Semi Symmetric non-metric connection on a Hsu r-contact metric structure manifold

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Abstract

In the present paper, we define a semi-symmetric non-metric connection $\widetilde{B}$ on a Hsu r-contact metric structure manifold M_n and define the curvature tensor of M_n with respect to semi-symmetric non-metric connection. It has been shown that if a Hsu r-contact metric structure manifold admits a semi-symmetric non-metric connection whose curvature tensor is locally isometric to the unit sphere, then the conformal and con-harmonic curvature tensor with respect to Riemannian connection are identical iff $n - \frac{a^r}{c}(n+2) = 0$ Also it has been shown that if a Hsu r-contact metric structure manifold admits a semi-symmetric non-metric connection whose curvature tensor is locally isometric to the unit sphere, then the con-circular curvature tensor coincides with curvature tensor with respect to the Riemannian connection if $n - \frac{a^r}{c}(n+2) = 0$ Some other useful results on projective curvature tensor W and con-circular curvature tensor C with respect to semi-symmetric non-metric connection have been obtained.

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1 Introduction

Consider a differentiable manifold M_n of differentiability class C^∞ . Let there exist in M_n a vector valued C^∞ - linear function Φ, a vector field η^p and a C^∞ -one form ξ_p such that

$$\Phi^2(X) = a^r X + c\xi_p(X)\eta^p, \tag{1.1}$$

$$\overline{\eta}^p = 0, \tag{1.2}$$

$$G(\overline{X},\overline{Y}) = -a^r G(X,Y) - c\xi_p(X)\xi_p(Y). \tag{1.3}$$

Where $\Phi(X) = \overline{X}$, is a non zero complex number and c is an integer, then the set $(\Phi, \eta^p, a^r, c, \xi_p, G)$ satisfying (1.1) to (1.3) is called a Hsu r-contact metric structure and M_n equipped with a Hsu r-contact metric structure will be called a Hsu r-contact metric structure manifold [4], [6].
It is easy to calculate in M_n

$$\xi_p(\eta^p) = \frac{a^r}{c}, \tag{1.4}$$

$$\xi_p(\overline{X}) = 0, \tag{1.5}$$

and

$$G(X,\eta^p) \overset{def}{=} \xi_p(X). \tag{1.6}$$

Where 'r' is no. of parameters used defining the Hsu contact metric structure manifold and 'G' is a set containing some parameters, it can be defined from equation (1.3) and (1.6).

Definition : A structure on an n-dimensional manifold M of class C^∞ given by a non-null tensor field F satisfying

$$F^2 = a^r I$$

is called π - structure or Hsu-structure, where $'a'$ is a non zero complex constant and I denotes the unit tensor field. Then M is called π - structure manifold or Hsu-structure manifold [1].
Remark 1.1: A Hsu r-contact metric structure manifold $(\Phi, \eta^p, a^r, c, \xi_p, G)$ gives an almost Norden r-contact metric manifold [2], an almost para norden r-contact metric manifold, an almost para r-contact metric manifold [6] or Lorentzian para contact metric manifold[4] according as
$(a^r = -1, c = 1)\,(a^r = -1, c = -1)\,(a^r = 1, c = 1)\, or\, (a^r = 1, c = -1)$.
Definition 1.1: A C^∞ - manifold M_n , satisfying

$$D_X\eta^p = \Phi(X) \overset{def}{=} \overline{X}, \tag{1.7}$$

will be denoted by M_n^*
In M_n^*, we can easily show that

$$(D_X\xi_p)(Y) = {}'\Phi(X,Y) = (D_Y\xi_p)(X), \tag{1.8}$$

Where

$${}'\Phi(X,Y) \overset{def}{=} G\left(\overline{X},Y\right) = G\left(X,\overline{Y}\right) = {}'\Phi(Y,X). \tag{1.9}$$

Definition 1.2: An affine connection $\widetilde{B}$ is said to be metric if

$$\widetilde{B}_X G = 0, \tag{1.10}$$

The affine metric connection $\widetilde{B}$ satisfying

$$\left(\widetilde{B}_X \Phi\right)(Y) = \xi_p(Y)\,X - G(X,Y)\,\eta^p, \tag{1.11}$$

is called metric connection. A metric connection $\widetilde{B}$ is called semi-symmetric non-metric connection if

$$\widetilde{B}_X Y = D_X Y - \xi_p(Y)\,X - G(X,Y)\,\eta^p, \tag{1.12}$$

Where D is the Riemannian connection.
Also

$$\left(\widetilde{B}_X G\right)(Y,Z) = 2\xi_p(Y)\,G(X,Z) + 2\xi_p(Z)G(X,Y),$$

which implies

$$S(X,Y) = \xi_p(Y)\overline{X} - \xi_p(X)\overline{Y}, \tag{1.13}$$

Where S is the torsion tensor of the connection $\widetilde{B}$. The curvature tensor with respect to the semi-symmetric non-metric connection is defined as

$$\widetilde{R}(X,Y,Z) \overset{def}{=} \widetilde{B}_X\widetilde{B}_Y Z - \widetilde{B}_Y\widetilde{B}_X Z - \widetilde{B}_{[X,Y]}Z, \tag{1.14}$$

Using (1.12) in (1.14), we get

$$\begin{aligned}\widetilde{R}(X,Y,Z) &= K(X,Y,Z) - \beta(X,Z)Y + \beta(Y,Z)X \\ &\quad - G(Y,Z)(D_X\eta^p - \xi_p(X)\eta^p) + G(X,Z)(D_Y\eta^p - \xi_p(Y)\eta^p),\end{aligned} \tag{1.15}$$

where

$$\beta(X,Y) = (D_X\xi_p)(Y) + \xi_p(X)\xi_p(Y) + G(X,Y)\xi_p(\eta^p), \tag{1.16}$$

and

$$K(X,Y,Z) \overset{def}{=} D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z, \tag{1.17}$$

where $\widetilde{R}$ and K be the curvature tensors with respect to the connection $\widetilde{B}$ and D respectively.
Using (1.7) in (1.15), we get

$$\begin{aligned}\widetilde{R}(X,Y,Z) &= K(X,Y,Z) - \beta(X,Z)Y + \beta(Y,Z)X \\ &\quad - G(Y,Z)(\overline{X} - \xi_p(X)\eta^p) + G(X,Z)(\overline{Y} - \xi_p(Y)\eta^p),\end{aligned} \tag{1.18}$$

If $\widetilde{R}(X,Y,Z)=0$, then above equation becomes

$$K(X,Y,Z)-\beta(X,Z)Y+\beta(Y,Z)X$$
$$-G(Y,Z)(\overline{X}-\xi_p(X)\eta^p)+G(X,Z)(\overline{Y}-\xi_p(Y)\eta^p)=0, \tag{1.19}$$

Contracting above equation with respect to X , we get

$$Ric(Y,Z)-\beta(Y,Z)+n\beta(Y,Z)+\frac{a^r}{c}G(Y,Z)$$
$$+G(\overline{Y},Z)-\xi_p(Y)\xi_p(Z)=0 \tag{1.20}$$

Using (1.16) in (1.20), we get

$$cRic(Y,Z)+c(n-1)\left[{}'\Phi(Y,Z)+\xi_p(Y)\xi_p(Z)+\frac{a^r}{c}G(Y,Z)\right]$$
$$+G(\overline{Y},\overline{Z})+cG(\overline{Y},Z)=0 \tag{1.21}$$

Contracting above equation with respect to Z , we get

$$\gamma Y+n\left(\frac{a^r}{c}Y+\overline{Y}\right)+(n-2)\xi_p(Y)\eta^p=0, \tag{1.22}$$

Contracting above equation with respect to Y , we get

$$\widetilde{R}=-\frac{a^r}{c}(n+2)(n-1), \tag{1.23}$$

Where Ric and $\widetilde{R}$ are Ricci tensor and scalar curvature of the manifold respectively.

The Projective curvature tensor W , Con-harmonic curvature tensor L , Conformal curvature tensor V and Con-circular curvature tensor C in a Riemannian manifold are given by [5], [3].

$$W(X,Y,Z)=K(X,Y,Z)-\frac{1}{(n-1)}\left[Ric(Y,Z)X-Ric(X,Z)Y\right], \tag{1.24}$$

$$L(X,Y,Z)=K(X,Y,Z)-\frac{1}{(n-2)}\Big[Ric(Y,Z)X-Ric(X,Z)Y$$
$$+G(Y,Z)\gamma(X)-G(X,Z)\gamma(Y)\Big], \tag{1.25}$$

$$V(X,Y,Z)=K(X,Y,Z)-\frac{1}{(n-2)}\Big[Ric(Y,Z)X-Ric(X,Z)Y$$
$$+G(Y,Z)\gamma(X)-G(X,Z)\gamma(Y)\Big]+\frac{\widetilde{R}}{(n-1)(n-2)}\left[G(Y,Z)X-G(X,Z)Y\right] \tag{1.26}$$

$$C(X,Y,Z) = K(X,Y,Z) - \frac{\widetilde{R}}{n(n-1)}\left[G(Y,Z)X - G(X,Z)Y\right], \tag{1.27}$$

Where;

$$'W(X,Y,Z,T) \overset{def}{=} G(W(X,Y,Z),T), \tag{1.28}$$

$$'L(X,Y,Z,T) \overset{def}{=} G(L(X,Y,Z),T), \tag{1.29}$$

$$'V(X,Y,Z,T) \overset{def}{=} G(V(X,Y,Z),T), \tag{1.30}$$

$$'C(X,Y,Z,T) \overset{def}{=} G(C(X,Y,Z),T). \tag{1.31}$$

2 Curvature Tensors:

Theorem 1. *If a Hsu r-contact metric structure manifold admits a semi-symmetric non-metric connection whose curvature tensor is locally isometric to the unit sphere, then the Conformal and Con-harmonic curvature tensors with respect to the Riemannian connection are identical iff*

$$n - \frac{a^r}{c}(n+2) = 0; c \neq a^r.$$

Proof. If the curvature tensor with respect to the semi-symmetric non-metric connection is locally isometric to the unit sphere, then

$$\widetilde{R}(X,Y,Z) = G(Y,Z)X - G(X,Z)Y, \tag{2.1}$$

Using (2.1) in (1.18), we get

$$\begin{aligned} G(Y,Z)X - G(X,Z)Y &= K(X,Y,Z) - \beta(X,Z)Y + \beta(Y,Z)X \\ &\quad -G(Y,Z)(\overline{X} - \xi_p(X)\eta^p) + G(X,Z)(\overline{Y} - \xi_p(Y)\eta^p), \end{aligned} \tag{2.2}$$

Contracting above equation with respect to X and using (1.4) and (1.9), we get

$$\begin{aligned} cRic(Y,Z) = c(n-1)\left[G(Y,Z) -' \Phi(Y,Z) - \xi_p(Y)\xi_p(Z) - \frac{a^r}{c}G(Y,Z)\right] \\ -G(\overline{Y},\overline{Z}) - cG(\overline{Y},Z); c \neq a^r, \end{aligned} \tag{2.3}$$

Contracting above equation with respect to Z ,we get

$$c\gamma Y = -cn(Y - \widetilde{Y}) - (n-2)c\xi_p(Y)\eta^p - (a^r n + c)Y, \tag{2.4}$$

Contracting above equation with respect to Y , we get

$$\widetilde{R} = (n-1)\left[n - \frac{a^r}{c}(n+2)\right]; c \neq a^r, \tag{2.5}$$

where Ric and $\widetilde{R}$ are Ricci tensor and scalar curvature of the manifold respectively.

From (2.5), (1.25) and (1.26), we obtain the necessary part of the theorem. Converse part is obvious from (1.25) and (1.26).

Now using (1.19) and (1.21) in (1.24), we get

$$\begin{aligned} W(X,Y,Z) &= \beta(X,Z)Y - \beta(Y,Z)X + G(Y,Z)\overline{X} - G(Y,Z)\xi_p(X)\eta^p \\ &\quad -G(X,Z)\overline{Y} + G(X,Z)\xi_p(Y)\eta^p + \xi_p(Y)\xi_p(Z)X - \xi_p(X)\xi_p(Z)Y \\ &\quad + \frac{n}{(n-1)}\left[G(\overline{Y},Z)X - G(\overline{X},Z)Y\right] + \frac{1}{c(n-1)}\left[G(\overline{Y},\overline{Z})X - G(\overline{X},\overline{Z})Y\right] \\ &\quad + \frac{a^r}{c}\left[G(X,Z)Y - G(Y,Z)X\right], \end{aligned} \tag{2.6}$$

Now operating G on both the sides of the above equation and using (1.6) and (1.28) we get

$$\begin{aligned} {}'W(X,Y,Z,T) &= \beta(X,Z)G(Y,T) - \beta(Y,Z)G(X,T) + G(Y,Z)G(\overline{X},T) \\ &\quad -G(Y,Z)\xi_p(X)\xi_p(T) - G(X,Z)G(\overline{Y},T) + G(X,Z)\xi_p(Y)\xi_p(T) \\ &\quad +\xi_p(Y)\xi_p(Z)G(X,T) - \xi_p(X)\xi_p(Z)G(Y,T) + \frac{n}{(n-1)}\Big[G(\overline{Y},Z)G(X,T) \\ &\quad -G(\overline{X},Z)G(Y,T)\Big] + \frac{1}{c(n-1)}\left[G(\overline{Y},\overline{Z})G(X,T) - G(\overline{X},\overline{Z})G(Y,T)\right] \\ &\quad + \frac{a^r}{c}\left[G(X,Z)G(Y,T) - G(Y,Z)G(X,T)\right]; c \neq a^r. \end{aligned} \tag{2.7}$$

□

Theorem 2. *On a C^∞ - manifold M_n , we have*

$$\begin{aligned} {}'W(X,Y,Z,\eta^p) &= \beta(X,Z)\xi_p(Y) - \beta(Y,Z)\xi_p(X) + \frac{n}{(n-1)} \\ &\quad \left[{}'\Phi(Y,Z)\xi_p(X) - {}'\Phi(X,Z)\xi_p(Y)\right] + \frac{1}{c(n-1)}\left[G(\overline{Y},\overline{Z})\xi_p(X) - G(\overline{X},\overline{Z})\xi_p(Y)\right], \end{aligned} \tag{2.8}$$

$$'W(\overline{X},\overline{Y},Z,\eta^p) = 0, \tag{2.9}$$

$$'W(\eta^p, Y, Z, \eta^p) = \beta(\eta^p, Z)\xi_p(Y) - \frac{a^r}{c}\beta(Y, Z) + \frac{a^r}{c}\left(\frac{n}{(n-1)}\right)G(\overline{Y}, Z) + \frac{a^r}{c^2}\left(\frac{n}{(n-1)}\right)G(\overline{Y}, \overline{Z}); \quad c \neq a^r, \tag{2.10}$$

$$'W(X, Y, \eta^p, \eta^p) = \beta(X, \eta^p)\xi_p(Y) - \beta(Y, \eta^p)\xi_p(X), \tag{2.11}$$

$$'W(\eta^p, Y, Z, T) = \beta(\eta^p, Z)G(Y, T) - \beta(Y, Z)\xi_p(T) - \xi_p(Z)G(\overline{Y}, T) + 2\xi_p(Z)\xi_p(Y) \xi_p(T) - \tfrac{2a^r}{c}G(Y, T)\xi_p(Z) + \tfrac{n}{(n-1)}G(\overline{Y}, Z)\xi_p(T) - \tfrac{1}{c(n-1)} G(\overline{Y}, \overline{Z})\xi_p(T); c \neq a^r, \tag{2.12}$$

$$'W(\eta^p, Y, Z, \eta^p) = \beta(\eta^p, Z)\xi_p(Y) - \frac{a^r}{c}\beta(Y, Z) + \frac{a^r}{c^2(n-1)} \left[ncG(\overline{Y}, Z) + G(\overline{Y}, \overline{Z})\right]; c \neq a^r, \tag{2.13}$$

Proof. Replacing T by η^p in (2.7) and using (1.4), (1.5), (1.6) and (1.9) we get (2.8).

Replacing X by $\overline{X}$ and Y by $\overline{Y}$ in (2.8) and using (1.5), we get (2.9).

Replacing X by η^p in (2.8) and using (1.2), (1.4), (1.5), (1.6) and (1.9) we get (2.10).

Replacing Z by η^p in (2.8) and using (1.2), (1.5) and (1.9) we get (2.11).

Replacing X by η^p in (2.7) and using (1.2), (1.4) and (1.6) we get (2.12).

Replacing T by η^p in (2.12) and using (1.4), (1.5) and (1.6) we get (2.13) □

Theorem 3. *If a Hsu r-contact metric structure manifold admits a semi-symmetric non-metric connection whose curvature tensor is locally isometric to the unit sphere, then the Con-circular curvature tensor coincides with curvature tensor with respect to the Riemannian connection if*

$$n - \frac{a^r}{c}(n+2) = 0; c \neq a^r.$$

Proof. Using (2.5) in (1.24), we get

$$C(X, Y, Z) = K(X, Y, Z) - \frac{\left[n - \frac{a^r}{c}(n+2)\right]}{n}\left[G(Y, Z)X - G(X, Z)Y\right]; \quad c \neq a^r. \tag{2.14}$$

which is required proof of the theorem. Now using (1.19) and (1.23) in (1.27), we get

$$C(X,Y,Z) = \beta(X,Z)Y - \beta(Y,Z)X + G(Y,Z)(\overline{X}) - G(Y,Z)\xi_p(X)\eta^p - G(X,Z)(\overline{Y}) + G(X,Z)\xi_p(Y)\eta^p + \tfrac{a^r}{c}\left(\tfrac{(n+2)}{n}\right)[G(Y,Z)X - G(X,Z)Y]\,; c \neq a^r. \tag{2.15}$$

Operating G on both sides of the above equation and using (1.6), (1.9) and (1.31), we get

$$\begin{aligned} {}'C(X,Y,Z,T) &= \beta(X,Z)G(Y,T) - \beta(Y,Z)G(X,T) + G(Y,Z)'\Phi(X,T) \\ &\quad - G(Y,Z)\xi_p(X)\xi_p(T) - G(X,Z)'\Phi(Y,T) + G(X,Z)\xi_p(Y)\xi_p(T) \\ &\quad + \frac{a^r}{c}\left(\frac{n+2}{n}\right)[G(Y,Z)G(X,T) - G(X,Z)G(Y,T)]\,; c \neq a^r. \end{aligned} \tag{2.16}$$

□

Theorem 4. *On C^∞ manifold, we have*

$$'C(\eta^p,Y,Z,T) = \beta(\eta^p,Z)G(Y,T) - \beta(Y,Z)\xi_p(T) - \frac{a^r}{c}G(Y,Z)\xi_p(T) - \xi_p(Z) G(\overline{Y},T) + \xi_p(Y)\xi_p(Z)\xi_p(T) + \frac{a^r}{c}\left(\frac{n+2}{n}\right)[G(Y,Z)\xi_p(T) - G(Y,T)\xi_p(Z)]\,; c \neq a^r, \tag{2.17}$$

$$'C(X,Y,Z,\eta^p) = \beta(X,Z)\xi_p(Y) - \beta(Y,Z)\xi_p(X) + \frac{a^r}{c}\left(\frac{n+2}{n}\right) [G(Y,Z)\xi_p(X) - G(X,Z)\xi_p(Y)] - \frac{a^r}{c}G(Y,Z)\xi_p(X) + \frac{a^r}{c}G(X,Z)\xi_p(Y); c \neq a^r, \tag{2.18}$$

$$\begin{aligned} 'C(\eta^p,Y,Z,\eta^p) &= \beta(\eta^p,Z)\xi_p(Y) - \frac{a^r}{c}\beta(Y,Z) + \frac{a^r}{c}\xi_p(Y)\xi_p(X) \\ &\quad + \frac{a^r}{c}\left(\frac{n+2}{n}\right)\left[\frac{a^r}{c}G(Y,Z) - \xi_p(Y)\xi_p(Z)\right]; c \neq a^r, \end{aligned} \tag{2.19}$$

$$'C(X,Y,\eta^p,\eta^p) = \beta(X,\eta^p)\xi_p(Y) - \beta(Y,\eta^p)\xi_p(X), \tag{2.20}$$

$$'C(\eta^p,Y,\overline{Z},\overline{T}) = \beta(\eta^p,\overline{Z})G(Y,\overline{T}), \tag{2.21}$$

$$'C(\overline{X},\overline{Y},Z,\eta^p) = 0. \tag{2.22}$$

Proof. Replacing X by η^p in (2.16) and using (1.2), (1.4), (1.5), (1.6) and (1.9) we get (2.17)
Replacing T by η^p in (2.8) and using (1.4), (1.5), (1.6) and (1.9) we get (2.18).
Replacing T by η^p in (2.17) and using (1.4)and (1.6) we get (2.19).
Replacing Z by η^p in (2.18) and using (1.6) we get (2.20).
Replacing Z by $\overline{Z}$ and T by $\overline{T}$ in (2.17) and using (1.4) we get (2.21).
Replacing X by $\overline{X}$ and Y by $\overline{Y}$ in (2.18) and using (1.4) we get (2.22).

□

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