Existence and controllability of Hilfer fractional differential inclusions

Deepak Dhiman [1a], Ashok Kumar [1b], Ashish Kumar [2] and Lakshmi Narayan Mishra [3,*]

[1]*Department of Mathematics, H. N. B. Garhwal University, Srinagar-246174, India.*
email: [a] deepakdhiman09@gmail.com , [b] ashrsdma@gmail.com

[2]*Department of Mathematics, Indian Institute of Technology, Roorkee-247667, India*
email: akumar2@ma.iitr.ac.in

[3]*Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology (VIT) University, Vellore, Tamil Nadu 632 014, India.*
email: lakshminarayanmishra04@gmail.com

Abstract

In this paper, we study the controllability for a class of Hilfer semilinear differential inclusions in Banach spaces by dropping the compactness of the evolution operator generated by the linear part and by assuming the regularity of the nonlinear part with respect to the weak topology. Also, we are not implementing any conditions on the multivalued nonlinearity expressed in terms of measures of noncompactness. An example is also given to illustrate the obtained theoretical results.

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1 Introduction

Fractional calculus is the generalization of the standard integer calculus to the arbitrary order which provides a great deal for the description of memory and hereditary properties of diversified materials and processes. In the past twenty years, the subject of the fractional calculus has attracted research attentions towards itself due to its applications in various branches of science like physics,viscoelasticity, fluid mechanics, heat conduction [1, 23, 24, 15]. One can refer to the monographs of Kilbas et al. [20], Miller and Ross [25] and Podlubny [36] for the fundamentals of fractional calculus.

Controllability is one of the important fundamental concepts in applied Mathematics. Exact controllability enables to steer the system from initial state to arbitrary final state, while approximate controllability means that system can be

[1] *Corresponding Author

steered to arbitrary small neighborhood of final state.On the finite dimensional systems, both the concepts are equivalent while in infinite dimensional systems, controllability of the system is sufficient for the approximate controllability of the system. For the basics and current research in the control theory, one can see the articles [6, 13, 21, 22, 37, 39, 42, 43].
In the literature, it has been seen that functional integral and fractional differential equations are closely related. For detailed work one can see the references [26, 28, 29, 30]. Fixed point theory is a great tool to study the existence and uniqueness of solutions of fractional differential equations. Theory and applications of fixed point theory can be found in [3, 8, 9, 27, 31, 32, 33, 34, 38] and the references therein.

Hilfer[16] proposed a generalized Riemann-Liouville fractional derivative, named Hilfer fractional derivative, which interpolates Riemann-Liouville fractional derivative and Caputo fractional derivative and later on presented the solution of such linear differential equations [17]. Subsequently, Furati et al.[12] proved existence and uniqueness for the nonlinear initial value problem in a weighted space of continuous function. Gu and Trujillo [14] discussed the existence of mild solution of evolution equation with Hilfer fractional derivative via measure of noncompactness. Ahmad et al.[2] studied the existence of mild solutions of Hilfer fractional stochastic integro-differential equations with nonlocal conditions via Sadovskii fixed point theorem. Approximate controllability of a class of semilinear Hilfer fractional differential control inclusions in Banach spaces is investigated in [7]. Yang et al.[44] established the approximate controllability of Hilfer fractional differential inclusions with nonlocal conditions.

Benedetti et al.[4] explored the controllability for a class of semilinear differential inclusions by assuming the regularity of the nonlinear part with respect to the weak topology and by dropping the compactness of the evolution operator generated by the linear part. By the same technique, Zhou et al.[45] discussed the existence and controllability for Caputo fractional evolution inclusions in Banach spaces. Wang et al.[41] studied the controllability for fractional non-instantaneous nonlinear impulsive differential inclusions by establishing weakly convergent criteria in the piecewise continuous functions spaces and by dropping the regularity conditions on the multivalued non-linearity expressed in terms of measures of noncompactness and by dropping the invertibility of the linear controllability operator satisfies a condition expressed in terms of measures of non-compactness. Vijayakumar and Murugesu[40] investigated the controllability of second-order non-autonomous differential inclusions in Banach spaces without compactness. Motivated by the same, the main purpose in this paper is to study the the existence and controllability of the following Hilfer fractional differential inclusions by dropping the compactness condition of the operator generated by linear part of the differential inclusion system which is not discussed so far to the best of our knowledge.

$$
\begin{aligned}
D^{\beta,\gamma}x(t) &\in Ax(t) + F(t, x(t)), \quad t \in J := (0, b] \\
I_{0+}^{(1-\beta)(1-\gamma)}x(0) &= x_0.
\end{aligned}
\tag{1.1}
$$

where $A : D(A) \subseteq \mathbb{X} \to \mathbb{X}$ is a closed linear operator on a Banach space $\mathbb{X}$ which is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators (i.e. C_0 semigroup) $\{Q(t)\}_{t\geq 0}$ in Banach space $\mathbb{X}$ and the state $x(\cdot)$ takes values in a Banach space $\mathbb{X}$, $x_0 \in \mathbb{X}$, $F : J \times \mathbb{X} \to \mathbb{X}$ is a multivalued map.

We also investigate the controllability of the following Hilfer fractional differential inclusions

$$\begin{aligned} D^{\beta,\gamma}x(t) &\in Ax(t)+Bu(t)+F(t,x(t)), t\in(0,b] \\ I_{0+}^{(1-\beta)(1-\gamma)}x(0) &= x_0. \end{aligned}$$

where the control function takes values in $L^2(J,\mathbb{U})$, a Banach space of admissible control functions and $\mathbb{U}$ is a Banach space and B is a bounded linear operator from $\mathbb{U}$ into $\mathbb{X}$

The paper is organized as follows. In Section 2, we present some basic definitions and some theorems as preliminaries. Section 3 is devoted to the sufficient conditions for existence of mild solution of the system (1.1) and in section 4, controllability of Hilfer differential inclusions (1.2) is proved. Finally in section 5, an example is given to illustrate the theory.

2 Preliminaries

Definition 1. *The Riemann-Liouville fractional integral of order $\alpha \in \mathbb{R}^+$ of a function $f:\mathbb{R}^+\to\mathbb{R}$ is defined by*

$$I^\alpha f(t)=\tfrac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)ds, \qquad t>0,$$

provided the right hand side is point wise defined on $\mathbb{R}^+$, where Γ is a gamma function.

Definition 2. *The Caputo fractional derivative of order α, where $\alpha\in(n-1,n)$ of a continuous function $f:\mathbb{R}^+\to E$ is defined by*

$$D^\alpha f(t)=\tfrac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}f^{(n)}(s)ds,\ t>0.$$

Definition 3. *The Riemann - Liouville fractional derivative of order α, where $\alpha\in(n-1,n)$, for a function $f:[0,\infty)\to\mathbb{R}$ is defined by*

$${}^LD^\alpha f(t)=\tfrac{1}{\Gamma(n-\alpha)}\tfrac{d^n}{dt^n}\int_0^t(t-s)^{n-\alpha-1}f(s)ds,\ t>0,$$

where the function f has absolutely continuous derivatives up to order $(n-1)$.

Definition 4. *[16] The left-sided Hilfer fractional derivative of order $0\le\alpha\le1$ and type $0<\beta<1$ of function $f(t)$ is defined as*

$$D_{a^+}^{\alpha,\beta}f(t)=I_{0^+}^{\alpha(1-\beta)}DI_{0^+}^{(1-\alpha)(1-\beta)}f(t),$$

where $D:=\frac{d}{dt}$.

Remark 1. *(i) For $\alpha=0$, $0<\beta<1$ and $a=0$, the Hilfer fractional derivative corresponds to the classical Riemann - Liouville fractional derivative: $D_{0^+}^{0,\beta}f(t)=DI_{0^+}^{1-\beta}f(t)={}^LD_{0+}^{\beta}f(t)$.*
(ii)For $\alpha=1$, $0<\beta<1$ and $a=0$, the Hilfer fractional derivative corresponds to the classical Caputo fractional derivative: $D_{0^+}^{0,\beta}f(t)=I_{0^+}^{1-\beta}Df(t)={}^cD_{0+}^{\beta}f(t)$.

Remark 2. *The Hilfer fractional derivative is considered as an interpolator between the Riemann - Liouville and Caputo derivative.*

Theorem 1. *[11]* ***(Glicksberg-Ky Fan fixed point theorem)*** *Let $\mathbb{X}$ be a Hausdorff locally convex linear topological space and let $\mathbb{Q} \subset \mathbb{X}$ be a nonempty compact convex subset. Suppose $G : \mathbb{Q} \to CC(\mathbb{Q})$ is a upper semi-continuous function. Then G has a fixed point; here $CC(\mathbb{Q})$ denoted the family of nonempty closed, convex subsets of $\mathbb{Q}$.*

Theorem 2. *[19] Let $\Omega \subseteq \mathbb{X}$, where $\mathbb{X}$ is a Banach space. Then Ω is relatively weakly compact iff Ω is relatively weakly sequentially compact.*

Corollary 1. *[19] Let $\Omega \subseteq \mathbb{X}$, where $\mathbb{X}$ is a Banach space. Then Ω is weakly compact iff Ω is weakly sequentially compact.*

Theorem 3. *[10]* ***(Krein-Smulian Theorem)*** *The convex hull of a weakly compact set in a Banach space $\mathbb{E}$ is weakly compact.*

Theorem 4. *[35]* ***(Pettis measurability Theorem)*** *Let $(\mathbb{E}, \Sigma)$ be a measure space, $\mathbb{X}$ be a separable Banach space. Then a function $\mathcal{F} : \mathbb{E} \to \mathbb{X}$ is measurable if and only if for every $g \in \mathbb{X}$, the function $go\mathcal{F} : \mathbb{E} \to \mathbb{R}$ is measurable with respect to Σ and the Borel σ-algebra in $\mathbb{R}$.*

3 Existence of mild solution

In this section, we study the existence of mild solutions for the Hilfer fractional inclusion (1.1) under the following assumptions:

$\mathbf{H_0}$ Q(t)is continuous in the uniform operator topology for $t > 0$, and $\{Q(t)\}_{t\geq 0}$ is uniformly bounded, i.e., there exists $M > 1$ such that $\sup_{t\in[0,\infty)} |Q(t)| < M$.

Assume that the multivalued nonlinearity $F : [0, b] \times \mathbb{X} \to \mathbb{X}$ has nonempty convex and weakly compact values and

$\mathbf{H_1}$ The multifunction $F(\cdot, x) : [0, b] \to \mathbb{X}$ has a measurable selection for every $x \in \mathbb{X}$, i.e. we can find a measurable function $f : [0, b] \to \mathbb{X}$ such that $f(t) \in F(t, x)$ for a.e. $t \in J$;

$\mathbf{H_2}$ The multimap $F(t, \cdot) : \mathbb{X} \to \mathbb{X}$ is weakly sequentially closed for a.e. $t \in [0, b]$, i.e. it has a weakly sequentially closed graph;

$\mathbf{H_3}$ There exists a function $\omega_n \in L^1([0, b]; \mathbb{R}^+)$ such that for each $c \in \mathbb{X}$, $\|c\| \leq n$,

$$\|F(t, c)\| = \sup\{\|y\| : y \in F(t, c)\} \leq \omega_n(t) \ for \ a.e. \ t \in [0, b].$$

Theorem 5. *[14] A function $x \in C(J, \mathbb{X})$ is said to be a mild solution of the inclusion system (1.1) if $x(0) = x_0$ and there exists $f(t) \in L^1(J, \mathbb{X})$ such that $f(t) \in F(t, x(t))$ and satisfies the integral equation*

$$x(t) = S_{\beta,\gamma}(t)x_0 + \int_0^t T_\gamma(t - s)f(s)ds, \tag{3.1}$$

where $\mathcal{P}_\gamma(t) = \int_0^\infty \gamma\theta M_\gamma(\theta)T_0(t^\gamma\theta)d\theta$, $S_{\beta,\gamma}(t) = I_{0^+}^{\beta(1-\gamma)}K_\gamma(t)$ *and* $K_\gamma(t) = t^{\gamma-1}\mathcal{P}_\gamma(t)$.

Theorem 6. *[14] Under assumption* $(\mathbf{H_0})$*, for any fixed* $t > 0$*,* $\{S_{\beta,\gamma}(t)\}_{t>0}$ *and* $\{T_\gamma(t)\}_{t>0}$ *are linear operators and for any* $x \in \mathbb{X}$

$$\|T_\gamma(t)x\| \leq \frac{Mt^{\gamma-1}}{\Gamma(\gamma)}\|x\|,$$

$$\|S_{\beta,\gamma}(t)x\| \leq \frac{Mt^{\nu-1}}{\Gamma(\nu)}\|x\|,$$

where

$$\nu = \beta + \gamma - \beta\gamma.$$

Given $q \in C([0,b];\mathbb{X})$, let us denote

$$\Psi_q = \{f \in L^1([0,b],\mathbb{X}) : f(t) \in F(t,x(t)) \ for\ a.e.\ t \in \ [0,b]\},$$

then Ψ_q is non-empty by [4].
Define the solution multi-operator $\Xi : C([0,b],\mathbb{X}) \multimap C([0,b],\mathbb{X})$ as

$$\Xi(q) = \begin{cases} x \in C([0,b],\mathbb{E}) & ; \\ x(t) = S_{\beta,\gamma}(t)x_0 + \int_0^t T_\gamma(t-s)f(s)ds, & f \in \Psi_q. \end{cases}$$

Fix $n \in \mathbb{N}$, consider Q_n the closed ball of radius n in $C([0,b];\mathbb{E})$ centered at the origin and denote by $\Xi_n := \Xi|_{Q_n} : Q_n \multimap C([0,b];\mathbb{E})$ the restriction of the multioperator Ξ on the set Q_n. We describe some properties of Ξ_n.

Proposition 1. *The multioperator* Ξ_n *has a weakly sequentially closed graph.*

Proof. Let $\{q_m\} \subseteq Q_n$ and $x_m \subseteq C([a,b];\mathbb{E})$ satisfying $x_m \subseteq \Xi(q_m)$ for all m and $q_m \rightharpoonup q$, $x_m \rightharpoonup x$ in $C([a,b];\mathbb{E})$; we will prove that $x \in \Xi_n(q)$.
Since $q_m \in Q_n$ for all m and $q_m(t) \rightharpoonup q(t)$ for every $t \in [0,b]$, it follows that $\|q(t)\| \leq \liminf_{\lambda\to\infty} \|q_m(t)\| \leq n$ for all t (see [5]). The fact that $x_m \in G_n(q_m)$ means that there exists a sequence $\{f_m\}$, such that for every $t \in [0,b]$,

$$x_m(t) = S_{\beta,\gamma}(t)x_0 + \int_0^t T_\gamma(t-s)f_m(s)ds.$$

We observe that, according to $(\mathbf{H_3})$, $\|f_m(t)\| \leq \eta_n(t)$ for a.a. t and every m, i.e. $\{f_m\}$ is bounded and uniformly integrable and $\{f_m(t)\}$ is bounded in $\mathbb{E}$ for a.a. $t \in [a,b]$. Hence, by the reflexivity of the space $\mathbb{E}$ and by the DunfordPettis Theorem (see [35]), we have the existence of a subsequence, denoted as the sequence, and a function g such that $f_m \rightharpoonup g$ in $L^1([0,b];\mathbb{E})$.

$$x_m(t) \rightharpoonup S_{\beta,\gamma}(t)x_0 + \int_0^t T_\gamma(t-s)g(s)ds, t \in [0,b].$$

To finish up, we have just to demonstrate that $g(t) \in F(t,x(t))$ for a.e. $t \in [0,b]$. The verification is very much alike to the second piece of [4], Proposition 4.2, along these lines, we leave off it. The proof is now completed. □

Proposition 2. *The multioperator Ξ_n is weakly compact.*

Proof. We first prove that $\Xi_n(Q_n)$ is weakly relatively sequentially compact. Let $\{q_m\} \subset Q_n$ and $\{x_m\} \subset C([a,b];\mathbb{E})$ satisfying $x_m \in \Xi_n(Q_m)$ for all m. By the definition of the multioperator Ξ_n, there exist a sequence $\{f_m\}$, $f_m \in S_{q_m}$, such that

$$x_m(t) = S_{\beta,\gamma}(t)x_0 + \int_0^t T_\gamma(t-s)f_m(s)ds, \forall t \in [0,b].$$

Further, reasoning as in Proposition 1, we have that there exists a subsequence, denoted as the sequence, and a function g such that $f_m \rightharpoonup g$ in $L^1([0,b];\mathbb{E})$. Therefore

$$x_m(t) \rightharpoonup l(t) = S_{\beta,\gamma}(t)x_0 + \int_0^t T_\gamma(t-s)f(s)ds, \forall t \in [0,b].$$

Furthermore, by the weak convergence of $\{f_m\}$, by (2.1), (3.1) and (3.3),we have

$$\begin{aligned}\|x_m(t)\| \leq & \|S_{\beta,\gamma}(t)x_0\| + \|\int_0^t T_\gamma(t-s)f(s)ds\| \\ \leq & \frac{Mt^{\nu-1}}{\Gamma(\nu)}\|x_0\| + \frac{M}{\Gamma(\gamma)}b^{\gamma-1}\|f\|_{L^1[0,b]}\end{aligned}$$

for all $m \in \mathbb{N}$ and for all $t \in [0,b]$. Reasoning again like in Proposition 1, it is then easy to prove that $x_m \rightharpoonup l$ in $C([0,b];\mathbb{E})$. Thus $\Xi_n(Q_n)$ is weakly relatively sequentially compact, hence weakly relatively compact by Theorem 2. □

Proposition 3. *The multioperator Ξ_n has convex and weakly compact values.*

Proof. Fix $p \in Q_n$, since F is convex valued, from the linearity of the integral and of the operators $S_{\beta,\gamma}(t)$ and $T_\gamma(t)$, it follows that the set $\Xi_n(p)$ is convex. The weak compactness of $\Xi_n(p)$ follows by Propositions 1 and 2. □

Theorem 7. *Assume that* ($\mathbf{H_0}$), ($\mathbf{H_1}$) *and* ($\mathbf{H_2}$) *holds and if there exists a sequence of functions* $\{w_n\} \subset L^1([0,b],\mathbb{R}^+)$ *such that*

$$\sup_{\|d\|\leq n} \|F(t,d)\| \leq \omega_n(t) \ for \ a.a. \ t \in [0,b], \ n \in \mathbb{N}$$

with

$$\lim_{n\to\infty} \frac{1}{n}\int_0^b \omega_n(s)ds = 0, \tag{3.2}$$

Then (1.1) has at least one mild solution.

Proof. We show that there exists $n \in \mathbb{N}$ such that the operator Ξ_n maps the ball Q_n into itself. Assume to the contrary, that there exist sequences $\{z_n\}, \{x_n\}$ such that $z_n \in Q_n$, $x_n \in \Xi_n(z_n)$ and $x_n \notin Q_n$, $\forall n \in \mathbb{N}$. Then there exists a sequence $\{f_n\} \subset L^1([0,a];X)$, $f_n(s) \in F(s, z_n(s))$, $\forall n \in \mathbb{N}$ and a.e. $s \in [0,a]$ such that

$$x_n(t) = S_{\beta,\gamma}(t)x_0 + \int_0^t T_\gamma(t-s)f(s)ds.$$

$$\|x_n\|_0 \le C_1 + C_2 \int_0^b \omega_n(s)ds$$

where

$$C_1 = \frac{Mb^{\nu-1}}{\Gamma(\nu)}\|x_0\|, \tag{3.3}$$

$$C_2 = \frac{Mb^{\gamma-1}}{\Gamma(\gamma)}. \tag{3.4}$$

But then

$$1 \le \frac{C_1}{n} + \frac{C_2}{n}\int_0^b \omega_n(s)ds = 0$$

as $n \to \infty$, which gives a contradiction to (4.3).
Now fix $n \in \mathbb{N}$ such that $\Xi_n(Q_n) \subseteq Q_n$. By Proposition 4.4 the set $V_n = \overline{\Xi_n(Q_n)}^w$ is weakly compact. Let $W_n = \overline{co}(V_n)$, where $\overline{co}$ denotes the closed convex hull of V_n. By Theorem 2.5, W_n is a weakly compact set. Moreover from the fact that $\Xi_n(Q_n) \subseteq Q_n$ and that Q_n is a convex closed set we have that $W_n \subseteq Q_n$ and hence

$$\Xi_n(W_n) = \Xi_n(\overline{co}(\Xi_n(Q_n))) \subseteq \Xi_n(Q_n) \subseteq \overline{\Xi_n(Q_n)}^w = V_n \subseteq W_n.$$

In view of Proposition 3.3, Ξ_n has a weakly sequentially closed graph. Thus from Theorem 1, the system (1.1) has a solution. The proof is now completed.

□

We are able to prove the existence result also under less restrictive growth assumptions, for instance sublinearity.

Remark 3. *Assume that (*$\mathbf{H_0}$*), (*$\mathbf{H_1}$*) and (*$\mathbf{H_2}$*) holds and suppose, there exist* $\zeta \in L^1([0,b];\mathbb{R}^+)$ *and a nondecreasing function* $\chi : [0,+\infty) \to [0,+\infty)$ *such that* $\|F(t,d)\| \le \zeta(t)\chi(\|d\|)$ *for a.e.* $t \in [0,b]$ *and every* $d \in \mathbb{E}$*. Then (1.1) has a solution provided that*

$$\lim_{n\to\infty} \frac{\chi(n)}{n} = 0.$$

Theorem 8. *Assume that (*$\mathbf{H_0}$*), (*$\mathbf{H_1}$*) and (*$\mathbf{H_2}$*) holds and if there exist* $\zeta \in L^1([0,b];\mathbb{R}^+)$ *such that*

$$\|F(t,d)\| \le \zeta(t)(1+\|d\|) \ for\ a.a.\ t \in [0,b],\ \forall\ c \in \mathbb{E}$$

and

$$\frac{Mb^{\gamma-1}}{\Gamma(\gamma)}\|\zeta\|_1 < 1, \tag{3.5}$$

then the existence problem (1.1) has a solution.

Proof. Reasoning as in Theorem 4.1 and assuming that there exist $\{z_n\}, \{y_n\}$ such that $z_n \in Q_n, y_n \in \Xi(z_n)$ and $y_n \notin Q_n$, $\forall n \in \mathbb{N}$, we would get

$$\begin{aligned} n < \|x_n\|_0 \leq & C_1 + C_2 \int_0^b \zeta(s)(1 + \|d\|)ds \\ \leq & C_1 + C_2(1+n)\|\zeta\|_1, n \in \mathbb{N}. \end{aligned} \tag{3.6}$$

As $n \to \infty$, (3.6) gives

$$C_2\|\zeta\|_1 \geq 1$$

i.e.

$$\frac{Mb^{\gamma-1}}{\Gamma(\gamma)}\|\zeta\|_1 \geq 1,$$

giving the contradiction with (3.5). □

Theorem 9. ***(Superlinear growth condition)*** *Suppose, there exist $\zeta \in L^1([0,b];\mathbb{R}^+)$ and a nondecreasing function $\chi : [0,+\infty) \to [0,+\infty)$ such that*

$$\|F(t,d)\| \leq \zeta(t)\chi(\|d\|) \ for \ a.a. \ t \in [0,b], \ \forall \ c \in \mathbb{E}$$

and $n_0 > 0$ such that

$$\frac{n_0}{C_1 + C_2\|\zeta\|_1\chi(n_0)} > 1, \tag{3.7}$$

where C_1 and C_2 are the positive constants defined in (3.3) and (3.4), and additionally, if ($\mathbf{H_0}$), ($\mathbf{H_1}$) and ($\mathbf{H_2}$) holds,then the existence problem (1.1) has a solution.

Proof. It is sufficient to prove that the operator Ξ maps the ball Q_{n_0} into itself. In fact, given any $z \in Q_{n_0}$ and $y \in \Xi(z)$, it holds

$$\begin{aligned} \|x\|_0 \leq & C_1 + C_2 \int_0^b \|\zeta(s)\|\chi(n_0)ds \\ \leq & C_1 + C_2\chi(n_0)\|\zeta\|_1 < n_0. \end{aligned}$$

The conclusion then follows by Theorem 1. □

4 Controllability Results

In this section, we discuss the controllability for Hilfer differential inclusions (1.2) in a reflexive Banach space and assume that

- $\mathbf{H_4}$ The control function $u(\cdot)$takes its value in $L^1([0,b];\mathbb{U})$, a Banach space of admissible control functions and $\mathbb{U}$ is a Banach space. $B : \mathbb{U} \to \mathbb{X}$ is a bounded linear operator with

$$\|B\| = M_1. \tag{4.1}$$

Theorem 10. *[14] A function $x \in C(J, \mathbb{X})$ is said to be a mild solution of the inclusion system (1.2) if $x(0) = x_0$ and there exists $f(t) \in L^1(J, \mathbb{X})$ such that $f(t) \in F(t, x(t))$ and satisfies the integral equation*

$$x(t) = S_{\beta,\gamma}(t)x_0 + \int_0^t T_\gamma(t-s)[Bu(s) + f(s)]ds \tag{4.2}$$

Assume that

H_5 The mapping $W : L^1([0,b];\mathbb{U}) \to \mathbb{X}$ given by

$$Wu = \int_0^b T_\gamma(t-s)Bu(s)ds$$

has a bounded inverse $W^{-1} : \mathbb{X} \to L^1([0,b];\mathbb{U})/ker(W)$ and there exists a real number M_2 such that $M_2 > 0$ and

$$\|W^{-1}\| \le M_2.$$

Consider the following integral operators

$$S_1 f(t) = \int_0^t T_\gamma(t-s)f(s)ds$$

$$S_2 f(t) = \int_0^t T_\gamma(t-s)BW^{-1}\Big(-\int_0^b T_\gamma(b-\eta)f(\eta)d\eta\Big)(s)ds, t \in [0,b]$$

where $S_1, S_2 : L^1([0,b];\mathbb{X}) \to C([0,b];\mathbb{X})$.

Define the solution multi-operator $\Xi : C([0,b],\mathbb{X}) \multimap C([0,b],\mathbb{X})$ as

$$\Xi(q) = \begin{cases} x \in C([0,b],\mathbb{E}) & ; \\ x(t) = S_{\beta,\gamma}(t)x_0 + S_1 f(t) + \int_0^t T_\gamma(t-s)BW^{-1}\Big(x_1 - S_{\beta,\gamma}(b)x_0\Big)(s)ds + S_2 f(t), & f \in \Psi_q. \end{cases}$$

Lemma 1. *The operators S_1 and S_2 are linear and continuous.*

Proof. The linearity of both the operators S_1 and S_2 follows from the linearity of the integral operator and of the operators B,W^{-1} and $T_\gamma(t)$ for every $t \in [0,b]$.

$$\begin{aligned} \|S_1 f(t)\| &\le \frac{M}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}|f(s)|ds \\ &\le \frac{M}{\Gamma(\gamma)} b^{\gamma-1}\|f\|_{L^1[0,b]}. \end{aligned}$$

$$\begin{aligned}
\|S_2 f(t)\| &\leq \frac{MM_1}{\Gamma(\gamma)} b^{\gamma-1} \int_0^t \Big\| W^{-1}\Big(-\int_0^b T_\gamma(b-\eta) f(\eta) d\eta \Big)(s) \Big\| ds \\
&\leq \frac{MM_1}{\Gamma(\gamma)} b^{\gamma-1} \Big\| W^{-1}\Big(-\int_0^b T_\gamma(b-\eta) f(\eta) d\eta \Big)(s) \Big\|_{L^1([0,b],U)} \\
&\leq \frac{MM_1}{\Gamma(\gamma)} b^{\gamma-1}\sqrt{b} \Big\| W^{-1}\Big(-\int_0^b T_\gamma(b-\eta) f(\eta) d\eta \Big)(s) \Big\|_{L^2([0,b],U)} \\
&\leq \frac{MM_1M_2}{\Gamma(\gamma)} b^{\gamma-1}\sqrt{b} \Big[\frac{M}{\Gamma(\gamma)} b^{\gamma-1} \int_0^b \|f(\eta)\| d\eta \Big] \\
&= \frac{M^2 M_1 M_2}{(\Gamma(\gamma))^2} b^{2\gamma-\frac{3}{2}} \|f\|_1.
\end{aligned}$$

□

Fix $n \in \mathbb{N}$, consider the closed ball $Q_n \subseteq C([a,b];\mathbb{E})$ of radius n centered at the origin and use a notation $\Xi_n := \Xi|_{Q_n} : Q_n \multimap C([0,b];\mathbb{E})$ i.e. Ξ_n is a restriction of the multioperator Ξ on the set Q_n. Here are some properties of the operator Ξ_n.

Lemma 2. *The multioperator Ξ_n has a weakly sequentially closed graph.*

Proof. Let $\{q_m\} \subseteq Q_n$ and $x_m \subseteq C([a,b];\mathbb{E})$ satisfying $x_m \subseteq \Xi(q_m)$ for all m and $q_m \rightharpoonup q$, $x_m \rightharpoonup x$ in $C([a,b];\mathbb{E})$; we will prove that $x \in \Xi_n(q)$.
Since $q_m \in Q_n$ for all m and $q_m(t) \rightharpoonup q(t)$ for every $t \in [0,b]$, it follows that $\|q(t)\| \leq \liminf_{\lambda\to\infty} \|q_m(t)\| \leq n$ for all t (see [5] Proposition III.5) i.e.there exists a sequence $\{f_m\}, f_m \in S_{q_m}$, such that for every $t \in [0,b]$,

$$x_m(t) = S_{\beta,\gamma}(t)x_0 + S_1 f_m(t) + \int_0^t T_\gamma(t-s) B W^{-1}\Big(x_1 - S_{\beta,\gamma}(b)x_0\Big)(s) ds + S_2 f_m(t).$$

By assumption (H_3), $\|f_m(t)\| \leq \eta_n(t)$ for a.a. t and for every m, i.e. $\{f_m\}$ is bounded and uniformly integrable and $\{f_m(t)\}$ is bounded in E for a.a. $t \in [0,b]$. By using the reflexivity of $\mathbb{E}$ and by the Dunford-Pettis Theorem (see [10], p. 294), there exist a subsequence, say f_m, and a function g such that $f_m \rightharpoonup g$ in $L^1([0,b];\mathbb{E})$. Therefore, $S_i f_m \rightharpoonup S_i g$ for $i = 1,2$. Thus

$$x_m(t) \rightharpoonup S_{\beta,\gamma}(t)x_0 + S_1 g(t) + \int_0^t T_\gamma(t-s) B W^{-1}\Big(x_1 - S_{\beta,\gamma}(b)x_0\Big)(s) ds + S_2 g(t) = x_0(t), t \in [0,b]$$

implying, for the uniqueness of the weak limit in $\mathbb{E}$, that $x_0(t) = x(t)$ for all $t \in [0,b]$. To conclude, we have only to prove that $g(t) \in F(t, q(t))$ for a.a. $t \in [0,b]$. By Mazurỹ convexity Theorem (see e.g. [16]) we have a sequence

$$\tilde{f}_m = \sum_{i=0}^{k_m} \lambda_{mi} f_{m+i}, \lambda_{mi} \geq 0, \sum_{i=0}^{k_m} \lambda_{mi} = 1$$

satisfying $\tilde{f}_m, g$ in $L^1([0,b];\mathbb{E})$ and, up to subsequence, there is $N_0 \subseteq [0,b]$ having Lebesgue measure zero and $\tilde{f}_m(t) \rightharpoonup g(t)$ for all $t \in [a,b]\ N_0$ (see [11], Chapter IV, Theorem 38). Now, the verification is very much alike to the second piece of [4], Proposition 4.2, along these lines, we leave off it. The proof is now completed. □

Lemma 3. *The multioperator Ξ_n is weakly compact.*

Proof. We first prove that $\Xi_n(Q_n)$ is weakly relatively sequentially compact. Let $\{q_m\} \subset Q_n$ and $\{x_m\} \subset C([a,b];\mathbb{E})$ satisfying $x_m \in \Xi_n(Q_m)$ for all m. By the definition of the multioperator Ξ_n, there exist a sequence $\{f_m\}$, $f_m \in S_{q_m}$, such that

$$x_m(t) = S_{\beta,\gamma}(t)x_0 + S_1 f_m(t) + \int_0^t T_\gamma(t-s)BW^{-1}\Big(x_1 - S_{\beta,\gamma}(b)x_0\Big)(s)ds + S_2 f_m(t), \forall t \in [0,b].$$

Further, reasoning as in Proposition (2), we have that there exists a subsequence, denoted as the sequence, and a function g such that $f_m \rightharpoonup g$ in $L^1([0,b];\mathbb{E})$. Therefore

$$x_m(t) \rightharpoonup l(t) = S_{\beta,\gamma}(t)x_0 + S_1 g(t) + \int_0^t T_\gamma(t-s)BW^{-1}\Big(x_1 - S_{\beta,\gamma}(b)x_0\Big)(s)ds + S_2 g(t), \forall t \in [a,b].$$

Furthermore, by the weak convergence of $\{f_m\}$, by (4.1), (2), and the continuity of the operators S_1 and S_2 we have

$$\begin{aligned}
\|x_m(t)\| \leq & \|S_{\beta,\gamma}(t)x_0\| + \|S_1 g(t)\| + \|\int_0^t T_\gamma(t-s)BW^{-1}\Big(x_1 - S_{\beta,\gamma}(b)x_0\Big)(s)ds\| + \|S_2 g(t)\| \\
\leq & \frac{Mt^{\nu-1}}{\Gamma(\nu)}\|x_0\| + \frac{M}{\Gamma(\gamma)}b^{\gamma-1}\|g\|_{L^1[0,b]} \\
& + \frac{MM_1M_2b^\gamma}{\Gamma(\gamma+1)}(\|x_1\| + \frac{Mb^{\nu-1}}{\Gamma(\nu)}\|x_0\|) + \frac{M^2M_1M_2}{(\Gamma(\gamma))^2}b^{2\gamma-\frac{3}{2}}\|g\|_1
\end{aligned}$$

for all $m \in \mathbb{N}$ and for all $t \in [0,b]$. Reasoning again like in Proposition (2), it is then easy to prove that $x_m \rightharpoonup l$ in $C([0,b];\mathbb{E})$. Thus $\Xi_n(Q_n)$ is weakly relatiively sequentially compact, hence weakly relatively compact by Theorem 2. □

Lemma 4. *The multioperator Ξ_n has convex and weakly compact values.*

Proof. Fix $p \in Q_n$, since F is convex valued, from the linearity of the integral and of the operators $S_{\beta,\gamma}(t)$ and $T_\gamma(t)$, it follows that the set $\Xi_n(p)$ is convex. The weak compactness of $\Xi_n(p)$ follows by Propositions (2) and (3). □

Theorem 11. *Suppose ($\mathbf{H_0}$), ($\mathbf{H_1}$), ($\mathbf{H_2}$), ($\mathbf{H_4}$) and ($\mathbf{H_5}$) holds and suppose there exists a sequence of functions $\{w_n\} \subset L^1([0,b],\mathbb{R}^+)$ such that*

$$\sup_{\|d\| \leq n} \|F(t,d)\| \leq \omega_n(t) for a.a. t \in [0,b], n \in \mathbb{N}$$

with

$$\lim_{n\to\infty}\frac{1}{n}\int_0^b \omega_n(s)ds = 0, \tag{4.3}$$

Then the controllability problem (1.2) has at least one mild solution.

Proof. We show that there exists $n \in \mathbb{N}$ such that the operator $\Xi_n(Q_n) \subseteq Q_n$. Let, if possible,there exist sequences $\{z_n\}, \{x_n\}$ such that $z_n \in Q_n$, $x_n \in \Xi_n(z_n)$ and $x_n \notin Q_n$, $\forall n \in \mathbb{N}$. Then $\exists$ a sequence $\{f_n\} \subset L^1([0,b];\mathbb{X})$, $f_n(s) \in F(s, z_n(s))$, $\forall n \in \mathbb{N}$ and a.e. $s \in [0,b]$ such that

$$x_n(t) = S_{\beta,\gamma}(t)x_0 + S_1 f_n(t) + \int_0^t T_\gamma(t-s)BW^{-1}\Big(x_1 - S_{\beta,\gamma}(b)x_0\Big)(s)ds + S_2 f_n(t).$$

$$\|x_n\|_0 \leq \tilde{C}_1 + \tilde{C}_2 + \tilde{C}_3 \int_0^b \omega_n(s)ds$$

where

$$\tilde{C}_1 = \left(1 + \frac{MM_1M_2b^\gamma}{\Gamma(\gamma+1)}\right)\frac{Mb^{\nu-1}}{\Gamma(\nu)}\|x_0\|, \tag{4.4}$$

$$\tilde{C}_2 = \frac{MM_1M_2b^\gamma}{\Gamma(\gamma+1)}\|x_1\|, \tag{4.5}$$

$$\tilde{C}_3 = \left(\frac{Mb^{\gamma-1}}{\Gamma(\gamma)} + \frac{M^2M_1M_2}{(\Gamma(\gamma))^2}b^{2\gamma-\frac{3}{2}}\right) \tag{4.6}$$

But then

$$1 \leq \frac{\tilde{C}_1}{n} + \frac{\tilde{C}_2}{n} + \frac{\tilde{C}_3}{n}\int_0^b \omega_n(s)ds = 0$$

as $n \to \infty$, which gives a contradiction to (4.3).
Fix $n \in \mathbb{N}$ such that $\Xi_n(Q_n) \subseteq Q_n$. Then the set $V_n = \overline{\Xi_n(Q_n)}^w$ is weakly compact by Proposition (3). Let $W_n = \overline{co}(V_n)$, where $\overline{co}$ denotes the closed convex hull of V_n. Then W_n is a weakly compact set by Theorem (3).By using the fact that $\Xi_n(Q_n) \subseteq Q_n$ and that Q_n is a closed convex set, we have $W_n \subseteq Q_n$ and hence

$$\Xi_n(W_n) = \Xi_n(\overline{co}(\Xi_n(Q_n))) \subseteq \Xi_n(Q_n) \subseteq \overline{\Xi_n(Q_n)}^w = V_n \subseteq W_n.$$

Hence Ξ_n has a weakly sequentially closed graph by Proposition (1). Hence the system (1.2) has a solution by Theorem (2). The proof is now completed.

□

Now, We will prove the controllability result under some less restrictive growth assumptions.

Remark 4. *Suppose, there exist $\zeta \in L^1([0,b];\mathbb{R}^+)$ and a monotonic increasing function $\chi : [0,+\infty) \to [0,+\infty)$ such that $\|F(t,d)\| \leq \zeta(t)\chi(\|d\|)$ for a.e. $t \in [0,b]$ and every $d \in \mathbb{E}$. Then (2) is equivalent to*

$$\lim_{n\to\infty} \frac{\chi(n)}{n} = 0.$$

Theorem 12. *Suppose, there exist $\zeta \in L^1([0,b];\mathbb{R}^+)$ such that*

$$\|F(t,d)\| \leq \zeta(t)(1+\|d\|) \ for \ a.a. \ t \in [0,b], \ \forall \ c \in \mathbb{E}$$

and

$$\left(\frac{Mb^{\gamma-1}}{\Gamma(\gamma)} + \frac{M^2M_1M_2}{(\Gamma(\gamma))^2}b^{2\gamma-\frac{3}{2}}\right)\|\zeta\|_1 < 1, \tag{4.7}$$

and if in addition, suppose ($\mathbf{H_0}$), ($\mathbf{H_1}$), ($\mathbf{H_2}$), ($\mathbf{H_4}$) and ($\mathbf{H_5}$) holds, then controllability problem (1.2) has a solution.

Proof. By using the same fact as that in Theorem (4.1) and by assuming that there exist $\{z_n\}, \{y_n\}$ such that $z_n \in Q_n, y_n \in \Xi(z_n)$ and $y_n \notin Q_n$, $\forall n \in \mathbb{N}$, we get

$$\begin{aligned} n < \|x_n\|_0 \leq & \tilde{C}_1 + \tilde{C}_2 + \tilde{C}_3 \int_0^b \zeta(s)(1+\|d\|)ds \\ \leq & \tilde{C}_1 + \tilde{C}_2 + \tilde{C}_3(1+n)\|\zeta\|_1, n \in \mathbb{N}. \end{aligned} \tag{4.8}$$

As $n \to \infty$, (4.8) gives

$$\tilde{C}_3\|\zeta\|_1 \geq 1$$

i.e.

$$\left(\frac{Mb^{\gamma-1}}{\Gamma(\gamma)} + \frac{M^2M_1M_2}{(\Gamma(\gamma))^2}b^{2\gamma-\frac{3}{2}}\right)\|\zeta\|_1 \geq 1, \tag{4.9}$$

giving the contradiction with (4.9). □

Theorem 13. ***(Superlinear growth condition)*** *Suppose ($\mathbf{H_0}$), ($\mathbf{H_1}$), ($\mathbf{H_2}$), ($\mathbf{H_4}$) and ($\mathbf{H_5}$) holds and suppose, there exist $\zeta \in L^1([0,b];\mathbb{R}^+)$ and a nondecreasing function $\chi : [0,+\infty) \to [0,+\infty)$ such that*

$$\|F(t,d)\| \leq \zeta(t)\chi(\|d\|) \ for \ a.a. \ t \in [0,b], \ \forall \ c \in \mathbb{E}$$

and $n_0 > 0$ such that

$$\frac{n_0}{\tilde{C}_1 + \tilde{C}_2 + \tilde{C}_3\|\zeta\|_1\chi(n_0)} > 1, \tag{4.10}$$

where $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3$ are the positive constants defined in (4.4), (4.5) and (4.6), then controllability problem (1.2) has a solution.

Proof. Sufficient to prove that the operator $\Xi(Q_{n_0}) \subseteq Q_{n_0}$.
For any $z \in Q_{n_0}$ and $y \in \Xi(z)$, we have

$$\begin{aligned}\|x\|_0 \leq & \tilde{C}_1 + \tilde{C}_2 + \tilde{C}_3 \int_0^b \|\zeta(s)\| \chi(n_0) ds \\ \leq & \tilde{C}_1 + \tilde{C}_2 + \tilde{C}_3 \chi(n_0) \|\zeta\|_1 < n_0.\end{aligned}$$

Hence the conclusion follows by Theorem 2. □

Consider the following differential inclusion

$$\begin{aligned} D^{\beta,\frac{1}{3}} w(t,x) &= w_{xx}(t,x) + u(t,x) + f\Big(t, \int_\Omega k(x,\xi) w(t,\xi) d\xi\Big), 0 \leq x \leq 1, 0 \leq t \leq 1 \\ w(t,0) &= w(t,1) = 0, t \in J \\ I^{\frac{2}{3}(1-\beta)} w(0,x) &= w_0(x), x \in [0,1]. \end{aligned} \tag{4.11}$$

where $D^{\beta,\frac{1}{3}}$ is the Hilfer fractional derivative of order $\frac{1}{3}$ and type β, $I^{\frac{2}{3}(1-\beta)}$ is the Riemann Liouville integral of order $\frac{2}{3}(1-\beta)$, $J = [0,1]$, $\mathbb{E} = \mathbb{X} = L^2[0,1]$. Define $w(t)(x) = w(t,x)$, $\mu(t) = u(t,\cdot)$ and let $F : J \times L^2[0,1] \to 2^{L^2[0,1]} - \{\phi\}$ is given by $F(t,w)(x) = f\Big(t, \int_\Omega k(x,\xi) w(\xi) d\xi\Big)$ and $\mathcal{A}v = v^{''}$ with the domain $D(\mathcal{A}) = \{\, v \in X : v, v^{'}$ are absolutely continuous, $v^{''} \in X$, $v(0) = v(1) = 0\}$
Assume the following hypotheses:

(i) f satisfies Carathedory property i.e. $f(\cdot, c) : [0,1] \times [0,1] \to \mathbb{R}$ is measurable, for all $c \in \mathbb{R}$; and $f(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is continuous, for a.a. $t \in [0,1]$;

(ii) There exist $\zeta : L^1([0,b];\mathbb{R})$ and an increasing function $\chi : [0,\infty) \to [0,\infty)$ such that, for a.a. $t \in [0,b]$ and every $x \in [0,1]$, $c \in \mathbb{R}$, $|f(t,c)| \leq \zeta(t)\chi(|c|)$;

(iii) $k : [0,1] \times [0,1] \to \mathbb{R}$ is measurable with $k(x,\cdot) \in L^2([0,1];\mathbb{R})$ and $\|k(x,\cdot)\| \leq 1$ for all $x \in [0,1]$;

(iv) The control $\mu(t) = u(t,\cdot)$ belongs to $L^2([0,1]; L^2([0,1];\mathbb{R}))$;

Then (4.11) can be reduced to

$$\begin{aligned} D^{\beta,\frac{1}{3}} w(t) &\in \mathcal{A}w(t) + B\mu(t) + F\Big(t, w(t)\Big), 0 \leq y \leq 1, 0 \leq t \leq 1 \\ w(0) &= w(1) = 0, \\ I^{\frac{2}{3}(1-\beta)} w(0) &= w_0. \end{aligned} \tag{4.12}$$

where B is identity operator. We now prove that Theorem 4.1 can be applied to controllability problem for (4.12), obtaining as the result a solution of the corresponding problem for (4.11). Notice first of all that Pettis measurability theorem (see [35]), the separability of $L^2([0,1];\mathbb{R})$ and conditions (i) and (ii) imply that F is measurable (see [18]) and, being single-valued, it satisfies $(\mathbf{H_1})$.

Fix now $t \in [0,1]$ satisfying (ii) and take $z_n \to z$ in $L^2([0,1];\mathbb{R})$. Then, for every $x \in [0,1]$, it holds

$$\int_0^t k(x,\xi)z_n(\xi)d\xi \to \int_0^t k(x,\xi)z(\xi)d\xi.$$

By using condition (ii), we have

$$f(t, \int_0^t k(x,\xi)z_n(\xi)d\xi) \to f(t, \int_0^t k(x,\xi)z(\xi)d\xi)$$

for a.a. $z \in L^2[0,1]$. Also Hölder's inequality and (iii) implies that

$$\| \int_0^1 k(\cdot,\xi)z(\xi)d\xi\| \leq \|k(\cdot,\xi)\|_2\|z\|_2 \leq \|z\|_2. \tag{4.13}$$

Therefore, since the weak convergence of $\{z_n\}$ yields its boundedness, according to (iii), there exists a constant $M > 0$ such that, for every $n \in \mathbb{N}$,

$$\|f(t, \int_0^t k(x,\xi)z_n(\xi)d\xi)\| \leq \zeta(t)\chi(\|z_n\|_2) \leq \zeta(t)\chi(M)$$

and hence F satisfies $(\mathbf{H_2})$ by Lebesque dominated convergence theorem.
By using (iii) and (4.13), we have, for a.a. $t \in [0,1]$ and every $z \in L^2([0,1];\mathbb{R})$,

$$\|F(t,z)\|_2^2 = \int_0^1 [f(t, \int_0^t k(x,\xi)z(\xi)d\xi)]^2 dx \leq \zeta(t)^2\chi(\|z_n\|_2)^2.$$

Also

$$\lim_{\lambda\to\infty} \frac{1}{n}\int_0^t w_n(t)dt = \lim_{\lambda\to\infty} \frac{\chi(n)}{n}\|\zeta\|_1 = 0$$

by using theorem (4.9) and (iii). $(\mathbf{H_4})$ holds trivially as $B = I$ and condition (iv) yields $(\mathbf{H_5})$ and hence we obtain a solution $z \in C([0,1]; L^2([0,1];\mathbb{R}))$ of the controllability problem for (4.11).

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