{M}$	768
S R G M D N $\hat{S}$	$\hat{S}$ N D P G R S	$\bar{P}\bar{M}$	4608
S R G P D N $\hat{S}$	$\hat{S}$ N D P G R S	$\bar{M}^2$	2304 (48)
S R M P D N $\hat{S}$	$\hat{S}$ N D P G R S	$\bar{G}\bar{M}$	2304
S G M P D N $\hat{S}$	$\hat{S}$ N D P G R S	$\bar{R}\bar{M}$	768
S R G M P D $\hat{S}$	$\hat{S}$ N D P M R S	$\bar{N}\bar{G}$	2304
S R G M P N $\hat{S}$	$\hat{S}$ N D P M R S	$\bar{D}\bar{G}$	768
S R G M D N $\hat{S}$	$\hat{S}$ N D P M R S	$\bar{P}\bar{G}$	4608
S R G P D N $\hat{S}$	$\hat{S}$ N D P M R S	$\bar{M}\bar{G}$	2304
S R M P D N $\hat{S}$	$\hat{S}$ N D P M R S	$\bar{G}^2$	2304 (48)
S G M P D N $\hat{S}$	$\hat{S}$ N D P M R S	$\bar{R}\bar{G}$	768
S R G M P D $\hat{S}$	$\hat{S}$ N D P M G S	$\bar{N}\bar{R}$	768
S R G M P N $\hat{S}$	$\hat{S}$ N D P M G S	$\bar{D}\bar{R}$	256
S R G M D N $\hat{S}$	$\hat{S}$ N D P M G S	$\bar{P}\bar{R}$	1536
S R G P D N $\hat{S}$	$\hat{S}$ N D P M G S	$\bar{M}\bar{R}$	768
S R M P D N $\hat{S}$	$\hat{S}$ N D P M G S	$\bar{G}\bar{R}$	768
S G M P D N $\hat{S}$	$\hat{S}$ N D P M G S	$\bar{R}^2$	256 (16)

Table III Pentatonic ragas (scale) of north Indian music. Only Arohana (ascent) are shown. The complete set is obtained by the combinatorial generating function (equation) in the text.

Polynomial	Arohana(ascent)	Number
RG	SMPDN	16
RM	SGPDN	8
RP	SGMDN	16
RD	SGMPN	2
RN	SGMPD	8
GM	SRPDN	24
GP	SRMDN	48
GD	SRMPN	8
GN	SRMPD	24
MP	SRGDN	48
MD	SRGPN	8
MN	SRGPD	24
PD	SRGMN	16
PN	SRGMD	48
DN	SRGMP	16

Table IV Enumeration of 381924 Combinations of Ragas With Pentatonic or higher scales.

Arohan(Ascent)	Avarohan(descent)	Numbera
Sampuran	Sampuran	1024(32)
Sampuran	Shadava	8704
Sampuran	Audava	10048
Shadava	Sampuran	8704(272)
Shadava	Shadava	73984
Shadava	Audava	85408
Audava	Sampuran	10048
Audava	Shadava	85408
Audava	Audava	98596(314)
	Grand Total:	381924

(ascent) Number in parentheses is the numbers of symmetrical ragas, wherein the ascent and descent exhibit a mirror symmetry and are thus non- bhshanka ragas.