Coefficient Bounds For New Subclasses Defined Using Frasin Differential Operator

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Abstract

In this paper we defined two new subclasses of the function class $\sum$ of bi-univalent and analytic functions in the unit disk which are associated with Frasin differential operator[6]. We find some coefficient bounds for the functions in the defined subclasses. As special cases, well-known results were obtained by varying parameters in the main results.

Keywords: coefficient bounds, bi-univalent, subclasses, differential operator.
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1 Introduction

We denote by A the class of regular functions defined in the open unit disk $\Delta = \{z : |z| < 1\}$ with the normalization conditions $f(0) = f'(0) - 1 = 0$ and the Taylor series expansion,

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

By the Koebe-one quater theorem[4](Theorem.2.3 pg.31) , we know that "The range of every function of the class S contains a disk $\{w : |w| < 1/4\}$". Hence there exists inverse f^{-1} for every function $f \in S$, defined by

$$f^{-1}(f(z)) = z, (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w, (|w| < r_0(f) : r_0(f) \geq 1/4).$$

Where the inverse of f is given by,

$$\begin{aligned} f^{-1}(w) &= w - a_2 w^2 + (2a_2^2 w^2 - a_3)w^3 - (5a_2^2 - 5a_2 a_3 + a_4)w^4 + ... \\ &=: g(w) \end{aligned}$$

A function $f \in A$ is said to be bi-univalent if both f and f^{-1}(its inverse) are univalent in Δ. We denote by $\sum$ the class of bi-univalent and analytic functions

in Δ of the form (1.1).
Using the binomial series,

$$(1-\lambda)^m = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \lambda^j, m \in \mathbb{N} = 1, 2, ... \ \textit{and} \ \ j \in \mathbb{N}_0 = 0, 1, 2, ...$$

Frasin [6] defined the following differential operator for function $f \in A$,

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1_{m,\lambda} f(z) &= (1-\lambda)^m f(z) + (1-(1-\lambda)^m) z f'(z) \\ &= D_{m,\lambda} f(z) \ \ , \ (\lambda > 0; m \in \mathbb{N}) \end{aligned}$$

In general,

$$\begin{aligned} D^n_{m,\lambda} f(z) &= D_{m,\lambda}(D^{n-1}_{m,\lambda} f(z)) \ \ , \ n \in \mathbb{N}_0 \\ &= z + \sum_{k=2}^{\infty} [1 + (k-1) c^m_j(\lambda)]^n a_k z^k \end{aligned}$$

where, $c^m_j(\lambda) = \sum\limits_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \lambda^j$.

Remark

1. For $m = 1$, we get the Al-oboudi differential operator, $D^n_{1,\lambda}$ [1].

2. For $m = \lambda = 1$, we get the Salagean differential operator, D^n [8].

We consider $\mathbb{P}$ to be the class of Caratheodary functions. *i.e.*, for $p(z) \in \mathbb{P}$, $\Re\{p(z)\} > 0$, $p(z)$ is analytic in Δ and have the series expansion

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \ \ , \ \ z \in \Delta$$

Lemma 1. *If $p(z) \in \mathbb{P}$, then $|p_n| \leq 2$ for each $n = 1, 2, ...$*

2 The function class $N^{\zeta,\mu}_{\sum}(m, \gamma, \lambda; \alpha)$

Definition 1. *A function $f(z)$ of the form (1.1) in the function class $\sum$ is said to be in the class $N^{\zeta,\mu}_{\sum}(m, \gamma, \lambda; \alpha)$ if and only if*

$$\left| arg\{(1-\gamma)\Big(\frac{D^{\zeta}_{m,\lambda} f(z)}{z}\Big)^{\mu} + \gamma [D^{\zeta}_{m,\lambda} f(z)]' \Big(\frac{D^{\zeta}_{m,\lambda} f(z)}{z}\Big)^{\mu-1}\} \right| < \frac{\alpha\pi}{2}$$

$(0 < \alpha \leq 1; \lambda > 0, \gamma \geq 1; m \in \mathbb{N}; \zeta \in \mathbb{N}_0; \mu \geq 0; z \in \Delta)$
and

$$\left| arg\{(1-\gamma)\Big(\frac{D^{\zeta}_{m,\lambda} g(w)}{w}\Big)^{\mu} + \gamma [D^{\zeta}_{m,\lambda} g(w)]' \Big(\frac{D^{\zeta}_{m,\lambda} g(w)}{w}\Big)^{\mu-1}\} \right| < \frac{\alpha\pi}{2}$$

$(0 < \alpha \leq 1; \lambda > 0, \gamma \geq 1; m \in \mathbb{N}; \zeta \in \mathbb{N}_0; \mu \geq 0; w \in \Delta)$, *where* $g := f^{-1}$.

Remark 1. *1 If $m = 1$ in Definition.1 we have, $N_{\sum}^{\zeta,\mu}(1,\gamma,\lambda;\alpha) = N_{\sum}^{\lambda,\mu}(\zeta,\alpha,\gamma)$ the class introduced and studied by Sarep Bulut[9].*

2 If $m = \lambda = 1$ in Definition.1 we have, $N_{\sum}^{\zeta,\mu}(1,\gamma,1;\alpha) = N_{\sum}^{\zeta,\mu}(\alpha,\gamma)$ the class introduced and studied by Bilal Seker et al.[10].

3 If $\zeta = 0$ in Definition.1 we have, $N_{\sum}^{0,\mu}(m,\gamma,\lambda;\alpha) = N_{\sum}^{\mu}(\alpha,\gamma)$ the class introduced and studied by Caglar et al.[3].

4 If $\zeta = 0$ and $\mu = 1$ in Definition.1 we have, $N_{\sum}^{0,1}(m,\gamma,\lambda;\alpha) = \mathfrak{B}_{\sum}(\alpha,\gamma)$ the class introduced and studied by Frasin and Aouf[5].

5 If $\zeta = 0$ and $\mu = \gamma = 1$ in Definition.1 we have, $N_{\sum}^{0,1}(m,1,\lambda;\alpha) = H_{\sum}^{\alpha}$ the class introduced and studied by Srivastava et al.[11].

6 If $\zeta = \mu = 0$ and $\gamma = 1$ in Definition.1 we have, $N_{\sum}^{0,0}(m,1,\lambda;\alpha) = S_{\sum}^{*}[\alpha]$ the class of strongly bi-starlike functions of order α, studied by Brannan and Taha[2].

7 If $\mu = 1$ and $m = \lambda = 1$ in Definition.1 we have, $N_{\sum}^{\zeta;1}(1,\gamma,1;\alpha) = \mathfrak{B}_{\sum}(\zeta,\alpha,\gamma)$ the class introduced and studied by Porwal and Darus [7]

3 Coefficient bounds for the function class $N_{\sum}^{\zeta,\mu}(m,\gamma,\lambda;\alpha)$

Theorem 1. *Let the function $f(z)$ given by (1.1) is in the class $N_{\sum}^{\zeta,\mu}(m,\gamma,\lambda;\alpha)$ with $(0 < \alpha \le 1; \lambda > 0, \gamma \ge 1; m \in \mathbb{N}; \zeta \in \mathbb{N}_0; \mu \ge 0 z \in \Delta)$, then*

$$|a_2| \le \frac{2\alpha}{\sqrt{(1 + c_j^m(\lambda))^{2\zeta}(\gamma+\mu)^2 + \alpha A}}$$

and

$$|a_3| \le \frac{2\alpha}{(1 + 2c_j^m(\lambda))^{\zeta}(2\gamma+\mu)} + \frac{4\alpha^2}{(\gamma+\mu)^2(1 + c_j^m(\lambda))^{2\zeta}}$$

where

$$A = [2(1 + 2c_j^m(\lambda))^{\zeta}(2\gamma+\mu) - (1 + c_j^m(\lambda))^{2\zeta}(\gamma^2 + 2\gamma + \mu)]$$

Proof. Definition.1 can be writtern as,

$$(3.1) \quad (1-\gamma)\left(\frac{D_{m,\lambda}^{\zeta} f(z)}{z}\right)^{\mu} + \gamma[D_{m,\lambda}^{\zeta} f(z)]'\left(\frac{D_{m,\lambda}^{\zeta} f(z)}{z}\right)^{\mu-1} = [c(z)]^{\alpha}, \quad (z \in \Delta)$$

$$(3.2) \quad (1-\gamma)\left(\frac{D_{m,\lambda}^{\zeta} g(w)}{w}\right)^{\mu} + \gamma[D_{m,\lambda}^{\zeta} g(w)]'\left(\frac{D_{m,\lambda}^{\zeta} g(w)}{w}\right)^{\mu-1} = [d(w)]^{\alpha}, \quad (w \in \Delta)$$

where $c(z)$ and $d(w)$ are in $\mathbb{P}$ and have the series expansions

$$c(z) = 1 + c_1 z + c_2 z^2 + ...$$

and

$$d(z) = 1 + d_1 z + d_2^2 w^2 + ...$$

Equating the coefficients of z, z^2, w and w^2 by using the above equations in (3.1) and (3.2), we get

$$(1 + c_j^m(\lambda))^\zeta(\gamma + \mu)a_2 = \alpha c_1 \tag{3.3}$$

$$(1+2c_j^m(\lambda))^\zeta(2\gamma+\mu)a_3+(1+c_j^m(\lambda))^{2\zeta}(\mu-1)(\gamma+\frac{\mu}{2})a_2^2 = \alpha c_2+\frac{\alpha(\alpha-1)}{2}c_1^2 \tag{3.4}$$

$$-(1 + c_j^m(\lambda))^\zeta(\gamma + \mu)a_2 = \alpha d_1 \tag{3.5}$$

(3.6)

$$(1+2c_j^m(\lambda))^\zeta(2\gamma+\mu)(2a_2^2-a_3)+(1+c_j^m(\lambda))^{2\zeta}(\mu-1)(\gamma+\frac{\mu}{2})a_2^2 = \alpha d_2+\frac{\alpha(\alpha-1)}{2}d_1^2$$

From (3.3) and (3.5), we obtain

$$c_1 = -d_1 \tag{3.7}$$

and

$$2(1 + c_j^m(\lambda))^{2\zeta}(\gamma + \mu)^2 a_2^2 = \alpha^2(c_1^2 + d_1^2) \tag{3.8}$$

Also from (3.4), (3.6) and (3.8), we have

$$[2(1+2c_j^m(\lambda))^\zeta + (1+c_j^m(\lambda))^{2\zeta}(\mu-1)](2\gamma+\mu)a_2^2 = \alpha(c_2+d_2)+\frac{\alpha(\alpha-1)}{2}(c_1^2+d_1^2)$$

We use (3.8) in the above equation so that

$$a_2^2 = \frac{\alpha^2(c_2 + d_2)}{2\alpha(1 + 2c_j^m(\lambda))^\zeta(2\gamma + \mu) + (1 + \gamma)^{2\zeta}(\gamma + \mu)^2 - \alpha(\gamma^2 + 2\gamma + \mu)} \tag{3.9}$$

Using Lemma.1 from the above equation we get the required inequality for $|a_2|$. Now to find the bound for $|a_3|$, first we subtract (3.6) from (3.4). Thus we get (3.10)

$$2(1+2c_j^m(\lambda))^\zeta(2\gamma+\mu)a_3-2(1+2c_j^m(\lambda))^\zeta(2\gamma+\mu)a_2^2 = \alpha(c_2-d_2)+\frac{\alpha(\alpha-1)}{2}(c_1^2-d_1^2)$$

From (3.7), (3.8) and (3.10), it follows that

$$a_3 = \frac{\alpha^2(c_1^2 + d_1^2)}{2(1 + c_j^m(\lambda))^{2\zeta}(\gamma + \mu)^2} + \frac{\alpha(c_2 - q_2)}{2(1 + 2c_j^m(\lambda))^\zeta(2\gamma + \mu)} \tag{3.11}$$

By applying Lemma.1 in the above equation we get the desired estimate for a_3. □

By considering $m = 1$ in Theorem.1 we have the following corollary, a result by Serap[9]

Corollary 1. *[9] Let the function $f(z)$ given by (1.1) is in the class $N_{\sum}^{\zeta,\mu}(\gamma,\lambda;\alpha)$ with $(0 < \alpha \leq 1; \lambda \geq 0, \gamma \geq 1; \zeta \in \mathbb{N}_0; \mu \geq 0, z \in \Delta)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(1+\lambda)^{2\zeta}(\gamma+\mu)^2 + \alpha[2(1+2\lambda)^{\zeta}(2\gamma+\mu) - (1+\lambda)^{2\zeta}(\gamma^2+2\gamma+\mu)]}}$$

$$|a_3| \leq \frac{2\alpha}{(1+2\lambda)^{\zeta}(2\gamma+\mu)} + \frac{4\alpha^2}{(\gamma+\mu)^2(1+\lambda)^{2\zeta}}$$

By considering $m = \lambda = 1$ in Theorem.1 we have the following corollary, a result by Seker *et al.*[10]

Corollary 2. *[10] Let the function $f(z)$ given by (1.1) is in the class $N_{\sum}^{\zeta,\mu}(\gamma;\alpha)$ with $(0 < \alpha \leq 1; \gamma \geq 1; \zeta \in \mathbb{N}_0; \mu \geq 0, z \in \Delta)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2^{2\zeta}(\gamma+\mu)^2 + \alpha[2.3^{\zeta}(2\gamma+\mu) - 2^{2\zeta}(\gamma^2+2\gamma+\mu)]}}$$

$$|a_3| \leq \frac{2\alpha}{3^{\zeta}(2\gamma+\mu)} + \frac{4\alpha^2}{(\gamma+\mu)^2 2^{2\zeta}}$$

By considering $\zeta = 0$ in Theorem.1 we have the following corollary, a result by Caglar *et al.*[3]

Corollary 3. *[3] Let the function $f(z)$ given by (1.1) is in the class $N_{\sum}^{\mu}(\gamma;\alpha)$ with $(0 < \alpha \leq 1; \gamma \geq 1; \mu \geq 0, z \in \Delta)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(\gamma+\mu)^2 + \alpha(\mu+2\gamma-\gamma^2)}}$$

$$|a_3| \leq \frac{2\alpha}{(2\gamma+\mu)} + \frac{4\alpha^2}{(\gamma+\mu)^2}$$

By considering $\zeta = 0$ and $\mu = 1$ in Theorem.1 we have the following corollary, a result by Frasin *et al.*[5]

Corollary 4. *[5] Let the function $f(z)$ given by (1.1) is in the class $\mathfrak{B}_{\sum}(\gamma;\alpha)$ with $(0 < \alpha \leq 1; \gamma \geq 1; z \in \Delta)$, then*

$$|a_2| \leq \alpha \frac{2}{\sqrt{(\gamma+1)^2 + \alpha(1+2\gamma-\gamma^2)}}$$

$$|a_3| \leq \frac{2\alpha}{(2\gamma+1)} + \frac{4\alpha^2}{(\gamma+1)^2}$$

By considering $\zeta = 0$ and $\mu = \gamma = 1$ in Theorem.1 we have the following corollary, a result by Srivastava *et al.*[11]

Corollary 5. *[11] Let the function $f(z)$ given by (1.1) is in the class $H^{\alpha}_{\sum}$ with $(0 < \alpha \leq 1; z \in \Delta)$, then*

$$|a_2| \leq \sqrt{\alpha \frac{2}{\alpha + 2}}$$

$$|a_3| \leq \frac{\alpha(3\alpha + 2)}{3}$$

By considering $\zeta = \mu = 0$ and $\gamma = 1$ in Theorem.1 we have the following corollary, a result by Brannan *et al.*[2]

Corollary 6. *[2] Let the function $f(z)$ given by (1.1) is in the class $S^{*}_{\sum}[\alpha]$ with $(0 < \alpha \leq 1; z \in \Delta)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha + 1}}$$

$$|a_3| \leq 4\alpha^2 + \alpha$$

By considering $m = \lambda = \gamma = 1$ in Theorem.1 we have the following corollary, a result by Seker *et al.*[10]

Corollary 7. *[10] Let the function $f(z)$ given by (1.1) is in the class $N^{\zeta,\mu}_{\sum}(\alpha, \gamma)$ with $(0 < \alpha \leq 1; \gamma \geq 1; \zeta \in \mathbb{N}_0; z \in \Delta)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2^{2\zeta}(1 + \gamma)^2 - \alpha(\gamma(2 + \gamma) + 1) + 2\alpha.3^{\zeta}(1 + 2\gamma)}}$$

$$|a_3| \leq \frac{2\alpha}{3^{\zeta}(1 + 2\gamma)} + \frac{4\alpha^2}{2^{2\zeta}(1 + \gamma)^2}$$

By considering $m = \lambda = \gamma = \mu = 1$ in Theorem.1 we have the following corollary, a result by Seker *et al.*[10]

Corollary 8. *[10] Let the function $f(z)$ given by (1.1) is in the class $N^{\zeta,\mu}_{\sum}(\alpha, \gamma)$ with $(0 < \alpha \leq 1; \gamma \geq 1; \zeta \in \mathbb{N}_0; z \in \Delta)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2^{2\zeta+2}(1 - \alpha) + 2\alpha.3^{\zeta+1}}}$$

$$|a_3| \leq \frac{2\alpha}{3^{(\zeta+1)}} + \frac{\alpha^2}{2^{2\zeta}}$$

By considering $m = \lambda = \mu = 1$ in Theorem.1 we have the following corollary, a result by Porwal *et al.*[7]

Corollary 9. *[7] Let the function $f(z)$ given by (1.1) is in the class $B_{\sum}(\zeta,\alpha,\gamma)$ with $(0<\alpha\leq 1;\gamma\geq 1;\zeta\in\mathbb{N}_0;z\in\Delta)$, then*

$$|a_2|\leq \frac{2\alpha}{\sqrt{4^{\zeta}(\gamma+1)^2+\alpha[2.3^{\zeta}(2\gamma+1)-4^{\zeta}(\gamma+1)^2]}}$$

$$|a_3|\leq \frac{2\alpha}{3^{\zeta}(2\gamma+1)}+\frac{4\alpha^2}{4^{\zeta}(\gamma+1)^2}$$

4 The function class $N_{\sum}^{\zeta,\mu}(m,\gamma,\lambda;\beta)$

Definition 2. *A function $f(z)$ of the form (1.1) in the function class $\sum$ is said to be in the class $N_{\sum}^{\zeta,\mu}(m,\gamma,\lambda;\beta)$ if and only if*

$$\left|arg\{(1-\gamma)\Big(\frac{D_{m,\lambda}^{\zeta}f(z)}{z}\Big)^{\mu}+\gamma[D_{m,\lambda}^{\zeta}f(z)]'\Big(\frac{D_{m,\lambda}^{\zeta}f(z)}{z}\Big)^{\mu-1}\}\right|>\beta$$

$(0\leq\beta<1;\lambda>0,\gamma\geq 1;m\in\mathbb{N};\zeta\in\mathbb{N}_0;\mu\geq 0;z\in\Delta)$
and

$$\left|arg\{(1-\gamma)\Big(\frac{D_{m,\lambda}^{\zeta}g(w)}{w}\Big)^{\mu}+\gamma[D_{m,\lambda}^{\zeta}g(w)]'\Big(\frac{D_{m,\lambda}^{\zeta}g(w)}{w}\Big)^{\mu-1}\}\right|>\beta$$

$(0\leq\beta<1;\lambda>0,\gamma\geq 1;m\in\mathbb{N};\zeta\in\mathbb{N}_0;\mu\geq 0;w\in\Delta)$, *where $g:=f^{-1}$.*

Remark 2. *1 If $m=1$ in Definition.1 we have, $N_{\sum}^{\zeta,\mu}(1,\gamma,\lambda;\beta)=N_{\sum}^{\lambda,\mu}(\zeta,\beta,\gamma)$ the class introduced and studied by Sarep Bulut[9].*

2 If $m=\lambda=1$ in Definition.1 we have, $N_{\sum}^{\zeta,\mu}(1,\gamma,1;\beta)=N_{\sum}^{\zeta,\mu}(\alpha,\beta)$ the class introduced and studied by Bilal Seker et al.[10].

3 If $\zeta=0$ in Definition.1 we have, $N_{\sum}^{0,\mu}(m,\gamma,\lambda;\beta)=N_{\sum}^{\mu}(\alpha,\beta)$ the class introduced and studied by Caglar et al.[3].

4 If $\zeta=0$ and $\mu=1$ in Definition.1 we have, $N_{\sum}^{0,1}(m,\gamma,\lambda;\beta)=\mathfrak{B}_{\sum}(\alpha,\beta)$ the class introduced and studied by Frasin and Aouf[5].

5 If $\zeta=0$ and $\mu=\gamma=1$ in Definition.1 we have, $N_{\sum}^{0,1}(m,1,\lambda;\beta)=H_{\sum}^{\beta}$ the class introduced and studied by Srivastava et al.[11].

6 If $\zeta=\mu=0$ and $\gamma=1$ in Definition.1 we have, $N_{\sum}^{0,0}(m,1,\lambda;\beta)=S_{\sum}^{*}[\beta]$ the class of strongly bi-starlike functions of order α, studied by Brannan and Taha[2].

7 If $\mu=1$ and $m=\lambda=1$ in Definition.1 we have, $N_{\sum}^{\zeta,1}(1,\gamma,1;\beta)=\mathfrak{B}_{\sum}(\zeta,\beta,\gamma)$ the class introduced and studied by Porwal and Darus [7]

5 Coefficient bounds for the function class $N_{\Sigma}^{\zeta,\mu}(m,\gamma,\lambda;\beta)$

Theorem 2. *Let the function $f(z)$ given by (1.1) is in the class $N_{\Sigma}^{\zeta,\mu}(m,\gamma,\lambda;\beta)$ with $(0 \leq \beta < 1; \lambda > 0, \gamma \geq 1; m \in \mathbb{N}; \zeta \in \mathbb{N}_0; \mu \geq 0; z \in \Delta)$, then*

$$|a_2| \leq \min\{\frac{2(1-\beta)}{(1+c_j^m(\lambda))^{\zeta}(\gamma+\mu)}, \sqrt{\frac{4(1-\beta)}{|2(1+2c_j^m(\lambda))^{\zeta}+(1+c_j^m(\lambda))^{2\zeta}(\mu-1)|(2\gamma+\mu)}}\}$$

and

$$|a_3| \leq \min\{ \frac{4(1-\beta)^2}{(1+c_j^m(\lambda))^{2\zeta}(\gamma+\mu)^2} + \frac{2(1-\beta)}{(1+2c_j^m(\lambda))^{\zeta}(2\gamma+\mu)}, \frac{(1-\beta)[|H|+(1+c_j^m(\lambda))^{2\zeta}|\mu-1|]}{(2\gamma+\mu)(1+2c_j^m(\lambda))^{\zeta}|H-2(1+2c_j^m(\lambda))^{\zeta}|}\}$$

where $H = 4(1+2c_j^m(\lambda))^{\zeta}+(1+c_j^m(\lambda))^{2\zeta}(\mu-1)$

Proof. Definition.2 can be writtern as,

(5.1)
$$(1-\gamma)(\frac{D_{m,\lambda}^{\zeta}f(z)}{z})^{\mu}+\gamma[D_{m,\lambda}^{\zeta}f(z)]'(\frac{D_{m,\lambda}^{\zeta}f(z)}{z})^{\mu-1} = \beta+(1-\beta)c(z), \quad (z \in \Delta)$$

(5.2)
$$(1-\gamma)(\frac{D_{m,\lambda}^{\zeta}g(w)}{w})^{\mu}+\gamma[D_{m,\lambda}^{\zeta}g(w)]'(\frac{D_{m,\lambda}^{\zeta}g(w)}{w})^{\mu-1} = \beta+(1-\beta)d(w), \quad (w \in \Delta)$$

where $c(z)$ and $d(w)$ are in $\mathbb{P}$ and have the series expansions

$$c(z) = 1+c_1z+c_2z^2+...$$

and

$$d(z) = 1+d_1z+d_2^2w^2+...$$

Equating the coefficients of z, z^2, w and w^2 by using the above equations in (5.1) and (5.2), we get

$$(1+c_j^m(\lambda))^{\zeta}(\gamma+\mu)a_2 = (1-\beta)c_1 \tag{5.3}$$

$$(1+2c_j^m(\lambda))^{\zeta}(2\gamma+\mu)a_3+(1+c_j^m(\lambda))^{2\zeta}(\mu-1)(\gamma+\frac{\mu}{2})a_2^2 = (1-\beta)c_2 \tag{5.4}$$

$$-(1+c_j^m(\lambda))^{\zeta}(\gamma+\mu)a_2 = (1-\beta)d_1 \tag{5.5}$$

$$(5.6)\quad (1+2c_j^m(\lambda))^\zeta(2\gamma+\mu)(2a_2^2-a_3)+(1+c_j^m(\lambda))^{2\zeta}(\mu-1)(\gamma+\frac{\mu}{2})a_2^2=(1-\beta)d_2$$

From (5.3) and (5.4), we obtain

$$c_1=-d_1 \tag{5.7}$$

and

$$2(1+c_j^m(\lambda))^{2\zeta}(\gamma+\mu)^2a_2^2=\alpha^2(c_1^2+d_1^2) \tag{5.8}$$

Also by adding (5.4) and (5.6) we have

$$[2(1+2c_j^m(\lambda))^\zeta+(1+c_j^m(\lambda))^{2\zeta}(\mu-1)](2\gamma+\mu)a_2^2=(1-\beta)(c_2+d_2) \tag{5.9}$$

From (5.8) and (5.9) we can find the values of a_2^2. Then we apply triangular inequality and Lemma.1.1. By taking square root we obtain the desired results. Now to find the bound for $|a_3|$, first we subtract (5.6) from (5.4). Thus we get

$$2(1+2c_j^m(\lambda))^\zeta(2\gamma+\mu)a_3-2(1+2c_j^m(\lambda))^\zeta(2\gamma+\mu)a_2^2=(1-\beta)(c_2-d_2) \tag{5.10}$$

Using the values of a_2^2 from (5.8) and (5.9) we get two different values for a_3 respectively,

$$a_3=\frac{(1-\beta)^2(c_1^2+d_1^2)}{2(1+c_j^m(\lambda))^{2\zeta}(\gamma+\mu)^2}+\frac{(1-\beta)(c_2-d_2)}{2(1+2c_j^m(\lambda))^\zeta(2\gamma+\mu)} \tag{5.11}$$

(5.12)

$$a_3=\frac{(1-\beta)\{[4(1+2c_j^m(\lambda))^\zeta+(1+c_j^m(\lambda))^{2\zeta}(\mu-1)]c_2-(1-c_j^m(\lambda))^{2\zeta}(\mu-1)d_2\}}{2(2\gamma+\mu)(1+2c_j^m(\lambda))^\zeta[2(1+2c_j^m(\lambda))^\zeta+(1+c_j^m(\lambda))^{2\zeta}(\mu-1)]}$$

By applying triangular inequality along with the Lemma.1.1 in (5.11) and (5.12) we get the desired bound for $|a_3|$. □

By considering $m=1$ in Theorem.2 we have the following corollary, a result by Serap[9]

Corollary 10. *[9] Let the function $f(z)$ given by (1.1) is in the class $N_{\Sigma}^{\zeta,\mu}(\gamma,\lambda;\beta)$ with $(0\le\beta<1;\lambda\ge0,\gamma\ge1;\zeta\in\mathbb{N}_0;\mu\ge0,z\in\Delta)$, then*

$$|a_2|\le\min\{\frac{2(1-\beta)}{(1+\lambda)^\zeta(\gamma+\mu)},\sqrt{\frac{4(1-\beta)}{|2(1+2\lambda)^\zeta+(1+\lambda)^{2\zeta}(\mu-1)|(2\gamma+\mu)}}\}$$

$$|a_3|\le\min\{\frac{4(1-\beta)^2}{(1+\lambda)^{2\zeta}(\gamma+\mu)^2}+\frac{2(1-\beta)}{(1+2\lambda)^\zeta(2\gamma+\mu)},\ \frac{(1-\beta)[|L|+(1+\lambda)^{2\zeta}|\mu-1|]}{(2\gamma+\mu)(1+2\lambda)^\zeta|L-2(1+2\lambda)^\zeta|}\}$$

where $L=4(1+2\lambda)^\zeta+(1+\lambda)^{2\zeta}(\mu-1)$

The following results can be considered as an improvement result for already proved results by several authors.

By considering $m = \lambda = 1$ in Theorem.2 we have the following corollary, an improvement result for Seker *et al.*[10]

Corollary 11. *[10] Let the function $f(z)$ given by (1.1) is in the class $N_{\sum}^{\zeta,\mu}(\gamma;\beta)$ with $(0 \leq \beta < 1; \gamma \geq 1; \zeta \in \mathbb{N}_0; \mu \geq 0, z \in \Delta)$, then*

$$|a_2| \leq \min\{\frac{2(1-\beta)}{2^{\zeta}(\gamma+\mu)}, \sqrt{\frac{2(1-\beta)}{|3^{\zeta}+2^{2\zeta-1}(\mu-1)|(2\gamma+\mu)}}\} \tag{5.13}$$

(5.14)

$$|a_3| \leq \min\{(\frac{2(1-\beta)}{2^{\zeta}(\gamma+\mu)})^2 + \frac{2(1-\beta)}{3^{\zeta}(2\gamma+\mu)}, \frac{2(1-\beta)|3^{\zeta}+2^{2\zeta-2}(\mu-1)|+2^{2\zeta-2}|\mu-1|}{(2\gamma+\mu)3^{\zeta}|3^{\zeta}+2^{2\zeta-2}(\mu-1)|}\}$$

By considering $\zeta = 0$ in Theorem.2 we have the following corollary, a result by Caglar *et al.*[3]

Corollary 12. *[3] Let the function $f(z)$ given by (1.1) is in the class $N_{\sum}^{\mu}(\gamma;\beta)$ with $(0 \leq \beta < 1; \gamma \geq 1; \mu \geq 0, z \in \Delta)$, then*

$$|a_2| \leq \min\{\sqrt{\frac{4(1-\beta)}{(\mu+1)(2\gamma+\mu)}}, \frac{2(1-\beta)}{\gamma+\mu}\} \tag{5.15}$$

$$|a_3| \leq \begin{cases} \min\{\frac{4(1-\beta)}{(\mu+1)(2\gamma+\mu)}, \frac{4(1-\beta)^2}{(\gamma+\mu)^2} + \frac{2(1-\beta)}{2\gamma+\mu}\} & , \ 0 \leq \mu < 1 \\ \frac{2(1-\beta)}{(2\gamma+\mu)} & ,\mu \geq 1 \end{cases} \tag{5.16}$$

By considering $\zeta = 0$ and $\mu = 1$ in Theorem.2 we have the following corollary, an improvement of the result by Frasin *et al.*[5]

Corollary 13. *[5] Let the function $f(z)$ given by (1.1) is in the class $\mathfrak{B}_{\sum}(\gamma;\beta)$ with $(0 \leq \beta < 1; \gamma \geq 1; z \in \Delta)$, then*

$$|a_2| \leq \min\{\sqrt{\frac{2(1-\beta)}{(2\gamma+1)}}, \frac{2(1-\beta)}{\gamma+1}\} \tag{5.17}$$

$$|a_3| \leq \frac{2(1-\beta)}{2\gamma+1} \tag{5.18}$$

By considering $\zeta = 0$ and $\mu = \gamma = 1$ in Theorem.1 we have the following corollary, an improvement of the result by Srivastava *et al.*[11]

Corollary 14. *[11] Let the function $f(z)$ given by (1.1) is in the class $H^{\alpha}_{\sum}$ with $(0 < \alpha \leq 1; z \in \Delta)$, then*

$$|a_2| \leq \begin{cases} \sqrt{\frac{2(1-\beta)}{3}} & ,0 \leq \beta \leq \frac{1}{3} \\ 1-\beta & ,\frac{1}{3} \leq \beta < 1, \end{cases} \tag{5.19}$$

$$|a_3| \leq \frac{2(1-\beta)}{3} \tag{5.20}$$

By considering $\zeta = \mu = 0$ and $\gamma = 1$ in Theorem.2 we have the following corollary, an improvement of the result by Brannan *et al.*[2]

Corollary 15. *[2] Let the function $f(z)$ given by (1.1) is in the class $S^{*}_{\sum}[\beta]$ with $(0 \leq \beta < 1; z \in \Delta)$, then*

$$|a_2| \leq \sqrt{2(1-\beta)} \tag{5.21}$$

$$|a_3| \leq \begin{cases} \sqrt{2(1-\beta)} & ,0 \leq \beta \leq \frac{3}{4} \\ (1-\beta)(5-4\beta) & ,\frac{3}{4} \leq \beta < 1, \end{cases} \tag{5.22}$$

By considering $m = \lambda = \mu = 1$ in Theorem.2 we have the following corollary, a result by Porwal *et al.*[7]

Corollary 16. *[7] Let the function $f(z)$ given by (1.1) is in the class $B_{\sum}(\zeta, \alpha, \beta)$ with $(0 \leq \beta < 1; \gamma \geq 1; \zeta \in \mathbb{N}_0; z \in \Delta)$, then*

$$|a_2| \leq \min\{\frac{2(1-\beta)}{2^{\zeta}(\gamma+1)}, \sqrt{\frac{2(1-\beta)}{3^{\zeta}(2\gamma+1)}}\} \tag{5.23}$$

$$|a_3| \leq \frac{2(1-\beta)}{3^{\zeta}(2\gamma+1)} \tag{5.24}$$

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