# Analytical Solution of Flow in a Composite Cylindrical Channel using Brinkman model

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**Abstract**

This study concerns steady flow of viscous, incompressible fluid in a composite cylindrical channel. Inner and outer part of the cylinder is of different permeability. Brinkman equation is used as a governing equation of motion in the porous cylinder. Analytical expressions for the two important cases, Poiseuille and Couette flow are obtained. Velocity profiles, rate of volume flow and shear stress on the boundaries are obtained and exhibited graphically. Effect of permeability variation parameter and gap parameter on the flow characteristics has been discussed.

**Subject class:** 76Dxx, 76Sxx.
**Keywords:** Composite channel, Porous Medium, Brinkman equation, Modified Bessel function, Poiseuille flow, Couette flow.

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## 1 Introduction:

Flow in porous media appear in many scientific, engineering and environmental applications. The Darcy-Brinkman equation is a governing equation for flow through a porous medium with an extra term (Brinkman term) added to the classical Darcy equation. The equation has been used widely to analyze high porosity porous media. There are number of authors you have investigated the flow through porous channels of various shapes. Surface covered with variable porosity porous medium, the catalyzer grains produced by calcination consist of layers with different porosities, tissues in human body, spherical particles having sticky or hairy surfaces are few examples of composite porous materials. The pioneer in this field are Kaviany (1985), Nakayama (1988), Vafai and Kim (1989), Plumb (1988), Vadasz (1993)and Al-Hadhrami et al. (2001) etc. Recently, Hooman and Gurgenci (2007) studied forced convection inside a circular tube filled with saturated porous medium and with uniform heat flux at the wall. Wang (2008) consider fully developed laminar forced convection inside a semi-circular channel filled with a Brinkman-Darcy porous medium and found analytical solutions for the flow. Pantokratoras and Fang (2010) found exact solutions for electromagnetic flow in a parallel plate channel filled with a porous medium using a Darcy-Brinkman flow model with a no-slip condition on boundaries. Wang (2010) investigate a fully developed flow and constant flux heat transfer in super-elliptic ducts filled with a porous medium. He obtained numerical solution using Ritz method for Darcy-Brinkman flow to determine the velocity and temperature fields. Verma and Datta (2012) found analytical solution for fully developed laminar flow of a viscous incompressible fluid in an annular region between two coaxial cylindrical tubes filled with a porous medium of variable permeability when the permeability of the porous medium varies with the radial distance. Wang (2016) studied oscillatory flow in ducts filled with a Darcy–Brinkman medium. He found analytic solutions for the annular, rectangular and the sector duct. Deo, Yadav and Tiwari (2010) found hydrodynamic permeability of membrane from porous particles. They presented variation of hydrodynamic permeability with different parameters. They found some new results for flow pattern in the porous region. Yadav (2018) studied comparative study for hydrodynamic permeability of membrane built up by non-homogeneous porous cylindrical particles and porous cylindrical shell enclosing a cylindrical cavity. Singh and Verma (2019) considered the steady flow of viscous, incompressible fluid in a composite cylindrical channel. The inner and outer parts of the cylindrical channel is of different permeability. The porous channel consists of two parts. The inner porous cylinder is of uniform permeability which is covered by an outer porous layer of variable permeability. They assumed two cases of permeability variation of the outer porous cylinder: (i) linear variation (ii) quadratic variation and concluded that permeability variation parameter and gap parameter have a very strong effect on the flow. Singh and Verma (2020) studied fully developed laminar flow of a viscous incompressible fluid in a long composite cylindrical channel. Channel consist of three regions. Outer and inner regions are of uniform permeability and mid region is a clear region. They found analytical expressions for velocity profiles, rate of volume flow and shear stress on the boundaries surface.

In the present paper we have considered steady flow of a viscous, incompressible fluid in a porous composite cylindrical channel. Inner and outer part of the cylinder is of different permeability. Analytical solutions are obtained by using Brinkman equation for the two special cases; (i) Poiseuille flow and (ii) Couette flow.

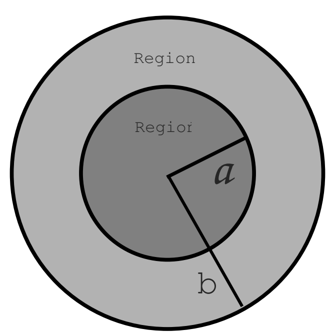


Fig. 1: Cross-sections of the porous channel.

## 2 Mathematical formulation:

Fully developed steady flow of a viscous, incompressible fluid in a composite cylinder is considered. The inner porous cylinder is of radius $a$ and is embedded in another porous cylinder of radius $b$, $(b > a)$. The geometry of the problem is shown in figure 1. The outer porous cylindrical region (region I) is of permeability $k_1$ and inner porous cylindrical region (region II) is of permeability $k_2$. Both regions have common constant pressure gradient. The Brinkman (1947) momentum equation for a fully developed flow is

$$\mu_e \left( \frac{d^2u^*}{dr^{*2}} + \frac{1}{r^*}\frac{du^*}{dr^*} \right) - \frac{\mu}{k}u^* = \frac{\partial p^*}{\partial z^*} \tag{2.1}$$

where $u^*$ is the fluid velocity, $\mu_e$ is the effective viscosity of porous medium, $\mu$ is the fluid viscosity, $k$ is the permeability of porous medium and $\partial p^*/\partial z^*$ is the applied pressure gradient. For the present problem we follow Brinkman (1947) and Chikh et al. (1995) and assume that $\mu_e = \mu$ (for high porosity cases). Therefore, eq.(2.1) becomes

$$\frac{d^2u^*}{dr^{*2}} + \frac{1}{r^*}\frac{du^*}{dr^*} - \frac{u^*}{k} = \frac{1}{\mu}\frac{\partial p^*}{\partial z^*} \tag{2.2}$$

Now we introduce dimensionless variables as follows

$$r = \frac{r^*}{a} \quad \text{and} \quad u = \frac{\mu u^*}{a^2(-\partial p^*/\partial z^*)}$$

the characteristic velocity being determined by $\frac{a^2}{\mu}(-\partial p^*/\partial z^*)$. Using these dimensionless variables in eq.(2.2) and after dropping the star index for our convenience, we get governing Brinkman equation of motion as

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{a^2}{k}u = -1 \tag{2.3}$$

Thus for inner porous cylinder of permeability $k_1$ , the governing equation of motion is

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \alpha^2 u = -1; \qquad (0 \le r \le 1) \tag{2.4}$$

where $a^2/k_1 = \alpha^2$ is called permeability variation parameter for region I and $u$ is the velocity in region I.

Similarly, for outer porous cylinder is of permeability $k_2$, the governing equation of motion is

$$\frac{d^2v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \beta^2 v = -1; \qquad (1 \le r \le q = b/a) \tag{2.5}$$

where $a^2/k_2 = \beta^2$ is called permeability variation parameter for region II and $v$ is the velocity in region II. Eq.(2.4) and (2.5) are modified Bessel's equations of order zero.

## 3 Solution and Results:

We will obtain solutions for two most applicable cases, Poiseuille flow and Couette-Poiseuille flow.

### 3.1 Poiseuille Flow

For Poiseuille flow both the inner and outer cylinders are stationary and the flow is due to applied pressure gradient. We have no slip condition on the surface of outer cylinder and continuity of velocity and tangential stress on the interface of the porous region I and region II. Also velocity is either maximum or minimum at $r = 0$ depending on the permeability of the two regions. Thus we have following boundary conditions

$$\begin{aligned} v(r) &= 0 && \text{at } r = q \\ u(r) &= v(r) && \text{at } r = 1 \\ u'(r) &= v'(r) && \text{at } r = 1 \\ u'(r) &= 0 && \text{at } r = 0 \end{aligned} \tag{3.1}$$

where, $q = b/a$. The general solutions of eq.(2.4) and (2.5) are given by

$$u(r) = A_1 I_0(\alpha r) + A_2 K_0(\alpha r) + \frac{1}{\alpha^2}, \quad (0 \le r \le 1) \tag{3.1.2}$$

and

$$v(r) = B_1 I_0(\beta r) + B_2 K_0(\beta r) + \frac{1}{\beta^2}, \quad (1 \le r \le q) \tag{3.1.3}$$

where, $I_o$ and $K_o$ are the modified Bessel functions of zeroth order of first and second kind, respectively. Here $A_1$, $A_2$, $B_1$ and $B_2$ are constants of integration to be determined. Using boundary conditions (3.1), we get constants $A_1$, $A_2$, $B_1$ and $B_2$ as given below

(3.1.4)

$$A_1 = \left[ \frac{\alpha^2 I_1(\beta)K_0(\beta) + \alpha^2 I_0(\beta)K_1(\beta) - \alpha^2 I_1(\beta)K_0(q\beta) - \alpha^2 K_1(\beta)I_0(q\beta) + \beta^2 I_1(\beta)K_0(q\beta) + \beta^2 K_1(\beta)I_0(q\beta)}{\alpha^2\beta(-\alpha I_1(\alpha)K_0(\beta)I_0(q\beta) + \alpha I_1(\alpha)I_0(\beta)K_0(q\beta) - \beta I_0(\alpha)I_1(\beta)K_0(q\beta) - \beta I_0(\alpha)K_1(\beta)I_0(q\beta))} \right]$$

$$A_2 = 0 \tag{3.1.5}$$

(3.1.6)

$$B_1 = \left[ \frac{\alpha^2 I_1(\alpha)K_0(\beta) + \alpha\beta I_0(\alpha)K_1(\beta) - \alpha^2 I_1(\alpha)K_0(q\beta) + \beta^2 I_1(\alpha)K_0(q\beta)}{\alpha\beta^2(-\alpha I_1(\alpha)K_0(\beta)I_0(q\beta) + \alpha I_1(\alpha)I_0(\beta)K_0(q\beta) - \beta I_0(\alpha)I_1(\beta)K_0(q\beta) - \beta I_0(\alpha)K_1(\beta)I_0(q\beta))} \right]$$

(3.1.7)
$$B_2 = \left[ \frac{-\alpha^2 I_1(\alpha) I_0(\beta) + \alpha\beta I_0(\alpha) I_1(\beta) + \alpha^2 I_1(\alpha) I_0(q\beta) - \beta^2 I_1(\alpha) I_0(q\beta)}{\alpha\beta^2(-\alpha I_1(\alpha) K_0(\beta) I_0(q\beta) + \alpha I_1(\alpha) I_0(\beta) K_0(q\beta) - \beta I_0(\alpha) I_1(\beta) K_0(q\beta) - \beta I_0(\alpha) K_1(\beta) I_0(q\beta))} \right]$$

The dimensionless velocity of the fluid at any point within the region I and II is given by eq.(3.1.2) and (3.1.3) on insertion of the preceding values of $A_1$, $A_2$, $B_1$ and $B_2$. The graphical presentation of velocity profiles for $\alpha < \beta$ and $\alpha > \beta$ is given in fig.(2). In the limiting case, when $\alpha$ and $\beta \to 0$ (i.e., when permeability of the porous medium is infinite in both the regions) in eq.(3.1.2) and (3.1.3), we obtain

$$\lim_{\alpha\to 0} u = \lim_{\beta\to 0} v = \frac{\left(q^2 - r^2\right)}{4} \tag{3.1.8}$$

which is velocity profile for classical Hagen-Poiseuille flow of a clear fluid through a cylindrical channel of radius $q$. If we take limit $\alpha \to \infty$ and $\beta \to 0$ in eq.(3.1.3). We get

$$\lim_{\substack{\alpha\to\infty \\ \beta\to 0}} v = \left[ \frac{\left(q^2 - 1\right)\log r - \left(r^2 - 1\right)\log q}{4\log q} \right] \tag{3.1.9}$$

which is the classical velocity profile for the clear fluid flow within an annular region between two coaxial cylinders having impermeable walls.

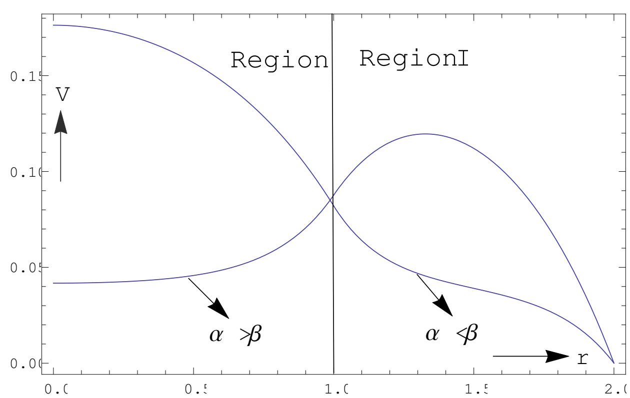


Fig. 2: Variation of velocity for poiseuille flow with radial distance $r$ for two different cases, when $\alpha > \beta$ ($\alpha = 5$ and $\beta = 2$) and when $\alpha < \beta$ ($\alpha = 2$ and $\beta = 5$) for fixed $q = 2$ .

## Rate of volume flow

The dimensionless rate of volume flow through cross-section of the inner cylinder is given by

$$Q_1 = 2\pi \int_0^1 u(r) \quad rdr \tag{3.1.10}$$

Substituting $u(r)$ from eq.(3.1.2) in the above equation, we obtain

$$Q_1 = 2\pi \left[ \frac{1}{2\alpha^2} + A_1 \left( \frac{I_1(\alpha)}{\alpha} \right) \right] \tag{3.1.11}$$

Similarly, the dimensionless rate of volume flow through cross-section of the outer cylinder is given by

$$Q_2 = 2\pi \int_1^q v(r)\; r\; dr \tag{3.1.12}$$

Substituting $v(r)$ from eq.(3.1.3) and after integration, we obtain

$$Q_2 = \frac{\pi}{\beta^2} \left[ 2\beta\{B_1(qI_1(q\beta) - I_1(\beta)) + B_2(K_1(\beta) - qK_1(q\beta))\} + q^2 - 1 \right] \tag{3.1.13}$$

where $I_1$ and $K_1$ are the modified Bessel functions of first and second kind of order one and constants $A_1$, $A_2$, $B_1$ and $B_2$ are given by eq.(3.1.4), (3.1.5), (3.1.6) and (3.1.7), respectively. In the evaluation of above integrals the following identity [Ref. Abramowitz and Stegun (1970)] has been used

$$\left( \frac{1}{z} \frac{d}{dz} \right)^m \{z^\nu \pounds_\nu(z)\} = z^{\nu - m} \pounds_{\nu - m}(z) \tag{3.1.14}$$

with $m = 1$ and $\nu = 1$. $\pounds_\nu$ denotes $I_\nu$ and $e^{\nu\pi i} K_\nu$. Now, the total dimensionless rate of volume flow through the channel is given by

$$Q = Q_1 + Q_2$$

(3.1.15)

$$Q = \frac{\pi}{\alpha^2\beta^2} [\{2\alpha A_1 I_1(\alpha) + 1\}\beta^2 + \{2\beta\{B_1(qI_1(q\beta) - I_1(\beta)) + B_2(K_1(\beta) - qK_1(q\beta))\} + q^2 - 1\}\alpha^2]$$

The dimensionless volume flow rate $Q_0$ for clear fluid flow (when permeability is infinite) can be obtained by taking limit $\alpha$ and $\beta \to 0$ in eq.(3.1.15). We get

$$Q_0 = \lim_{\substack{\alpha \to 0 \\ \beta \to 0}} Q = \frac{\pi q^4}{8} \tag{3.1.16}$$

This is a classical result for volume flow rate of Hagen-Poiseuilli flow. If we take limit $\alpha \to \infty$ and $\beta \to 0$ in eq.(3.1.15). We get

$$\lim_{\substack{\alpha \to \infty \\ \beta \to 0}} Q = \frac{\pi \left(q^2 - 1\right) \left[ \left(q^2 + 1\right) \log q + 1 - q^2 \right]}{8 \log q}$$

which is the dimensionless volume flow rate for clear fluid flow within an annular region between two coaxial cylinders having impermeable walls.

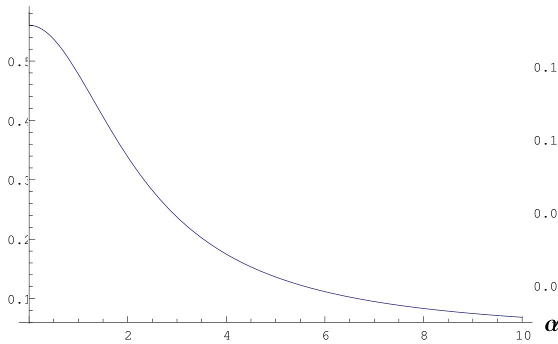


Fig. 3: Variation of volume flow rate with $\alpha$ for Poiseuilli flow when $q = 2$ and $\beta = 15$.

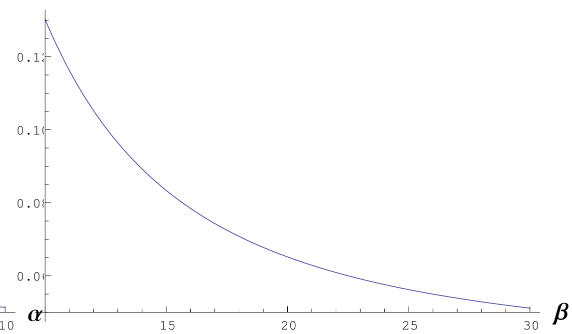


Fig. 4: Variation of volume flow rate with $\beta$ for Poiseuilli flow when $q = 2$ and $\alpha = 8$.

## Average velocity

The dimensionless average velocity of the flow is defined as

$$u_{avg} = \frac{Q}{\pi q^2} \tag{3.1.17}$$

Substituting $Q$ from the eq.(3.1.15) into above eq.(3.1.17), the average velocity of the flow within the composite channel is

$$\begin{aligned} u_{avg} &= \frac{1}{q^2}[\frac{1}{\alpha^2}(2\alpha A_1 I_1(\alpha)+1)+\frac{1}{\beta^2}((2\beta(B_1(-I_1(\beta))+B_1 q I_1(q\beta)+B_2(K_1(\beta) \\ &\quad -qK_1(q\beta)))+q^2-1))] \end{aligned} \tag{3.2}$$

where constants $A_1$, $A_2$, $B_1$ and $B_2$ are given by eq.(3.1.4), (3.1.5), (3.1.6) and (3.1.7), respectively. For clear fluid flow average velocity of the flow is obtained by taking limit $\alpha$ and $\beta \to 0$ in eq.(3.2), we get

$$\lim_{\substack{\alpha\to 0\\ \beta\to 0}} u_{avg} = \frac{q^2}{8} \tag{3.1.19}$$

which is well known average velocity for classical Hagen-Poiseuilli flow. If we take limit $\alpha \to \infty$ and $\beta \to 0$ in eq.(3.2), we get

$$\lim_{\substack{\alpha\to \infty\\ \beta\to 0}} u_{avg} = \left[\frac{-\left(q^2-1\right)^2+\left(q^4-1\right)\log q}{8q^2\log q}\right] \tag{3.1.20}$$

which is the dimensionless average velocity for clear fluid flow within an annular region between two coaxial cylinders having impermeable walls.

## Shearing stress on the surface of pipe

The dimensionless shearing stress at any point within inner porous cylinder is given by

$$\tau_{rz}(r) = -\frac{du}{dr} \tag{3.1.21}$$

Substituting $u$ from eq.(3.1.2) and differentiating the modified Bessel functions $I_o(\alpha r)$ and $K_o(\alpha r)$ with use of the identity $\frac{d}{dr}I_o(r) = I_1(r)$ and $\frac{d}{dr}K_o(r) = -K_1(r)$ [Ref. Abramowitz and Stegun (1970)], we get

$$\tau_{rz}(r) = -A_1\alpha\ I_1(\alpha r) \tag{3.1.22}$$

Similarly, the dimensionless shearing stress at any point within outer porous cylinder is given by

$$\tau_{rz}(r) = -\frac{dv}{dr} \tag{3.1.23}$$

Substituting $v$ from eq.(3.1.3) in above equation and after differentiation, we get

$$\tau_{rz}(r) = -\beta[B_1 I_1(\beta r) - B_2 K_1(\beta r)] \tag{3.1.24}$$

where $I_1$ and $K_1$ are modified Bessel function of order one. Shear stress on the surface of inner and outer cylinder is obtained by putting $r = 1$ and $r = q$ in eq.(3.1.22) and (3.1.24), respectively and using the appropriate sign. We obtain

$$\tau_{rz}(1) = -A_1\alpha I_1(\alpha) \tag{3.1.25}$$

and,

$$\tau_{rz}(q) = -\beta[B_1 I_1(\beta q) - B_2 K_1(\beta q)] \tag{3.1.26}$$

where $A_1$, $B_1$ and $B_2$ are given by eq.(3.1.4), (3.1.6) and (3.1.7), respectively. Dimensionless shearing stress on the surface of outer cylinder for clear fluid flow is obtained by taking limit $\alpha$ and $\beta \to 0$ in eq.(3.1.26), we get

$$\lim_{\substack{\alpha\to 0\\ \beta\to 0}} \tau_{rz}(q) = \frac{q}{2} \tag{3.1.27}$$

which is skin friction on the surface of cylinder for Hagen-Poiseuilli flow of clear fluid. If we take limit $\alpha \to \infty$ and $\beta \to 0$ in eq.(3.1.25), we obtain

$$\lim_{\substack{\alpha\to\infty\\ \beta\to 0}} \tau_{rz}(q) = \frac{(1 - q^2 + 2q^2 \log q)}{4q \log q} \tag{3.1.28}$$

which is the dimensionless skin friction on the impermeable wall of outer cylinder for clear fluid flow within an annular region between two coaxial cylinders.

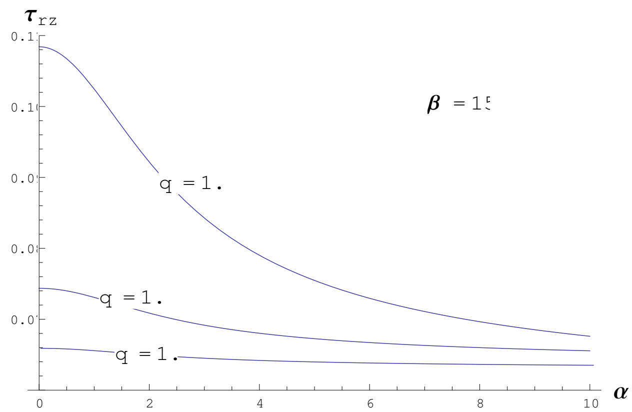


Fig. 5: Variation of shear stress for Poiseuille flow on the impermeable surface of cylinder with $\alpha$ for different values of gap parameter $q$ and fixed value of permeability parameter $\beta = 15$.

## 3.2 Couette-Poiseuille Flow

For Couette-Poiseuille flow the outer cylinder is moving with the constant velocity $U = \frac{a^2}{\mu}\left(-\frac{\partial p^*}{\partial z^*}\right)$ along the axis of cylinder and there is constant applied pressure gradient. The inner cylinder is stationary. We have no slip conditions at the boundary of outer cylinder, continuity of velocity and stress at the interface of the two region and $\frac{du}{dr} = 0$ at $r = 0$. In non dimensional form these conditions are

$$\begin{aligned} v(r) &= 1 \quad \text{at } r = q \\ u(r) &= v(r) \quad \text{at } r = 1 \\ u'(r) &= v'(r) \quad \text{at } r = 1 \\ u'(r) &= 0 \quad \text{at } r = 0 \end{aligned} \tag{3.3}$$

The general solutions of eq.(2.4) and (2.5) are given by

$$u(r) = C_1 I_0(\alpha r) + C_2 K_0(\alpha r) + \frac{1}{\alpha^2} \tag{3.2.2}$$

and,

$$v(r) = D_1 I_0(\beta r) + D_2 K_0(\beta r) + \frac{1}{\beta^2} \tag{3.2.3}$$

where, $I_o$ and $K_o$ are the modified Bessel functions of zeroth order of first and second kind respectively. Chikh et al. (1995) find similar solution for the forced convection in an annular duct partially filled with a porous medium.

here $C_1$, $C_2$, $D_1$ and $D_2$ are constants of integration determined by using above boundary conditions (3.3), we get constants $C_1$, $C_2$, $D_1$ and $D_2$ as

(3.2.4)
$$C_1 = \left[ \frac{\begin{array}{c}\alpha^2\left(-\beta^2\right) I_1(\beta)K_0(\beta) - \alpha^2\beta^2 I_0(\beta)K_1(\beta) + \alpha^2 I_1(\beta)K_0(\beta) + \alpha^2 I_0(\beta)K_1(\beta)\\ -\alpha^2 I_1(\beta)K_0(q\beta) - \alpha^2 K_1(\beta)I_0(q\beta) + \beta^2 I_1(\beta)K_0(q\beta) + \beta^2 K_1(\beta)I_0(q\beta)\end{array}}{\begin{array}{c}\alpha^2\beta(-\alpha I_1(\alpha)K_0(\beta)I_0(q\beta) + \alpha I_1(\alpha)I_0(\beta)K_0(q\beta) - \beta I_0(\alpha)I_1(\beta)K_0(q\beta)\\ -\beta I_0(\alpha)K_1(\beta)I_0(q\beta))\end{array}} \right]$$

(3.2.5)
$$C_2 = 0$$

(3.2.6)
$$D_1 = \left[ \frac{\begin{array}{c}-\alpha^2\beta^2 I_1(\alpha)K_0(\beta) + \alpha^2 I_1(\alpha)K_0(\beta) - \alpha\beta^3 I_0(\alpha)K_1(\beta) + \alpha\beta I_0(\alpha)K_1(\beta)\\ -\alpha^2 I_1(\alpha)K_0(q\beta) + \beta^2 I_1(\alpha)K_0(q\beta)\end{array}}{\begin{array}{c}\alpha\beta^2(-\alpha I_1(\alpha)K_0(\beta)I_0(q\beta) + \alpha I_1(\alpha)I_0(\beta)K_0(q\beta) - \beta I_0(\alpha)I_1(\beta)K_0(q\beta)\\ -\beta I_0(\alpha)K_1(\beta)I_0(q\beta))\end{array}} \right]$$

(3.2.7)
$$D_2 = \left[ \frac{\begin{array}{c}-\alpha^2\beta^2 I_1(\alpha)I_0(\beta) + \alpha^2 I_1(\alpha)I_0(\beta) + \alpha\beta^3 I_0(\alpha)I_1(\beta) - \alpha\beta I_0(\alpha)I_1(\beta)\\ -\alpha^2 I_1(\alpha)I_0(q\beta) + \beta^2 I_1(\alpha)I_0(q\beta)\end{array}}{\begin{array}{c}\alpha\beta^2(\alpha I_1(\alpha)K_0(\beta)I_0(q\beta) - \alpha I_1(\alpha)I_0(\beta)K_0(q\beta) + \beta I_0(\alpha)I_1(\beta)K_0(q\beta)\\ +\beta I_0(\alpha)K_1(\beta)I_0(q\beta))\end{array}} \right]$$

The dimensionless velocity of the fluid at any point within the region I and II is given by eq.(3.2.2) and (3.2.3) on insertion of the preceding values of $C_1$, $C_2$, $D_1$ and $D_2$. The graphical presentation of velocity profiles for $\alpha < \beta$ and $\alpha > \beta$ is given in fig.(6).

In the limiting case, when $\alpha$ and $\beta \to 0$ (i.e., when permeability of the porous medium is infinite in both the regions) in eq.(3.2.2) and (3.2.3), the classical velocity, $u_0$ (inner cylinder) and $v_0$ (outer cylinder) for clear fluid flow is obtained as

(3.2.8)
$$\lim_{\alpha\to 0} u = \lim_{\beta\to 0} v = \frac{1}{4}\left(q^2 - r^2 + 4\right)$$

which is velocity profile for classical flow of a clear fluid through a translating cylindrical channel of radius $q$. If we take limit $\alpha \to \infty$ and $\beta \to 0$ in eq.(3.2.3). We get

(3.2.9)
$$\lim_{\substack{\alpha\to\infty\\ \beta\to 0}} v = \frac{\left(q^2+3\right)\log r - \left(r^2-1\right)\log q}{4\log q}$$

which is the classical result for velocity profile of the clear fluid flow within an annular region between two coaxial cylinders when outer is moving and inner is fixed.

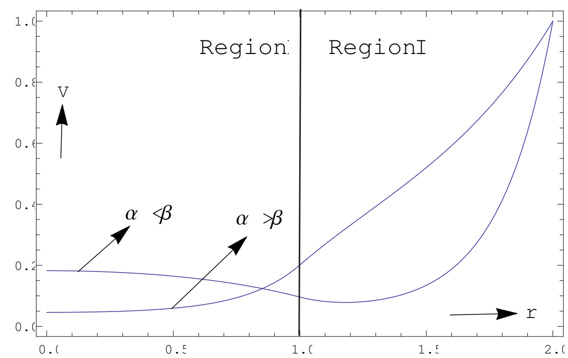


Fig. 6: Variation of velocity for Couette-Poiseuille flow with radial distance $r$ for two different cases, when $\alpha > \beta$ ($\alpha = 5$ and $\beta = 2$) and when $\alpha < \beta$ ($\alpha = 2$ and $\beta = 5$) for fixed $q = 2$ .

## Rate of volume flow

The dimensionless rate of volume flow through cross-section of the inner cylinder is given by

$$Q_1 = 2\pi \int_0^1 u(r)\ rdr \tag{3.2.10}$$

Substituting $u(r)$ from eq.(3.2.2) in above equation, we obtain

$$Q_1 = \frac{\pi}{\alpha^2}\left[(2\alpha C_1 I_1(\alpha) + 1)\right] \tag{3.2.11}$$

Similarly, the dimensionless rate of volume flow through cross-section of the outer cylinder is given by

$$Q_2 = 2\pi \int_1^q v(r)\ r\ dr \tag{3.2.12}$$

Substituting $v(r)$ from equation we obtain

$$Q_2 = \frac{\pi}{\beta^2}\left[2\beta\{D_1(qI_1(q\beta) - I_1(\beta)) + D_2(K_1(\beta) - qK_1(q\beta))\} + q^2 - 1\right] \tag{3.2.13}$$

where $I_1$ and $K_1$ are the modified Bessel functions of first kind of order one and constants $C_1$, $C_2$, $D_1$ and $D_2$ are given by eq.(3.2.4), (3.2.5), (3.2.6) and (3.2.7), respectively. Now the total dimensionless rate of volume flow through the channel is given by

$$Q = Q_1 + Q_2$$

(3.2.14)

$$Q = \frac{\pi}{\alpha^2\beta^2}[\{2\alpha C_1 I_1(\alpha) + 1\}\beta^2 + \{2\beta\{D_1(qI_1(q\beta) - I_1(\beta)) + D_2(K_1(\beta) - qK_1(q\beta))\} + q^2 - 1\}\alpha^2]$$

The dimensionless volume flow rate $Q_0$ for clear fluid flow (when permeability is infinite) can be obtained by taking limit $\alpha$ and $\beta \to 0$ in eq.(3.2.14). We get

$$Q_0 = \lim_{\substack{\alpha \to 0 \\ \beta \to 0}} Q = \frac{\pi q^2}{8}\left(q^2 + 8\right) \tag{3.2.15}$$

This is the classical result for clear fluid flow through moving cylinder of radius $q$. If we take limit $\alpha \to \infty$ and $\beta \to 0$ in eq.(3.2.14). We get

$$\lim_{\substack{\alpha \to \infty \\ \beta \to 0}} Q = \frac{\pi\left(3 - q^4 - 2q^2 + \left(q^4 + 8q^2 - 1\right)\log q\right)}{8 \log q} \tag{3.2.16}$$

which is the dimensionless volume flow rate for Couette-Poiseuille flow between two coaxial cylinders.

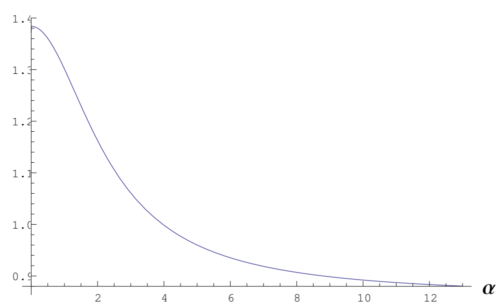


Fig. 7: Variation of volume flow rate with $\alpha$ for Couette-Poiseuille flow when $q = 2$ and $\beta = 15$.

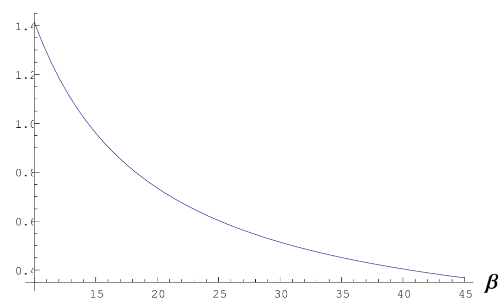


Fig. 8: Variation of volume flow rate with $\beta$ for Couette-Poiseuille flow when $q = 2$ and $\alpha = 5$.

## Average velocity

The dimensionless average velocity for Couette-Poiseuille flow is defined as

$$u_{avg} = \frac{Q}{\pi q^2} \tag{3.2.17}$$

Substituting $Q$ from the eq.(3.2.14) into above eq.(3.2.17), the average velocity of the flow within the composite channel is
(3.2.18)

$$u_{avg} = \frac{1}{q^2\alpha^2\beta^2}[\{2\alpha C_1 I_1(\alpha)+1\}\beta^2+\{2\beta\{D_1(qI_1(q\beta)-I_1(\beta))+D_2(K_1(\beta)-qK_1(q\beta))\}+q^2-1\}\alpha^2]$$

where constants $C_1$, $D_2$ and $D_2$ are given by eq.(3.2.4), (3.2.5), (3.2.6) and (3.2.7), respectively. For clear fluid flow average velocity of the flow is obtained by taking limit $\alpha$ and $\beta \to 0$ in eq.(3.2.18), we get

$$\lim_{\substack{\alpha \to 0 \\ \beta \to 0}} u_{avg} = \frac{\left(q^2 + 8\right)}{8} \tag{3.2.19}$$

which is well known average velocity for clear flow in a moving cylinder of radius $q$. If we take limit $\alpha \to \infty$ and $\beta \to 0$ in eq.(3.2.18), we get

$$\lim_{\substack{\alpha\to\infty \\ \beta\to 0}} u_{avg} = \frac{(3 - q^4 - 2q^2) + \left(q^4 + 8q^2 - 1\right)\log q}{8q^2 \log q} \tag{3.2.20}$$

which is the dimensionless average velocity for classical Couette-Poiseuille flow between two coaxial cylinders.

## Shearing stress on the surface of pipe

The dimensionless shearing stress at any point within inner porous cylinder is given by,

$$\tau_{rz}(r) = -\frac{du}{dr} \tag{3.2.21}$$

Substituting $u$ from eq.(3.2.2) and and differentiating the modified Bessel functions $I_o(\alpha r)$ and $K_o(\alpha r)$ with use of the identity $\frac{d}{dr}I_o(r) = I_1(r)$ and $\frac{d}{dr}K_o(r) = -K_1(r)$ [Ref. Abramowitz and Stegun (1970)], we obtain

$$\tau_{rz}(r) = C_1 \alpha I_1(\alpha r) \tag{3.2.22}$$

Similarly, the dimensionless shearing stress at any point within outer porous cylinder is given by,

$$\tau_{rz}(r) = -\frac{dv}{dr} \tag{3.2.23}$$

substituting $v$ from eq.(3.2.3) in above equation and after differentiation. we get

$$\tau_{rz}(r) = \beta[D_1 I_1(\beta r) - D_2 K_1(\beta r)] \tag{3.2.24}$$

where $I_1$ and $K_1$ are modified Bessel function of order one. Shear stress on the surface of inner and outer cylinder is obtained by putting $r = 1$ and $r = q$ in eq.(3.2.22) and (3.2.24), respectively and using the appropriate sign. We obtain

$$\tau_{rz}(1) = -C_1 \alpha I_1(\alpha) \tag{3.2.25}$$

and,

$$\tau_{rz}(q) = -\beta[D_1 I_1(\beta q) - D_2 K_1(\beta q)] \tag{3.2.26}$$

where $C_1$, $D_1$ and $D_2$ are given by eq.(3.2.4), (3.2.6) and (3.2.7), respectively. Dimensionless shearing stress on the surface of outer cylinder for clear fluid flow is obtained by taking limit $\alpha$ and $\beta \to 0$ in eq.(3.2.26), we get

$$\lim_{\substack{\alpha\to 0 \\ \beta\to 0}} \tau_{rz}(q) = \frac{q}{2} \tag{3.2.27}$$

which is skin friction on the surface of cylinder for clear fluid flow in a moving cylinder of radius $q$. If we take limit $\alpha \to \infty$ and $\beta \to 0$ in eq.(3.2.26), we obtain

$$\lim_{\substack{\alpha\to\infty \\ \beta\to 0}} \tau_{rz}(q) = \frac{q^2 - 2q^2 \log q + 3}{4q \log q} \tag{3.2.28}$$

which is the dimensionless skin friction on the impermeable wall of outer cylinder for clear Couette-Poiseuille flow between two coaxial cylinders.

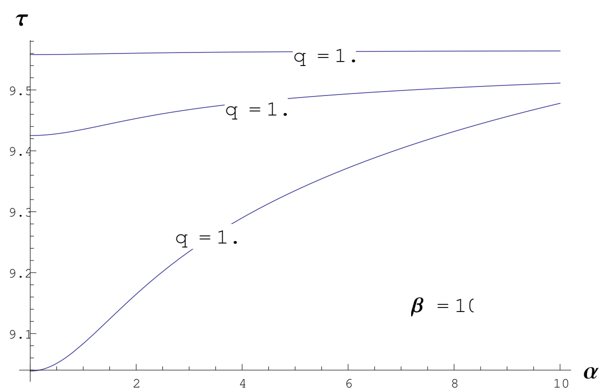


Fig. 9: Variation of shear stress for Couette - Poiseuille flow on the impermeable surface of cylinder with $\alpha$ for different values of gap parameter $q$ and fixed value of pereability parameter $\beta = 10$.

## 4 Discussion:

Fig.(2) shows the velocity profile for Poiseuille flow within the cylinder computed from eq.(3.1.2) and (3.1.3). The velocity profile is drawn for two cases for fixed value of $q = 2$; (i) when permeability of inner porous region is greater than that of outer region ($\alpha < \beta$ ) (ii) when permeability of outer porous region is greater than that of inner region ($\alpha > \beta$). We observe that velocity is greater in the region where permeability is large. When ($\alpha > \beta$) maximum velocity occur in outer region and when ($\alpha < \beta$) maximum velocity occur in the inner region at the centre of cylinder.

Fig.(6) shows the velocity profile for Couette-Poiseuille flow within the cylinder computed from eq.(3.2.2) and (3.2.3) for the two different cases; (i) when $\alpha < \beta$ (ii) when $\alpha > \beta$. We observe that motion of outer cylinder has great effect on the flow velocity in both the porous regions within the cylinder. Although the permeability of in inner region is large in the case ($\alpha < \beta$) than that of in the case ($\alpha > \beta$) but velocity low in the region near the interface.

Fig.(3), (4), (7) and (8) represents the variation of volume flow rate with permeability parameter $\alpha$ and $\beta$ for Poiseuille and Couette-Poiseuille flow. It is clear that with the increase of permeability parameter volume flow rate decreases in both the porous regions. This is due to the fact that increase in permeability parameter causes decrease in permeability of the porous medium.

Fig.(5) represents the variation of shear stress on the surface of impermeable cylinder with the permeability parameter $\alpha$ (permeability of inner region) for Poiseuille flow for different values of gap parameter $q$ and fixed value of $\beta = 15$. We observe that stress decreases with increase in the value of $\alpha$ i.e., decrease in the permeability on inner region. Also, stress on the cylinder increases with decrease in the gap parameter $q$. It is found that effect of permeability parameter $\alpha$ on the variation of stress large for small value of $q$ ($q = 1.2$) and this effect is almost negligible for large value of $q$ ($q = 1.5$).

Fig.(9) represents the variation of shear stress on the surface of impermeable cylinder with the permeability parameter $\alpha$ (permeability of inner region) for Couette-Poiseuille flow for different values of gap parameter $q$ and fixed value of $\beta = 10$. We observe that stress increases with increase in the value of $\alpha$ i.e., decrease in the permeability on inner region. This behaviour is opposite to that of in case of Poiseuille flow. Also, stress on the cylinder increases with increase in the gap parameter $q$. It is found that effect of permeability parameter $\alpha$ on the variation of stress large for small value of $q$ ($q = 1.2$) and this effect is almost negligible for large value of $q$ ($q = 1.5$) as in case of Poiseuille flow.

## 5 Conclusion:

The flow of a viscous incompressible fluid within a composite porous cylindrical channel has been investigated using the Brinkman model. Inner and outer part of the cylinder is of different permeability. Two special and useful flow namely, Poiseuille flow and Couette-Poiseuille flow has been considered. An analytical solution of the governing equations for the flow within the channel has been obtained. The exact expressions for velocity volume flow rate and sheer stress on the cylinder are obtained. In the limiting case when permeability of the porous regions tend to infinite the obtained results reduces to the classical results of Poiseuille and Couette-Poiseuille flow of clear fluid in the cylindrical channel. We found that variation of permeability and gap parameter has remarkable effect on the flow.

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