Matrix Summability of the Conjugate Series of Derived Fourier Series

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Abstract

In this paper, a new theorem on matrix summability of the conjugate series of a derived Fourier seires is proved, which generalizes some of previous known results.

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1. Introduction

There are so many known results Nörlund summability of Fourier series and derived series of Fourier series. In 1963, for the first time Tripathi [5] established a theorem for harmonic summability of conjugate series of derived Fourier series. In 1972, Tripathi and Prasad [6] extended Tripathi for (N, p_n) summability which is weaker than Harmonic summability. Several researchers like Tripathi and Singh [8], Tripathi and Prasad [7] and Sulaimann [3] studied (N, p_n) summability of conjugate series of derived Fourier series and conjugate series of Fourier series by matrix means. Our theorem generalizes all these results.

1. Definitions and Notations

Let f be 2πperiodic function in$[-\pi, \pi]$. The Fourier series associated with f at a point x is defined as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

The conjugate series of the above series is given by

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x)$$

Then the conjugate series of derived Fourier series is

$$-\sum_{n=1}^{\infty} n \ (a_n \cos nx + b_n \sin nx) \tag{0.1}$$

Let $\sum u_n$ be a given infinite series with sequence of partial sums$\{S_n\}$. Let $T = (a_{n,k})$ be an infinite triangular matrix of real Constant.The sequence- to- sequence transformation.

$$t_n = \sum_{k=0}^{n} a_{n,k} S_k = \sum_{k=0}^{n} a_{n,n-k} S_{n-k}$$

Defined the sequence $\{t_n\}$ of matrix means of the sequence of the coefficients $(a_{n,k})$.The series $\sum u_n$ is said to be summable to the sum S by matrix method if $\lim_{n\to\infty} t_n$ exists and is equal to$S$.We can write,
$t_n \to S(T)$, as $n \to \infty$.
The necessary and sufficient condition for T – transform to be regular i.e.,

$$\lim_{n\to\infty} S_n \to S \Rightarrow \lim_{n\to\infty} t_n \to S$$

is Silverman - Toeplitz condition. If $a_{n,k} = 0$for every $k > n$,then the matrix is called triangular (Toepitz [14]).

Particular Cases

Particular cases of matrix means are

1. $(C, 1)$ means, when $a_{n,k} = \frac{1}{n+1}$
2. Harmonic means, when $a_{n,k} = \frac{1}{(n-k+1)\log n}$
3. (C, δ) means, when $a_{n,k} = \dfrac{\binom{n-k+\delta+1}{\delta-1}}{\binom{n+\delta}{\delta}}$, $0 \le \delta \le 1$.
4. (H, P) means, when $a_{n,k} = \frac{1}{(\log)^{p-1}(n+1)} \prod_{q=0}^{p-1} \log^q (k+1)$.
5. Nörlund means, $a_{n,k} = \frac{p_{n-k}}{P_n}$, where $\{p_n\}$ is a real non – negative monotonic non – increasing sequence such that $P_n = \sum_{k=0}^{n} p_k \to \infty$,as $n \to \infty$.
6. Riesz means $\left(\bar{N}, p_n\right)$,when $a_{n,k} = \frac{p_k}{P_n}$,where $\{p_n\}$ is a positive and increasing sequence such that

$P_n = \sum_{k=0}^{n} p_k \to \infty$,as $n \to \infty$.

1. Generalized $N\ddot{o}$rlund means (N, p, q) ,when $a_{n,k} = \frac{p_{n-k} q_k}{R_n}$,where $\{p_n\}$ is positive, decreasing sequence and $\{q_n\}$ is a non – negative, non decreasing sequence such that

$R_n = \sum_{k=0}^{n} p_{n-k}\, q_k \rightarrow \infty$,as $n \rightarrow \infty$
We write,

$$h(t) = [f(x+t) - f(x-t) - 2f(x)]$$

$$H(x) = -\frac{1}{4\pi}\int_0^{\pi} h(t)\cos ec^2\, t\,\frac{1}{2}\,dt$$

$$M_n(t) = \frac{1}{2\pi}\sum_{k=0}^{n} a_{n,n-k}\frac{\cos\left(n-k+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \qquad (0.2)$$

τ = Integral part of $\frac{1}{t} = \left[\frac{1}{t}\right]$

1. **Previous Results**

Tripathi [5] has proved a theorem on conjugate derived series of Fourier series
Theorem: The conjugate derived series of the Fourier series of a function $f(x)$is summable by harmonic means to the sum

$$-\frac{1}{4\pi}\int_0^{\pi} h(t)\cos ec^2\, t\,\frac{1}{2}\,dt$$

at every point x where this integral exists and

$$H(t) = \int_0^t |dh(t)| = o\left(\frac{t}{\log\left({}^{1}\!/_{t}\right)}\right), \, as\ t \rightarrow +0$$

1. **Main theorem**

Tripathi [5] etc all has obtained many interesting results on summabilites of conjugate series of derived Fourier series. In this paper, we extend results of Lal [1] and generalize previous results. We prove the following theorem.
Theorem: If,

$$\int_0^t |dh(t)| = o\left(\frac{t\,\phi(t)}{\log\left({}^{1}\!/_{t}\right)}\right), \, as\ t \rightarrow 0, \qquad (4.1)$$

then the conjugate series of the derived Fourier series is summable (T) to the sum $-\frac{1}{4\pi}\int_0^{\pi} h(t)\cos ec^2\, t\,\frac{1}{2}\,dt$, where $\phi(t)$ is a positive monotonically decreasing function of t such that $\frac{t\,\phi(t)}{\log\left({}^{1}\!/_{t}\right)}$ increases monotonically as $t \rightarrow 0$ provided $\{a_{n,k}\}_{k=0}^{\infty}$ be a real non- negative and non- decreasing sequence with respect to $k$such that $T = (a_{n,k})$is an infinite triangular matrix with $a_{n,k} \geq 0$, $A_{n,\tau} = \sum_{k=0}^{r} a_{n,n-k}$, $A_{n,n} = 1$ for each$n \geq 0$.

1. **Lemma:** Our proof of the theorem requires some lemmas.

Lemma (??): Lal [1]
If $a_{n,k}$is non- negative and non- decreasing with k then for $0 \leq a \leq b \leq \infty$, $0 \leq t \leq \pi$ and for any n, we have

$$\left| \sum_{k=0}^{b} a_{n,n-k} \, e^{i(n-k)t} \right| \leq O\left(A_{n,\tau}\right)$$

Lemma (??):

For $\frac{1}{n+1} \leq t \leq \delta < \pi$, we can have

$$M_n(t) = O\left(\frac{A_{n,\tau}}{t}\right), \tau < n$$

Proof: Using equation (0.2)

$$M_n(t) \leq \left| \sum_{k=0}^{n} a_{n,n-k} \frac{\cos\left(n - k + \frac{1}{2}\right)t}{\sin\frac{t}{2}} \right|$$

$$\leq \frac{1}{\sin\left({}^{t}\!/_{2}\right)} \left| Re \sum_{k=0}^{n} a_{n,n-k} \, e^{i\left(n-k+\frac{1}{2}\right)t} \right|$$

$\leq \frac{\pi}{t} \left| Re \sum_{k=0}^{n} a_{n,n-k} \, e^{i(n-k)t} e^{i\frac{t}{2}} \right|$, using Jordan's lemma

$$\leq \frac{\pi}{t} \left| Re \sum_{k=0}^{n} a_{n,n-k} \, e^{i(n-k)t} \right|$$

$= \frac{\pi}{t} O\left(\frac{A_{n,\tau}}{t}\right)$ by lemma (**??**)

$$= O\left(\frac{A_{n,\tau}}{t}\right)$$

1. **Proof of the main theorem:**

It is well known that

$$S_r(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(u) \frac{d}{dx}\left(\sum_{\nu=0}^{r} \nu \sin \nu(u-x)\, du\right)$$

$$= -\frac{1}{\pi}\int_{0}^{\pi} \frac{d}{dt}\left[\frac{\cos\frac{t}{2} - \cos\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right]\{f(x+t) - f(x-t)\}\, dt$$

$$= -\frac{1}{\pi}\int_{0}^{\pi} \frac{\cos\frac{t}{2} - \cos\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\, dh(t)$$

$$= -\frac{1}{2\pi}\left[\int_0^{1/(n+1)} + \int_{1/(n+1)}^{\pi}\right]\left\{\cot\frac{t}{2}(1-\cos rt) + \sin rt\right\} dh(t) \quad , \; r < n.$$

$$= -\frac{1}{2\pi}\int_0^{1/(n+1)} \cot\frac{t}{2}(1-\cos rt)\, dh(t) - \frac{1}{2\pi}\int_{1/(n+1)}^{\pi} \cot\frac{t}{2} dh(t) + \frac{1}{2\pi}\int_{1/(n+1)}^{\pi}\left[\cot\frac{t}{2}\ \cos rt\ - \sin\right.$$

$$(0.3) \qquad = I_1 + I_2 + \frac{1}{2\pi}\int_{1/(n+1)}^{\pi}\left[\cot\frac{t}{2}\cos rt + \sin rt\right] dh(t)$$

We have,

$$I_1 \le \frac{1}{2\pi}\int_0^{1/(n+1)}\left|\left\{\cot\frac{t}{2}(1-\cos rt) + \sin rt\right\}\right| dh(t)$$

$$\le \frac{1}{2\pi}\int_0^{1/(n+1)}\left|\cot\frac{t}{2}\, 2\sin^2\frac{rt}{2}\right| |dh(t)| + \frac{1}{2\pi}\int_0^{1/(n+1)} |\sin rt|\, |dh(t)|$$

$$\le \frac{r^2}{2\pi}\int_0^{1/(n+1)}\left|\cot\frac{t}{2}\, 2\sin^2\frac{rt}{2}\right| |dh(t)| + \frac{r}{2\pi}\int_0^{1/(n+1)} |\sin t|\, |dh(t)|, \quad \because \sin(n\theta) \le n\sin\theta$$

$$= \frac{r^2}{2\pi}\int_0^{1/(n+1)} |\sin t|\, |dh(t)| + \frac{r}{2\pi}\int_0^{1/(n+1)} |\sin t|\, |dh(t)|$$

$$\le \frac{r^2}{2\pi}\int_0^{1/(n+1)} |t|\, |dh(t)| + \frac{r}{2\pi}\int_0^{1/(n+1)} |t|\, |dh(t)| \because |\sin\theta| \le |\theta|$$

$$\le \frac{r}{2\pi}\cdot\frac{r}{(n+1)}\int_0^{1/(n+1)} |dh(t)| + \frac{1}{2\pi}\cdot\frac{r}{(n+1)}\int_0^{1/(n+1)} |dh(t)|$$

$$\le \frac{r}{2\pi}\cdot\int_0^{1/(n+1)} |dh(t)| + \frac{1}{2\pi}\cdot\int_0^{1/(n+1)} |dh(t)| \because r < n$$

$= \frac{r}{2\pi}\, o\left(\frac{\xi(n+1)}{(n+1)\log(n+1)}\right) + \frac{1}{2\pi}\, o\left(\frac{\xi(n+1)}{(n+1)\log(n+1)}\right)$ by (**??**)
$= o(1) + o(1) \quad as \;\; n \to \infty$ by the hypothesis of the theorem

$$= o(1) \quad as \;\; n \to \infty$$

Next, consider I_2

$$I_2 = -\frac{1}{2\pi}\int_{1/(n+1)}^{\pi} \cot\frac{t}{2}\, dh(t)$$

$$I_2 = -\frac{1}{2\pi}\left[h(t)\cot\frac{t}{2}\right]_{1/(n+1)}^{\pi} - \frac{1}{4\pi}\int_{1/(n+1)}^{\pi} h(t)\cos ec^2\frac{t}{2}dt$$

$$I_2 = o(1) + \frac{1}{2\pi}\left[h\left(\frac{1}{n+1}\right)\cdot\frac{\cos\frac{1}{2(n+1)}}{\frac{1}{2(n+1)}}\cdot\frac{\frac{1}{2(n+1)}}{\sin\frac{1}{2(n+1)}}\right] - \frac{1}{4\pi}\int_{1/(n+1)}^{\pi} h(t)\cos ec^2\frac{t}{2}dt$$

$= o(\mathbf{??}) + \frac{1}{2\pi}2(n+1)h\left(\frac{1}{n+1}\right) - \frac{1}{4\pi}\int_{1/(n+1)}^{\pi} h(t)\cos ec^2\frac{t}{2}dt$ by $(\mathbf{??})$

$= o(\mathbf{??}) + o(\mathbf{??}) - \frac{1}{4\pi}\int_0^{\pi} h(t)\cos ec^2\frac{t}{2}dt + \frac{1}{4\pi}\int_0^{1/(n+1)} h(t)\cos ec^2\frac{t}{2}dt$ by 6.3)

Therefore from (6.1, (6.2) and $(\mathbf{??})$, we get

$$S_r(x) - \left(-\frac{1}{4\pi}\int_{-\pi}^{\pi} h(t)\cos ec^2\,dt\right)$$

$$= o(1) + \frac{1}{4\pi}\int_0^{1/(n+1)} h(t)\cos ec^2\frac{t}{2}dt + \frac{1}{2\pi}\int_{1/(n+1)}^{\pi}\frac{\cos\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\,dh(t).$$

$$\sum_{k=0}^{n} a_{n,n-k}\left[S_r(x) - \left(-\frac{1}{4\pi}\int_{-\pi}^{\pi} h(t)\cos ec^2\,dt\right)\right]$$

$$= \sum_{k=0}^{n} a_{n,n-k}\left[o(1) + \frac{1}{4\pi}\int_0^{1/(n+1)} h(t)\cos ec^2\frac{t}{2}dt + \frac{1}{2\pi}\int_{1/(n+1)}^{\pi}\frac{\cos\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\,dh(t)\right].$$

$$t_n(x) - H(x) = o(1) + \frac{1}{4\pi}\int_0^{1/(n+1)} h(t)\cos ec^2\frac{t}{2}dt + \int_{1/(n+1)}^{\pi}\left[\frac{1}{2\pi}\sum_{k=0}^{n}\frac{\cos\left(r+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right]dh(t)$$

Since the conjugate series of the derived Fourier series is summable to $H(x)$,therefore

$$-\frac{1}{4\pi}\int_0^{\pi} h(t)\cos ec^2 t\frac{1}{2}dt = o(1)\ \ as\, n \to \infty$$

Thus

$$t_n(x) - H(x) = o(1) + o(1) + \int_{1/(n+1)}^{\pi} M_n(t)\,dh(t)$$

$$= o(1) + \left[\int_{1/(n+1)}^{\delta} + \int_{\delta}^{\pi}\right] M_n(t)\,dh(t),$$

Where $\delta$is a fixed positive number such that for$t \le \delta$,condition $(\mathbf{??})$ holds
$= o(\mathbf{??}) + I_3 + I_4$, say
Let us consider I_3,using lemma $(\mathbf{??})$, we have

$$I_3 \le \int_{1/(n+1)}^{\delta} |M_n(t)|\,|dh(t)|$$

$$= O\left[\int_{1/(n+1)}^{\delta} \frac{A_{n,\tau}}{t}\,|dh(t)|\right]$$

$$= O\left[\frac{A_{n,\tau}}{t}\, o\left(\frac{t\,\xi\left(1/t\right)}{\log 1/t}\right)\right]_{1/(n+1)}^{\delta} + O\left[\int_{1/(n+1)}^{\delta} \frac{A_{n,\tau}}{t^2}\, o\left(\frac{t\,\xi\left(1/t\right)}{\log 1/t}\right) dt\right] + O\left[\int_{1/(n+1)}^{\delta} \frac{1}{t}\, o\left(\frac{t\,\xi\left(1/t\right)}{\log 1/t\right.}\right.$$

Integration by parts and using (**??**)

$$= O\left[A_{n,\tau}\,\frac{\xi\left(1/t\right)}{\log 1/t}\right]_{1/(n+1)}^{\delta} + o\left[\left(\frac{\xi(n)}{n\log n}\right)\int_{1/(n+1)}^{\delta} \frac{A_{n,\tau}}{t^2}\,dt\right] + o\left[\left(\frac{\xi(n)}{n\log n}\right)\int_{1/(n+1)}^{\delta} \frac{1}{t}\,d\left(A_{n,\tau}\right)\right]$$

By hypothesis of the theorem

$= o(\mathbf{??}) + o\left[A_{n,n}\,\frac{\xi(n)}{\log n}\right]_{1/(n+1)}^{\delta} + o\left[\left(\frac{\xi(n)}{n\log n}\right)\int_{1/\delta}^{(n+1)} A_{n,u}\,du\right] + o\left[\left(\frac{\xi(n)}{n\log n}\right)\int_{1/\delta}^{(n+1)} u\,d\left(A_{n,u}\right)\right]$, *take* $\frac{1}{t}$

$u = o(\mathbf{??}) + o\left[O(\mathbf{??})\,\frac{\xi(n)}{\log n}\right] + o\left[\left(\frac{\xi(n)}{n\log n}\right)\left(u\,A_{n,u}\right)_{1/\delta}^{(n+1)}\right] + o\left[\left(\frac{\xi(n)}{n\log n}\right)\int_{1/\delta}^{(n+1)} u\,d\left(A_{n,u}\right)\right] +$
$o\left[\left(\frac{\xi(n)}{n\log n}\right)\int_{1/\delta}^{(n+1)} u\,d\left(A_{n,u}\right)\right]$ Integration by parts

$$= o(1) + o\left[\frac{\xi(n)}{\log n}\right] + o\left[\frac{\xi(n)}{n\log n}\,n\,A_{n,n}\right] + o(1) + o\left[\left(\frac{\xi(n)}{n\log n}\right)\int_{1/\delta}^{(n+1)} u\,d\left(A_{n,u}\right)\right]$$

$$= o(1) + o\left[\frac{\xi(n)}{\log n}\right] + o\left[\frac{\xi(n)}{\log n}\right] + o(1) + o\left[\left(\frac{\xi(n)}{n\log n}\right) n\int_{1/\delta}^{(n+1)} d\left(A_{n,u}\right)\right]$$

$$= o(1) + o\left[\frac{\xi(n)}{\log n}\right] + o\left[\left(\frac{\xi(n)}{n\log n}\right)\sum_{k=0}^{n} a_{n,k}\right]$$

$$= o(1) + o\left[\frac{\xi(n)}{\log n}\right] + o\left[\left(\frac{\xi(n)}{\log n}\right) A_{n,n}\right]$$

$$= o(1) + o(1) + o\left[\frac{\xi(n)}{\log n}\,O(1)\right]$$

$$= o(1) + o(1) + o(1)$$

$$= o(1), \quad as\ n \to \infty \tag{0.4}$$

Lastly, by the Riemann- Lebesque theorem and the regularity condition of matrix summability, we obtain

$$I_4 \leq \int_{\delta}^{\pi} |M_n(t)| \, |dh(t)| = o(1), \; as\; n \to \infty. \tag{6.6}$$

Thus from (**??**) , (0.4) and (**??**), we get

$$t_n - \left(-\frac{1}{4\pi} \int_0^{\pi} h(t) \cos ec^2 t \frac{1}{2} dt \right) = o(1) \;\; as\, n \to \infty.$$

Complete the proof of the theorem.
Particular cases:

1. If we take $a_{n,k} = \frac{1}{\log(n-k+1)}$ and $\xi(t) = 1$then the result of Tripathi [5] become a particular case of our theorem.

2. If we take, $a_{n,k} = \frac{p^{\alpha}_{n-k}}{P^{\alpha}_n}$ and $\phi(t) = \frac{t\, p^{\alpha}_1}{P^{\alpha}_t}$, $\alpha \geq -1$ where $\{p^{\alpha}_n\}$ be a real non – negative sequence such that $p^{\alpha}_n \to \infty$ as$n \to \infty$, then Tripathi and Prasad [6] becomes a particular case of out theorem.

3. If $a_{n,k} = \frac{p_{n-k}}{P_n}$ and$\xi(t) = \frac{\log t\, \lambda(t)}{P_t}$, then the result of Tripathi and Prasad [7] becomes a particular case of main theorem.

4. If $a_{n,k} = \frac{p_{n-k}}{P^{\varepsilon}_n}$ and$\xi(t) = \log t$, where $\{p_n\}$ is a sequence with $p_0 > 0$ and $p_n \geq 0$ for $n > 0$ .For$\varepsilon > -1$,we define$P^{\varepsilon}_n = p^{\varepsilon}_0 + p^{\varepsilon}_1 + p^{\varepsilon}_2 + p^{\varepsilon}_3 + \ldots + p^{\varepsilon}_n$,then the main theorem reduces to Tripathi and Singh [8] theorem.

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