On the Diophantine Equation $3^x + 117^y = z^2$

Shivangi Asthana[1] **and Madan Mohan Singh**[2]

[1] *Department of Mathematics,*
North-Eastern Hill University, Shillong-793022, Meghalaya, India.
email: shivangiasthana.1@gmail.com

[2]*Department of Basic Sciences and Social Sciences,*
North - Eastern Hill University, Shillong-793022, Meghalaya, India.
email: mmsingh2004@gmail.com

Abstract

In this paper we prove that the Diophantine equation $3^x + 117^y = z^2$ has exactly four non-negative integer solutions for x, y and z . The solutions are $(1, 0, 2)$, $(3, 1, 12)$, $(7, 1, 48)$ and $(7, 2, 126)$ respectively.

Keywords: Exponential Diophantine equation, integer solutions.
Subject Classification: 11D61

1 Introduction

Diophantine equations of the form $a^x + b^y = c^z$ have been studied by numerous mathematicians for many years and by a variety of methods [1, 2, 3, 4, 5]. It was proved by Cao [6] that this equation has at most one solution with $z > 1$. The Diophantine equation $2^x + 5^y = z^2$ was studied by D. Acu [7] in 2007, who established that this equation has exactly two solutions in non-negative integers i.e $(x, y, z) \in \{(3, 0, 3), (2, 1, 3)\}$. In 2011, Suvarnamani, Singta and Chotchaisthit [8] solved two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ and showed that these two equations have no non-negative integer solutions where x, y and z are non-negative integers. In 2013, B. Sroysang [9] showed that $(1, 0, 2), (3, 0, 3)$ and $(4, 2, 5)$ are only three solutions (x, y, z) for the Diophantine equation $2^x + 3^y = z^2$ where x, y and z are non-negative integers. In 2012, the same author [10, 11] solved two Diophantine equations $3^x + 5^y = z^2$ and $3^x + 17^y = z^2$ and established that they have unique solution $(1, 0, 2)$ in non-negative integers (x, y, z). In 2013, Rabago [12] solved two Diophantine equations $3^x+19^y = z^2$ and $3^x+91^y = z^2$ where x, y and z are non-negative integers. He found two solutions for each of the equations i.e.$\{(1, 0, 2), (4, 1, 10)\}$ and $\{(1, 0, 2), (2, 1, 10)\}$ respectively. The Diophantine equation $3^x + 85^y = z^2$ was studied by B. Sroysang in 2014 [13] who found that $(1, 0, 2)$ is a unique solution in non-negative integers x, y and z for this equation.
In this paper, we have attempted to solve the Diophantine equation $3^x + 117^y = z^2$ and have found that $(1, 0, 2)$, $(3, 1, 12)$, $(7, 1, 48)$ and $(7, 2, 126)$ are exactly four solutions for this equation in non-negative integers (x, y, z).

2 Preliminaries

In 1844, Catalan [14] posed the following conjecture:

Proposition 1. *(Catalan's Conjecture)* $(3, 2, 2, 3)$ *is a unique solution* (a, b, x, y) *for the Diophantine equation* $a^x - b^y = 1$ *where* a, b, x *and* y *are integers such that* $min\{a, b, x, y\} > 1$.

We present two lemmas to prove the main result.

Lemma 1. *[10]* $(1, 2)$ *is a unique solution* (x, z) *for the Diophantine equation* $3^x + 1 = z^2$ *where* x *and* z *are non-negative integers.*

Proof. Suppose that there are non-negative integers x and z such that $3^x+1 = z^2$. If $x = 0$, then $z^2 = 2$ which is impossible. Then $x \geq 1$. Thus $z^2 = 3^x+1 \geq 3^1+1 = 4$. Then $z \geq 2$. Now we consider on the equation $z^2 - 3^x = 1$. By Proposition 1, we have $x = 1$. Thus $z = 2$. Hence $(1, 2)$ is a unique solution (x, z) for the equation $3^x + 1 = z^2$ where x and z are non-negative integers. □

Lemma 2. *The Diophantine equation* $1 + 117^y = z^2$ *has no non-negative integers solution.*

Proof. Suppose that there are non-negative integers y and z such that $1+117^y = z^2$. If $y = 0$ then $z^2 = 2$ which is impossible. Then $y \geq 1$. Thus $z^2 = 1 + 117^y \geq 1 + 117^1 = 118$. Then $z \geq 11$. Now we consider on the equation $z^2 - 117^y = 1$. By Proposition 1, we have $y = 1$. Thus $z^2 = 118$, which is not possible to get solution. Hence the equation $1 + 117^y = z^2$ has no non-negative integers solution.

□

3 Main Result

Theorem 1. *The Diophantine equation* $3^x + 117^y = z^2$ *has exactly four solutions in non-negative integers* $(x, y, z) \in \{(1, 0, 2), (3, 1, 12), (7, 1, 48), (7, 2, 126)\}$.

Proof. Let x, y and z be non-negative integers such that $3^x+117^y = z^2$. By lemma 2, we have $x \geq 1$. We have three cases for y:

Case (i) $y = 0$, From lemma (1), we have $(x, y, z) = (1, 0, 2)$.

Case (ii) y is even. If $y = 2l$, $l \in \mathbb{N}$, then we have

$$3^x = z^2 - 117^{2l}$$
$$or, \quad 3^x = (z - 117^l)(z + 117^l)$$

where $z - 117^l = 3^u$ and $z + 117^l = 3^{x-u}$, $x > 2u$. From here, we obtain

$$3^{x-u} - 3^u = z + 117^l - z + 117^l = 2 \cdot 117^l$$

$$or, \quad 3^u(3^{x-2u} - 1) = 2 \cdot 117^l$$

For $l = 1$, we have $3^u(3^{x-2u} - 1) = 2 \cdot 13 \cdot 9$ or, $3^u(3^{x-2u} - 1) = 3^2 \cdot 26$. This implies that $u = 2$ and $3^{x-4} - 1 = 26$ or $3^{x-4} = 27$ or $x = 7$. This gives us values $x = 7$, $y = 2$ and $z = 126$. Hence the solution is $(x, y, z) = (7, 2, 126)$.

Case(iii) y is odd. Let $y = 2l + 1$ where l is a non-negative integer. We will divide this case into two parts.

Part (i) $3^x + 117^y = z^2$ becomes $3^x + 117^{2l+1} = z^2$ or $3^x + 117 \cdot 117^{2l} = z^2$. So $3^x - 4 \cdot 117^{2l} = z^2 - 121 \cdot 117^{2l} = (z - 11 \cdot 117^l)(z + 11 \cdot 117^l)$.

$$\begin{cases} z - 11 \cdot 117^l = 1 \text{.............}(i) \\ z + 11 \cdot 117^l = 3^x - 4 \cdot 117^{2l} \text{................}(ii) \end{cases}$$

Subtracting equation (i) from (ii), we get $z + 11 \cdot 117^l - z + 11 \cdot 117^l = 3^x - 4 \cdot 117^{2l} - 1$ which implies $117^l(22 + 4 \cdot 117^l) = 3^x - 1$. This implies $l = 0$ and $3^x = 27$. This gives us values $x = 3$, $y = 1$ and $z = 12$. Hence the solution is $(x, y, z) = (3, 1, 12)$.

Part(ii) Again $3^x + 117^y = z^2$ becomes $3^x + 117^{2l+1} = z^2$. So $3^x + (2209 - 2092) \cdot 117^{2l} = z^2$ or $3^x - 2092 \cdot 117^{2l} = z^2 - 2209 \cdot 117^{2l}$. Hence, $3^x - 2092 \cdot 117^{2l} = (z - 47 \cdot 117^l)(z + 47 \cdot 117^l)$.

$$\begin{cases} z - 47 \cdot 117^l = 1 \text{..............}(iii) \\ z + 47 \cdot 117^l = 3^x - 2092 \cdot 117^{2l} \text{............}(iv) \end{cases}$$

Subtracting equation (iii) from (iv), we get $z + 47 \cdot 117^l - z + 47 \cdot 117^l = 3^x - 2092 \cdot 117^{2l} - 1$ which implies $117^l(94 + 2092 \cdot 117^l) = 3^x - 1$. This implies $l = 0$ and $3^x = 2187$. This gives values $x = 7$, $y = 1$ and $z = 48$. Hence, the solution is $(x, y, z) = (7, 1, 48)$.

□

Corollary 1. *The Diophantine equation $3^x + 117^y = w^4$ has no non-negative integers solution where x, y and w are non-negative integers.*

Proof. Suppose that x, y and w be non-negative integers such that $3^x + 117^y = w^4$. Let $z = w^2$. Then $3^x + 117^y = z^2$. By Theorem 1, we have $(x, y, z) \in \{(1, 0, 2), (3, 1, 12), (7, 1, 48), (7, 2, 126)\}$. Then $w^2 = z \in \{2, 12, 48, 126\}$. Here z is a square of some integer while $2, 12, 48, 126$ are not square of any integer. Hence, the Diophantine equation $3^x + 117^y = w^4$ has no non-negative integers solution. □

Corollary 2. $(3, 1, 2), (7, 1, 8)$ *and* $(7, 2, 21)$ *are exactly three solutions in positive integers (x, y, u) for the Diophantine equation $3^x + 117^y = 36u^2$ where x, y and u are positive integers.*

Proof. Let x, y and u be positive integers such that $3^x + 117^y = 36u^2$. Let $z = 6u$. Then $3^x + 117^y = z^2$. By Theorem 1, we have $(x, y, z) \in \{(3, 1, 12), (7, 1, 48), (7, 2, 126)\}$. Then, $u \in \{2, 8, 21\}$. Hence, $(3, 1, 2), (7, 1, 8)$ and $(7, 2, 21)$ are solutions in positive integers (x, y, u) for the Diophantine equation $3^x + 117^y = 36u^2$ where x, y and u are positive integers. □

Corollary 3. $(3, 1, 1)$ *and* $(7, 1, 2)$ *are exactly two positive integers solutions (x, y, v) for the Diophantine equation $3^x + 117^y = 144v^4$ where x, y and v are positive integers.*

Proof. Let x, y and v be positive integers such that $3^x + 117^y = 144v^4$. Let $z = 12v^2$. Then $3^x + 117^y = z^2$. By Theorem 1, we have $(x, y, z) \in \{(3, 1, 12), (7, 1, 48)\}$. Then $v \in \{1, 2\}$. Hence, $(3, 1, 1)$ and $(7, 1, 2)$ are exactly two positive integers solutions (x, y, v) for the Diophantine equation $3^x + 117^y = 144v^4$ where x, y and v are positive integers. □

Corollary 4. $(1, 0, 1)$ *is a unique non-negative integers solution* (x, y, m) *for the Diophantine equation* $3^x + 117^y = 4m^4$ *where* x, y *and* m *are non-negative integers.*

Proof. Let x, y and m be non-negative integers such that $3^x + 117^y = 4m^4$. Let $z = 2m^2$. Then $3^x + 117^y = z^2$. By Theorem 1, we have $(x, y, z) = (1, 0, 2)$. Then $2m^2 = 2$ or $m = 1$. Hence, $(1, 0, 1)$ is a unique non-negative integer solution (x, y, m) for the Diophantine equation $3^x + 117^y = 4m^4$ where x, y and m are non-negative integers. □

Corollary 5. $(1, 0, 1)$, $(3, 1, 6)$, $(7, 1, 24)$ *and* $(7, 2, 63)$ *are exactly four non-negative integers solutions* (x, y, m) *for the Diophantine equation* $3^x + 117^y = 4m^2$ *where* x, y *and* m *are non-negative integers.*

Proof. Let x, y and m be positive integers such that $3^x + 117^y = 4m^2$. Let $z = 2m$. Then $3^x + 117^y = z^2$. By Theorem 1, we have $(x, y, z) \in \{(1, 0, 2), (3, 1, 12), (7, 1, 48), (7, 2, 126)\}$. Then $m \in \{1, 6, 24, 63\}$. Hence, $(1, 0, 1)$, $(3, 1, 6)$, $(7, 1, 24)$ and $(7, 2, 63)$ are exactly four non-negative integers solutions (x, y, m) for the Diophantine equation $3^x + 117^y = 4m^2$. □

Corollary 6. $(3, 1, 2)$ *and* $(7, 1, 4)$ *are exactly two positive integers solutions* (x, y, n) *for the Diophantine equation* $3^x + 117^y = 9n^4$ *where* x, y *and* n *are positive integers.*

Proof. Let x, y and n be positive integers such that $3^x + 117^y = 9n^4$. Let $z = 3n^2$. Then $3^x + 117^y = z^2$. By Theorem 1, we have $(x, y, z) \in \{(3, 1, 12), (7, 1, 48)\}$. Then $n \in \{2, 4\}$. Hence, $(3, 1, 2)$ and $(7, 1, 4)$ are exactly two positive integers solutions (x, y, n) for the Diophantine equation $3^x + 117^y = 9n^4$. □

Corollary 7. $(7, 2, 3)$ *is a unique positive integers solution* (x, y, t) *for the Diophantine equation* $3^x + 117^y = 196t^4$ *where* x, y *and* t *are positive integers.*

Proof. Let x, y and t be positive integers such that $3^x + 117^y = 196t^4$. Let $z = 14t^2$. Then $3^x + 117^y = z^2$. By Theorem 1, we have $(x, y, z) = (7, 2, 126)$. Then $14t^2 = 126$. So $t = 3$. Hence, $(7, 2, 3)$ is a unique positive integers solution (x, y, t) for the Diophantine equation $3^x + 117^y = 196t^4$ where x, y and t are positive integers. □

4 Conclusion

In this paper we have shown that the Diophantine equation $3^x + 117^y = z^2$ has exactly four non-negative integer solutions where x, y and z are non-negative integers. The solutions are $(1, 0, 2)$, $(3, 1, 12)$, $(7, 1, 48)$ and $(7, 2, 126)$ respectively.

References

[1] B. He, A. Togbe and S. Yang, On the solutions of the exponential Diophantine equation $a^x + b^y = (m^2+1)^z$, *Quaestiones Mathematicae*, **36**, 2013, 119-135.

[2] N. Terai, On the Exponential Diophantine equation $(4m^2+1)^x + (5m^2-1)^y = (3m)^z$, *Int. J. Algebra*, **6**, 2012, 1135-1146.

[3] T. Miyazaki and N. Terai, On the exponential Diophantine equation $(m^2+1)^x + (cm^2-1)^y = (am)^z$, *Bull. Australian Math. Soc.*, **90**, 2014, 9-19.

[4] T. Miyazaki, P. Yuan and D. Wu, Generalizations of classical results on Jesmanowicz conjecture concerning Pythagorean triples II, *J. Number Theory*, **141**, 2014, 184-201.

[5] J. Su and X. Li, The exponential Diophantine equation $(4m^2+1)^x + (5m^2-1)^y = (3m)^z$, *Abstract and Applied Analysis*, 2014, 1-5.

[6] Z. Cao, A note on the Diophantine equation $a^x + b^y = c^z$, *Acta Arith.*, **XCI(1)**, 1999, 85-89.

[7] D. Acu, On a Diophantine equation $2^x + 5^y = z^2$, *Gen. Math*, **15**, 2007, 145-148.

[8] S. Suvarnamani, A. Singta and S. Chotchaisthit, On two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$, *Sci. Tech. RMUTT. J.*, **1**, 2011, 25-28.

[9] B. Sroysang, More on the Diophantine equation $2^x + 3^y = z^2$, *Int. J. Pure and Appl. Math.*, **84(2)**, 2013, 133-137.

[10] B. Sroysang, On the Diophantine Equation $3^x + 5^y = z^2$, *Int. J. Pure and Appl. Math.*, **81(4)**, 2012, 605-608.

[11] B. Sroysang, On the Diophantine Equation $3^x + 17^y = z^2$, *Int. J. Pure and Appl. Math.*, **89(1)**, 2013, 111-114.

[12] J. F. T. Rabago, On two Diophantine equations $3^x + 19^y = z^2$ and $3^x + 91^y = z^2$, *Int. J. of Math. and Scientific Computing*, **3(1)**, 2013, 28-29.

[13] B. Sroysang, More on the Diophantine equation $3^x + 85^y = z^2$, *Int. J. Pure and Appl. Maths.*, **91**, 2014, 131-134.

[14] P. Mihailescu, Primary cyclotomic units and a proof of Catalan's conjecture, *J. Reine Angew. Math.*, **27**, 2004, 167-195