On Skew Strong McCoy Rings

Pratibha, Amit B.Singh and Deepa Arora

Department of Mathematics,
Manav Rachna University, Faridabad-121004 (INDIA)
gptbhu@gmail.com

Abstract

In this paper we have introduced the notion of α- Skew Strong McCoy Rings. Hong et al.[4] extended the McCoys theorem to non - commutative rings through the concept of strongly right McCoyness. The paper discusses and extends the scope of these results to α- skew strong McCoy Rings.

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1 Introduction

Throughout this paper, all rings are associative with identity. For an endomorphism α of a ring R, $R[x;\alpha]$ denotes a skew polynomial ring with an indeterminate x over R. In [18]Nielsen, defined a ring R to be a right McCoy if whenever $f(x), g(x) \in R[x]\backslash\{0\}$ satisfy $f(x)g(x) = 0$, then there exists a non zero element $r \in R$ with $rg(x) = 0$. Left McCoy rings are defined similarly. If a ring is both left and right McCoy, then the ring is called a McCoy ring. Some properties of McCoy rings have been studied in Camillo and Nielsen[2,7,8,9,10,11], Yang et al.[20,21]. According to Krempa[10], an endomorphism α of a ring R is called rigid if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$, and a ring is called $\alpha - rigid$ if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is a monomorphism, and $\alpha - rigid$ rings are reduced rings by Hong et al. in [8,Proposition 5]. Moreover, R is a $\alpha - rigid$ ring if and only if $R[x;\alpha]$ is reduced [8,Proposition 3]. A ring R is reduced if it has no nilpotent elements, R is reversible if $ab = 0$ implies $ba = 0$ for $a, b \in R$, and R is right(left) duo, if every right(left) ideal is two sided. A ring is called right ore if given $a, b \in R$ with b regular, there exists $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$.

Several authors [2-16] have studied the McCoy rings and their properties. McCoy [17] proved in 1942 that if two polynomials annihilates each other over a commutative ring, then each polynomial has a non-zero annihilator in the base ring. In 1957, he further proved that if a polynomial annihilates an ideal of polynomials over any ring, then the ideal has a non-zero annihilator in the base ring[16]. Hong et al. in [4] elaborated the McCoy's famous theorem further to non-commutative rings via conceptualization of strongly McCoy rings. A ring R will be called strongly right McCoy provided that $f(x)g(x) = 0$ implies $f(x)r = 0$ for some non zero r in the right ideal of R generated by the coefficient of $g(x)$, where $f(x)$ and $g(x)$ are non zero polynomials in $R[x]$. Strongly left McCoy rings are defined similarly.

If a ring is both strongly left and strongly right McCoy then the ring is called a strongly McCoy ring.

Motivated by this, we introduce the concept of $\alpha - skew$ strong McCoy ring $R[x;\alpha]$ w.r.t endomorphism α of R. This concept extends both, McCoy rings and strongly right McCoy rings. We have evaluated the McCoy condition on polynomials in $R[x;\alpha]$ instead of $R[x]$ and R. Thus, scope to study strongly McCoy rings in a general settings are established.

2 α-Skew Strong McCoy.

Definition 1. *If $f(x), g(x) \in R[x;\alpha]$, where α be an endomorphism of R such that $f(x)g(x) = 0$ then for some $r \in I$, $f(x)r = 0$ where I be an ideal generated by coefficient of $g(x)$.*
So, clearly a ring R is strongly McCoy if R is id_R-skew strong McCoy, where id_R is the identity endomorphism of R.

Theorem 2.1. *Let α be an endomorphism of a ring R and R be $\alpha - rigid$. If $R[x;\alpha]$ be a right duo ring and $p(x) = \sum_{i=0}^{m} a_i x^j$, $q(x) = \sum_{j=0}^{n} b_j x^j$ be nonzero polynomials over $R[x;\alpha]$ with $p(x)q(x) = 0$. Thus, there exists $r \in R[x;\alpha]$ with $q(x)r = 0$ and $a_i b_j r = 0$ for all i and j.*

Proof. Let $R[x;\alpha]$ be a right duo ring, and $p(x), q(x) \in R[x;\alpha]$ are non-zero polynomials with $p(x)q(x) = 0$

$$p(x) = \sum_{i=0}^{m} a_i x^j \qquad and \qquad q(x) = \sum_{j=0}^{n} b_j x^j$$

where we assume $a_0, \ldots, a_m, b_0, \ldots, b_n \in R$.
We will show, there exists $r \in R[x;\alpha]$ with $q(x)r \neq 0$ and $p(x)\alpha^j(b_j)r = 0$ implies $p(x)b_j r = 0 \quad \forall\, j \in \{0 \ldots n\}$.
If deg $p(x) = 0$ then $p(x)\alpha^j(b_j) = 0 \quad \forall \quad j \in \{0 \ldots n\}$.

Suppose deg $p(x) \geq 1$. Let $a_0 g(x) \neq 0$. Assume j be minimal in such a way that $a_0 b_j \neq 0$.
By[2, Lemma 5.4], there exists an integer $t > 0$ satisfying ${a_0}^t b_j \neq 0$ and ${a_0}^{t+1} b_j = 0$. Since $R[x;\alpha]$ is right duo, there exists $r_1 \in R$ with ${a_0}^t b_j = b_j r_1$.
Suppose $q_1(x) = q(x)r_1$, we get $p(x)q_1(x) = 0$ implies $p(x)q(x)r_1 = 0$ with $q_1(x) \neq 0$.
Note,

$$a_0 q_1(x) = a_0 q(x) r_1$$

$$= a_0 (\sum_{j=0}^{n} b_j \alpha^j(r_1) x^j)$$

$$= \sum_{j=o}^{n} a_0 b_j \alpha^j(r_1) x^j$$

with $a_0 b_j \neq 0$ and $j = 0, 1 \ldots h-1$.
Hence, $a_0 b_j \alpha^j(r_1) = 0$ for $j = 0, 1 \ldots h-1$.

Next we assume either $a_0q_1(x) = 0$ or not. If not, then we apply the same procedure on $q_1(x)$, getting $r_2 \in R[x;\alpha]$ such that $q_1(x)r_2 \neq 0$ with $j = 0, 1 \ldots, l-1$.
and hence,

$$a_0 b_j \alpha^j (r_1 r_2) = 0 \quad for \quad j = 0, 1 \ldots, l-1$$

Now, after continual the same process upto a finite number of times, we ultimately get $r_1, \ldots, r_s \in R[x;\alpha]$ such that

$$q(x)(r_1 \ldots r_s) \neq 0, \quad a_0 q(x)(r_1 \ldots r_s) = 0$$

and

$$a_0 b_j \alpha^j (r_1, r_2, \ldots, r_s) = 0 \quad for \quad j = 0, \ldots, n.$$

In Consequence, for any type of $a_0q(x) = 0$ and $a_0q(x) \neq 0$, there exists $t \in R$ with $a_0 b_j \alpha^j(t) = 0$ for all j=0,..., n and thus, we obtain,

$$0 = p(x)q(x)t = \sum_{y=0}^{n} a_0 b^j \alpha^j(t) x^j + a_1 x (\sum_{i=0}^{n} b_j x^j) t$$

$$0 \quad = (\sum_{j=0}^{n} a_1 \alpha(b_j) x^{j+1}) t = 0$$

$$\Longrightarrow \sum_{j=o}^{n} a_1 \alpha(b_j) \alpha^{j+1}(t) x^{j+1} = 0$$

Next involving this operation on $a_1q(x)t$, we obtain $v \in R[x;\alpha]$ based on a_1 such that

$$a_1 \alpha(b_j) \alpha^{j+1}(t) \alpha^{j+1}(v) = 0$$

$$\Longrightarrow a_1 \alpha(b_j) \alpha^{j+1}(tv) \quad = 0$$

for all $j = 0, \ldots, n$ and $q(x)tv \neq 0$, since $a_0 b_j \alpha^j(t) = 0 \ \forall \ j = 0, \ldots, n$, it follows that

$$a_i \alpha^j(b_j) \alpha^{j+i}(tv) = 0$$

$\forall \ i, j$ with $i = 0, 1$ and $j = 0, \ldots, n$.

Continuing the same calculation. Subsequently, we get $w \in R$ such that $g(x)(tv \ldots w) \neq 0$ and $a_1 \alpha^j(b_j) \alpha^{j+1}(tv \ldots w) = 0$ for all i, j with $i = 0, \ldots, m$ and $j = 0, \ldots, n$.
Since R is $\alpha - rigid$, implies

$$a_1 b^j (tv \ldots w) = 0 \qquad \forall \ i = 0, \ldots, m \quad and \quad j = o, \ldots, n$$

. Let $r = tv \ldots w$, this finalized the proof. □

Theorem 2.2. *Let $R[x;\alpha]$ be a right Ore ring with the classical right quotient ring $Q_r(R)$.*

(i) $R[x;\alpha]$ is strongly right McCoy iff so is $Q_r(R)$.

(ii) $R[x;\alpha]$ *is right McCoy iff so is* $Q_r(R)$.

Proof. Let $Q = Q_r(R)$ .Suppose $P(x)Q(x) = 0$ for $P(x), Q(x) \in Q[x]$. Then, $P(x) = a_0u^{-1} + a_1u^{-1}x + \ldots + a_mu^{-1}x^m$ and $Q(x) = b_0v^{-1} + b_1v^{-1}x + \ldots + b_nv^{-1}x^n$, where u, v are regular. Since $P(x)Q(x) = 0$, $(a_0u^{-1}+a_1u^{-1}x+\ldots+a_mu^{-1}x^m)(b_0+b_1x+\ldots+b_nx^n) = 0$ and so

$$a_0u^{-1}b_0 = 0, a_0u^{-1}\alpha(b_1) + a_1u^{-1}\alpha(b_0) = 0, \ldots, a_mu^{-1}\alpha^n(b_n) = 0 \qquad (1)$$

Now for $u^{-1}b_0$, $u^{-1}\alpha(b_1), \ldots, u^{-1}\alpha^n(b_n)$, there exist $c_0, c_1, \ldots, c_n$ and s regular such that $u^{-1}(b_i) = c_is^{-1}$ for all i. Then from eq.(1), we have $a_0c_0 = 0$, $a_0c_1 + a_1c_0 = 0, \ldots, a_mc_n = 0$, and so $p(x)q(x) = 0$, where $p(x) = a_0 + a_1x + \ldots + a_mx^m$ and $q(x) = c_0 + c_1x + \ldots + c_nx^n$ in $R[x;\alpha]$.

Since $R[x;\alpha]$ is strongly right McCoy, there exists nonzero $r \in c_0R + c_1R + \ldots + c_nR$ such that $p(x)r = 0$. Note that,

$$r \in c_0R + c_1R + \ldots + c_nR \subset c_0Q + c_1Q + \ldots + c_nQ$$

$$= c_0s^{-1}Q + c_1s^{-1}Q + \ldots + c_ns^{-1}Q = u^{-1}b_0Q + u^{-1}b_1Q + \ldots + u^{-1}b_nQ.$$

Thus $ur \in b_0 + b_1Q + \ldots + b_nQ = b_0v^{-1}Q + b_1v^{-1}Q + \ldots + b_nv^{-1}Q$ and $ur \neq 0$. Since $p(x)r = 0$, we have

$$(\sum_{i=0}^{m} a_ix^i)r \quad = 0$$

$$\sum_{i=0}^{m} a_i\alpha^i(r)x^i = 0$$

$$a_i\alpha^i(r) = 0$$

Since R is $\alpha - rigid$. So, $a_ir = 0$ implies $a_iu^{-1}ur = 0 \implies a_iu^{-1}\alpha^{-1}(ur) = 0$. Also we can write,

$$\sum_{i=o}^{m} a_iu^{-1}\alpha^{-1}(ur)x^i \qquad = 0$$

$$(\sum_{i=0}^{m} a_iu^{-1}x^i)\alpha^{-1}\alpha^i(ur) = 0$$

$$P(x)ur \qquad = 0$$

Thus, $Q(R)$ is strongly right McCoy.
Conversely, let $p(x) = a_0 + a_1x + \ldots + a_mx^m$ and $q(x) = b_0 + b_1x + \ldots + b_nx^n \in R[x;\alpha]$ such that $p(x)q(x) = 0$. Then $p(x), q(x) \in Q[x]$. As $Q(R)$ is strongly right McCoy, there exists $0 \neq cu^{-1} \in Q(R)$ such that $f(x)cu^{-1} = 0$. Now theorem 22 [14] will work. □

Theorem 2.3. *For a ring* $R[x;\alpha]$ *the following conditions are eqvivalent:*

(i) $R[x;\alpha]$ *is strongly right(resp.left) McCoy;*

(ii) $D_2(R[x;\alpha])$ *is strongly right(resp.left) McCoy;*

(iii) $V_n(R[x;\alpha])$ *is strongly right(resp.left) McCoy for some* $n \geq 2$.

Proof. The proof of this is similar to Hong et al.[4]. □

Theorem 2.4. *Suppose* R *be a ring and* α *be an endomorphism of* R. *If* $R[x;\alpha]$ *is strongly right McCoy, then so is* R.

Proof. Assume that $R[x;\alpha]$ is strongly right McCoy. Let $R[x;\alpha][y]$ be the polynomial ring with an indetermined y over $R[x;\alpha]$. Let $p(x)q(x) = 0$. where,

$$p(x) = \sum_{i=0}^{m} a_i x^i \quad and \quad q(x) = \sum_{j=0}^{n} b_j x^j$$

in $R[x;\alpha]$ such that $p(x)q(x) = 0$.

Suppose $f(y) = \sum_{i=0}^{m} a_i y^i$ and $g(y) = \sum_{j=0}^{n} b_j y^j$ be non-zero polynomials in $R[x;\alpha][y]$ with $f(y)g(y) = 0$. Certainly, $f(y) \neq 0$ and $g(y) \neq 0$ from $f(x) \neq 0$ and $g(x) \neq 0$.
Since $R[x;\alpha]$ is strongly right McCoy, there exists $r \in c_0 + c_1 x + \ldots + c_k x^k \in (\sum_{y=0}^{k} b_j)R[x;\alpha]$ such that $f(y)r = 0$. Then $a_i x^i r = 0$ and so $a_i \alpha^i(r_t) = 0$ for all $0 \leq i \leq m$ and $0 \leq k \leq t$. Since R is $\alpha - rigid$, implies $a_i r_t = 0$. Since r is non-zero, there exists a non -zero r_{t_0}, and thus $p(x)r_{t_0} = 0$.

As $r \in \sum_{j=0}^{k} b_j R[x;\alpha]$ so every r_{t_0} is contained in $\sum_{j=0}^{k} b_j R$. Therefore, R is strongly right McCoy. □

Theorem 2.5. *Let* α *be an endomorphism of a ring* R. *If* R *is right duo or reversible and* $\alpha - rigid$, *then* $R[x;\alpha]$ *is strongly right McCoy.*

Proof. Let R be the right duo ring and suppose $R[x;\alpha]$ denote the skew polynomial ring with an indetermined t over $R[x;\alpha]$. Let $p(t) = \sum_{i=0}^{m} p_i(x)t^i$, $q(t) = \sum_{j=0}^{n} q_j(x)t^j$ be the nonzero polynomials in $R[x;\alpha][t]$ with $p(t)q(t) = 0$, where $p_i(x) = \sum_{h=0}^{n_i} a(i)_h x^h$ and $q_j(x) = \sum_{k=0}^{m_j} b(j)_k x^k$.
Now,

$$l = \sum_{i=0}^{m} deg(p_i(x)) + \sum_{j=0}^{n} deg(q_j(x))$$

where the degree used in the zero polynomial will be 0.

Suppose $F(x) = f_0 + f_1 x^l + \ldots + f_m x^{lm}$, $G(x) = g_0 + g_1 x^l + \ldots + g_n x^{ln} \in R[x;\alpha]$. As the coefficients of the $p_i' s(resp.q_j' s)$ corresponds to the coefficients of $F(x)(resp.G(x)$, we get$F(x)G(x) = 0$ from $p(t)q(t) = 0$.
Now as R is right duo, therefore Theorem 2.2 results out nonzero $r \in R$ such that $G(x)r = 0$ and $a_{(i)_h} \alpha^j (b_{(j)_h}) \alpha^{j+1}(r) = 0$.
Since R is $\alpha - rigid$, implies $a_{(i)_h} b_{(j)_l} r = 0$ for all $i, h, j \quad and \quad l$. This indicates that $\sum_{j=0}^{n} q_j(x)r \neq 0$ and $\sum_{j=0}^{n} q_j(x)r = 0$.
And hence $R[x;\alpha]$ is strongly right McCoy. □

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