Remarks on some fixed point theorems of Ćirić

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Abstract

Motivated by Ćirić [2] we establish some coincidence and fixed point results for a class of pair of mappings in Banach space which gives at least one coincidence point under certain conditions. Some examples are also given to illustrate the results.

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1 Introduction

Fixed point theory has emerged as an interesting area of mathematics since many real world problems can be modeled into the problem of finding fixed point of some nonlinear mapping. In 1922, Banach established a theorem known as Banach contraction principle and that became the foundation of this ultimate theory. Many mathematicians contributed in developing the theory by generalizing, extending and improving Banach contraction principle in many ways. In this direction Ćirić [2] has investigated a class of self-mappings which has atleast one fixed point satisfying a new condition.
Ćirić [2] established the following result.

Theorem 1.1. *[2] Let K be a closed and convex subset of a Banach space with the norm $||x|| = d(x, 0)$ and $f : K \to K$ a mapping which satisfies the condition*

$$d(x, fx) + d(y, fy) \leq qd(x, y), \text{where } 2 \leq q < 4, \tag{1.1}$$

for all x, y in K. Then f has at least one fixed point.

Ćirić, while establishing the above result, defined a sequence

$$x_{n+1} = (x_n + fx_n)/2, \quad n = 0, 1, \dots , \tag{1.2}$$

using the convex property of closed subset K of a Banach space. We remark that if we redefine the sequence defined in (1.2) then we can establish some general results where the choice of q (see, (1.1)) can vary.

Goebel [3], proved coincidence theorems for a pair of mappings and obtained Banach fixed point theorem as a corollary. Jungck [4] studied the potential of commuting pair of mappings and obtained fixed point theorems. Singh-Kulshrestha[6] obtained coincidence theorems for three mappings, one of them commuting with the other two satisfying generalized contraction condition. In this paper motivated by [1], [2] and [5], we first prove some coincidence point theorems for a class of pair of self mappings on a closed subset of a Banach space and then some more coincidence theorems on a closed convex subset of a Banach space for a pair of mappings by redefining (1.2).

2 Results

Now we state our results.

Theorem 2.1. *Let f and g be two self mappings defined on a closed subset K of a Banach space with the norm $||x|| = d(x, 0)$ satisfying*

$$fK \subseteq gK, \tag{2.1}$$

$$d(gx, fx) + d(gy, fy) \leq qd(gx, gy), \text{ where } 1 \leq q < 2, \tag{2.2}$$

for all x, y in K. Then f and g have at least one coincidence point.

Proof. Let x_0 be an arbitrary point in K. Since $fK \subseteq gK$, we may choose a point x_1 in K such that $fx_0 = gx_1$. In general choose x_n such that

$$gx_{n+1} = fx_n, \quad n = 0, 1, \dots . \tag{2.3}$$

For this sequence, we have

$$d(gx_n, fx_n) = ||gx_n - fx_n|| = d(gx_n, gx_{n+1}), \quad n = 0, 1, \dots . \tag{2.4}$$

Therefore, for $x = x_{n-1}$ and $y = x_n$ the condition (2.2) states

$$d(gx_{n-1}, fx_{n-1}) + d(gx_n, fx_n) \leq qd(gx_{n-1}, gx_n).$$

Hence, we have

$$d(gx_n, gx_{n+1}) \leq cd(gx_{n-1}, gx_n), \quad \text{where } 0 \leq c (= q - 1) < 1,$$

as $1 \leq q < 2$. Therefore, $\{gx_n\}$ defined by (2.3) is a Cauchy sequence in K and hence converges to some u in K. Since

$$d(u, fx_n) \leq d(u, gx_n) + d(gx_n, fx_n) = d(u, gx_n) + d(gx_n, gx_{n+1}). \tag{2.5}$$

Using (2.4), we conclude that sequence $\{fx_n\}$ also converges to u in K. Now $fK \subseteq gK \subseteq K$ implies $\{fx_n\}$ and $\{gx_n\}$ converges to same u in K, therefore there exists a point z in K such that $gz = u$. For $x = z$ and $y = x_n$, (2.2) becomes

$$d(gz, fz) + d(gx_n, fx_n) \leq qd(gz, gx_n) = qd(u, u) = 0$$

and we get $d(u, fz) \leq 0$, as $d(gx_n, fx_n) \to 0$, which implies $u = gz = fz$ and the proof of the theorem is complete. □

Now we give an example to illustrate Theorem 2.1.

Let $K = \{0, \frac{1}{2}, 1\}$ and a metric defined on K is given by $d(x, y) = |x - y|$. Also for all $x \in K$, functions $f, g : K \to K$ are defined as $f(x) = 0$ and $g(x) = [x]$, where $[x]$ represent the greatest integer value not greater than x.

It is easy to see that $f(K) \subseteq g(K)$. Now, we investigate the contraction condition (2.2) holds for all $x, y \in K$.

(i) For $x = 0, y = \frac{1}{2}$, we have

$$d(g(0), 0) + d\left(g(\frac{1}{2}), 0\right) = 0 \leq qd\left(0, g(\frac{1}{2})\right) = qd(0, 0).$$

(ii) For $x = 1, y = \frac{1}{2}$, we have

$$d(g(1), 0) + d(g(\frac{1}{2}), 0) = 1 \leq qd\left(g(1), g(\frac{1}{2})\right) = qd(1, 0) \implies 1 \leq q.$$

(iii) For $x = 0, y = 1$, we have

$$d(g(0), 0) + d(g(1), 0) = 1 \leq qd(g(0), g(1)) = qd(0, 1) \implies 1 \leq q.$$

Hence all the conditions of Theorem 2.1 are satisfied for all $x, y \in K$. Here $x = 0$ and $\frac{1}{2}$ are two coincidence points of f and g i.e., $f(0) = g(0) = 0$ and $f(\frac{1}{2}) = g(\frac{1}{2}) = 0$.

Theorem 2.2. *Let f and g be two self mappings defined on a closed subset K of a Banach space with the norm $||x|| = d(x, 0)$ satisfying the condition (2.1) and*

$$d(fx, fy) + d(gx, fx) + d(gy, fy) \leq qd(gx, gy), \text{ where } 1 \leq q < 3, \tag{2.6}$$

for all x, y in K. Then f and g have at least one coincidence point.

Proof. Let x_0 be an arbitrary point in K. Considering a sequence $\{gx_n\}$ in K defined by (2.3). For this sequence the inequality (2.4) holds. Now, if we put $x = x_{n-1}$ and $y = x_n$ in (2.6) and use (2.4), we get

$$d(gx_n, gx_{n+1}) \leq c\ d(gx_{n-1}, gx_n), \text{ where } 0 \leq c\{= (q-1)/2\} < 1,$$

as $1 \leq q < 3$. Therefore, $\{gx_n\}$ is a Cauchy sequence in K and hence converges to some u in K. Since (2.5) holds, $\{fx_n\}$ also converges to same u in K. Now $fK \subseteq gK \subseteq K$ implies $\{fx_n\}$ and $\{gx_n\}$ converges to same u in K, therefore there exists a point z in K such that $gz = u$. For $x = z$ and $y = x_n$, (2.6) becomes $d(u, fz) \leq 0$, which implies $fz = u = gz$ and the proof of the theorem is complete. □

Now we give a general form of the inequality (2.6) which includes (2.2) and (2.6).

Theorem 2.3. *Let f and g be two self mappings defined on a closed subset K of a Banach space with the norm $||x|| = d(x, 0)$. If there exist real numbers a, b and q such that $fK \subseteq gK$,*

$$0 \leq (q - b) < (a + b), \tag{2.7}$$

$$ad(fx, fy) + b\{d(gx, fx) + d(gy, fy)\} \leq qd(gx, gy), \tag{2.8}$$

for all x, y in K, then f and g have at least one coincidence point.

Proof. Let x_0 be an arbitrary point in K. Considering a sequence $\{gx_n\}$ in K defined by (2.3). For this sequence the inequality (2.4) holds. Now, if we put $x = x_{n-1}$ and $y = x_n$ in (2.8) and use (2.4), we get

$$d(gx_n, gx_{n+1}) \leq c\; d(gx_{n-1}, gx_n), \text{ where } 0 \leq c\{= (q - b)/a + b\} < 1,$$

as $0 \leq (q - b) < a + b$. Therefore, sequence $\{gx_n\}$ is a Cauchy sequence in K and hence converges to some u in K. Since (2.5) holds, $\{fx_n\}$ also converges to same u in K. Now $fK \subseteq gK \subseteq K$ implies $\{fx_n\}$ and $\{gx_n\}$ converges to same u in K, therefore there exists a point z in K such that $gz = u$. For $x = z$ and $y = x_n$, (2.8) becomes $(a + b)d(u, fz) \leq 0$, which implies $fz = u = gz$ as $(a + b) > 0$ and the proof of the theorem is complete. □

Now we deduce some corollaries from above established results by considering g as an identity mapping.

Corollary 1. *Let K be a closed subset of a Banach space with the norm $||x|| = d(x, 0)$ and $f : K \to K$ a mapping which satisfies the condition*

$$d(x, fx) + d(y, fy) \leq qd(x, y), \textit{where } 1 \leq q < 2, \tag{2.9}$$

for all x, y in K. Then f has at least one fixed point.

Now we present an example for the validation of Corollary 1.

Let $K = \{1, \frac{3}{2}, 2\}$ and a metric defined on K is given by $d(x, y) = |x - y|$. Also $f : K \to K$ is defined as $f(x) = \frac{3}{2}$ for all $x \in K$.
Now, we shall show that the contraction condition (2.9) holds for all $x, y \in K$.

(i) For $x = 1, y = \frac{3}{2}$, we have

$$d\left(1, \frac{3}{2}\right) + d\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{1}{2} \leq q\; d\left(1, \frac{3}{2}\right) = q\frac{1}{2} \implies 1 \leq q.$$

(ii) For $x = 2, y = \frac{3}{2}$, we have

$$d\left(2, \frac{3}{2}\right) + d\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{1}{2} \leq q\; d\left(\frac{3}{2}, 2\right) = q\frac{1}{2} \implies 1 \leq q.$$

(iii) For $x = 1, y = 2$, we have

$$d\left(1, \frac{3}{2}\right) + d\left(2, \frac{3}{2}\right) = 1 \leq q\ d(1,2) = q \implies 1 \leq q.$$

Hence all the conditions of Corollary 1 are satisfied for all $x, y \in K$ and $x = \frac{3}{2}$ is a fixed point of f. Also note that fixed point is unique in this case.

Corollary 2. *Let K be a closed subset of a Banach space with the norm $||x|| = d(x,0)$ and $f : K \to K$ a mapping which satisfies the condition*

$$d(fx, fy) + d(x, fx) + d(y, fy) \leq qd(x,y), \text{ where } 1 \leq q < 3, \tag{2.10}$$

for all x, y in K. Then f has at least one fixed point.

Corollary 3. *Let K be a closed subset of a Banach space with the norm $||x|| = d(x,0)$ and $f : K \to K$ a mapping. If there exist real numbers a, b and q such that*

$$0 \leq q - b < a + b, \tag{2.11}$$

$$ad(fx, fy) + b\{d(x, fx) + d(y, fy)\} \leq qd(x,y), \tag{2.12}$$

for all x, y in K, then f has at least one fixed point.

Ćirić [2] investigated a class of self mappings on a closed convex subset of a Banach space which have at least one fixed point. We extend these results for a pair of self mappings and establish the following coincidence theorems.

Theorem 2.4. *Let f and g be two self mappings defined on a closed and convex subset K of a Banach space with the norm $||x|| = d(x,0)$ satisfying (2.1) and*

$$d(gx, fx) + d(gy, fy) \leq qd(gx, gy), \text{ where } 2 \leq q < 4, \tag{2.13}$$

for all x, y in K. Then f and g have at least one coincidence point.

Proof. Let x_0 be an arbitrary point in K. Since K is convex, we may choose a point x_1 in K such that $gx_1 = (gx_0 + fx_0)/2$. In general, choose x_n such that

$$gx_{n+1} = (gx_n + fx_n)/2, \quad n = 0, 1, \ldots . \tag{2.14}$$

For this sequence, we have

$$gx_n - fx_n = 2\{gx_n - (gx_n + fx_n)/2\} = 2(gx_n - gx_{n+1})$$

and hence

$$d(gx_n, fx_n) = ||gx_n - fx_n|| = 2\ d(gx_n, gx_{n+1}), \ n = 0, 1, \ldots . \tag{2.15}$$

Therefore, for $x = x_{n-1}$ and $y = x_n$ the condition (2.13) states

$$2d(gx_{n-1}, gx_n) + 2d(gx_n, gx_{n+1}) \leq qd(gx_{n-1}, gx_n).$$

Hence, we have, $d(gx_n, gx_{n+1}) \leq cd(gx_{n-1}, gx_n)$, where $0 \leq c\{= (q-2)/2\} < 1$, as $2 \leq q < 4$. Therefore, $\{gx_n\}$ defined by (2.14) is a Cauchy sequence in K and hence converges to some u in K. Since

$$d(u, fx_n) \leq d(u, gx_n) + d(gx_n, fx_n) = d(u, gx_n) + 2d(gx_n, gx_{n+1}), \tag{2.16}$$

we conclude that sequence $\{fx_n\}$ also converges to same u in K. Now $fK \subseteq gK \subseteq K$ implies $\{fx_n\}$ and $\{gx_n\}$ converge to same u in K, therefore there exists a point z in K such that $gz = u$. For $x = z$ and $y = x_n$, (2.13) becomes

$$d(gz, fz) + d(gx_n, fx_n) \leq qd(gz, gx_n) = qd(u, u) = 0$$

and we get $d(u, fz) \leq 0$, as $d(gx_n, fx_n) \to 0$, which implies $u = gz = fz$ and the proof of the theorem is complete. □

Corollary 4. *Let K be as in Theorem 2.4 and let $f, g : K \to K$ be two self mappings satisfying (2.1) and*

$$d(gx, fy) + d(gy, fx) \leq pd(gx, gy), \text{ where } 0 \leq p < 2. \tag{2.17}$$

Then f and g have at least one coincidence point.

Proof. Using triangular inequality, we have

$$d(gx, fx) + d(gy, fy) \leq d(gx, gy) + d(gy, fx) + d(gx, gy) + d(gx, fy).$$

By (2.17), we get

$$d(gx, fx) + d(gy, fy) \leq pd(gx, gy) + 2d(gx, gy).$$

So, f and g satisfy (2.13) with $q = (p+2)$ and further proof follows from Theorem 2.4. □

Theorem 2.5. *Let f and g be two self mappings defined on a closed and convex subset K of a Banach space with the norm $||x|| = d(x, 0)$ satisfying (2.1) and*

$$d(fx, fy) + d(gx, fx) + d(gy, fy) \leq qd(gx, gy), \tag{2.18}$$

for all x, y in K, where $1 \leq q < 5$. Then f and g have at least one coincidence point.

Proof. Consider a sequence $\{gx_n\}$ in K defined by (2.14). For this sequence the inequality (2.15) and

(2.19)
$$gx_n - fx_{n-1} = \{(gx_{n-1} + fx_{n-1})/2 - fx_{n-1}\} = (gx_{n-1} - fx_{n-1})/2 = d(gx_{n-1}, gx_n),$$

hold. Then by using triangle inequality

$$d(gx_n, fx_n) \leq d(gx_n, fx_{n-1}) + d(fx_{n-1}, fx_n),$$

we get

$$2d(gx_n, gx_{n+1}) - d(gx_{n-1}, gx_n) \leq d(fx_{n-1}, fx_n). \tag{2.20}$$

Now, if we put $x = x_{n-1}$ and $y = x_n$ in (2.18) and use (2.15), then from (2.18) and (2.20), we get

$$d(gx_n, gx_{n+1}) \le cd(gx_{n-1}, gx_n), \text{ where } c = (q-1)/4.$$

Since $1 \le q < 5$, it follows that $\{gx_n\}$ defined by (2.14) is a Cauchy sequence in K and hence converges to some u in K. Since

$$d(u, fx_n) \le d(u, gx_n) + d(gx_n, fx_n) = d(u, gx_n) + 2d(gx_n, gx_{n+1}),$$

we conclude that sequence $\{fx_n\}$ also converges to same u in K. Now $fK \subseteq gK \subseteq K$ implies $\{fx_n\}$ and $\{gx_n\}$ converges to same u in K, therefore there exists a point z in K such that $gz = u$. For $x = z$ and $y = x_n$, (2.18) becomes

$$d(fz, u) + d(u, fz) + d(u, u) \le qd(u, u) = qd(u, u) = 0$$

and we get $d(u, fz) \le 0$, as $d(gx_n, fx_n) \to 0$, which implies $u = gz = fz$ and the proof of the theorem is complete. □

Now, we give a general form of inequality (2.18) which includes (2.13) and (2.18).

Theorem 2.6. *Let f and g be two self mappings defined on a closed and convex subset K of a Banach space with the norm $||x|| = d(x, 0)$. If there exist real numbers a, b and q such that $fK \subseteq gK$ and*

$$0 \le q + |a| - 2b < 2(a + b), \tag{2.21}$$

$$ad(fx, fy) + b\{d(gx, fx) + d(gy, fy)\} \le qd(gx, gy), \tag{2.22}$$

for all x, y in K. Then f and g have at least one coincidence point.

Proof. Consider a sequence $\{gx_n\}$ in K defined by (2.14). For this sequence the inequalities (2.15) and (2.20) holds. Put $x = x_{n-1}$ and $y = x_n$ in (2.22). If $a \ge 0$, then by (2.20) and (2.22), we obtain

(2.23)
$$2ad(gx_n, gx_{n+1}) - |a|d(gx_{n-1}, gx_n) + 2b\{d(gx_{n-1}, gx_n) + d(gx_n, gx_{n+1})\} \le qd(gx_{n-1}, gx_n),$$

since then $-a = -|a|$. If $a < 0$, then we use the inequality

$$d(gx_n, fx_n) + d(gx_n, fx_{n-1}) \ge d(fx_{n-1}, fx_n), \tag{2.24}$$

instead of (2.20). Then by (2.22) and (2.24), we obtain (2.23), because in this case we can write $-|a|$ instead of a. Therefore, (2.23) holds for all values of a, b and q. From (2.23), we get

$$d(gx_n, gx_{n+1}) \le d(gx_{n-1}, gx_n), \text{ where } k = (|a| - 2b + q)/2(a + b).$$

Since (2.21) implies $0 \le k < 1$, it follows that $\{gx_n\}$ defined by (2.14) is a Cauchy sequence in K and hence converges to some u in K. Since

$$d(u, fx_n) \le d(u, gx_n) + d(gx_n, fx_n) = d(u, gx_n) + 2d(gx_n, gx_{n+1}),$$

we conclude that sequence $\{fx_n\}$ also converges to same u in K. Now $fK \subseteq gK \subseteq K$ implies $\{fx_n\}$ and $\{gx_n\}$ converge to u in K. Therefore, there exists a point z in K such that $gz = u$. For $x = z$ and $y = x_n$, (2.22) becomes

$$(a+b)d(fz,u) \leq 0,$$

which implies $u = gz = fz$ as $(a+b) > 0$ and the proof of the theorem is complete. □

Recall the results established by Ćirić [2]. If we redefine the sequence on closed and convex subset of a Banach space, we can have some interesting results. In view of a sequence defined by (2.14) and as K being closed and convex, we can define a generalized sequence for a pair of self maps and establish some coincidence point results.

Theorem 2.7. *Let f and g be two self mappings defined on a closed and convex subset K of a Banach space with the norm $||x|| = d(x,0)$ and satisfying the following conditions:*

$$g(K) \supseteq (1-h)g(K) + hf(K), \text{ where } h \in (0,1), \tag{2.25}$$

$$d(gx,fx) + d(gy,fy) \leq qd(gx,gy), \tag{2.26}$$

for all x, y in K, where

$$1/h \leq q < 2/h. \tag{2.27}$$

Then f and g have at least one coincidence point.

Proof. Let x_0 be an arbitrary point in K and in view of (2.25), let a sequence $\{gx_n\}$ be defined by

$$gx_{n+1} = (1-h)gx_n + hfx_n, \; h \in (0,1), \; n = 0,1,\ldots . \tag{2.28}$$

For this sequence, we have

$$(gx_n - gx_{n+1}) = h(gx_n - fx_n) = hd(gx_n, fx_n), \quad n = 0,1,\ldots . \tag{2.29}$$

Therefore, for $x = x_{n-1}, y = x_n$ and using (2.29), condition (2.26) states

$$1/h\{d(gx_{n-1},gx_n) + d(gx_n,gx_{n+1})\} \leq qd(gx_{n-1},gx_n).$$

Hence, we get, $d(gx_n,gx_{n+1}) \leq cd(gx_{n-1},gx_n)$, where $0 \leq c = (qh-1) < 1$, as $1/h \leq q < 2/h$. Therefore, $\{gx_n\}$ is a Cauchy sequence in K and hence converges to some u in K. Since

$$d(u,fx_n) \leq d(u,gx_n) + d(gx_n,fx_n) = d(u,gx_n) + 1/hd(gx_n,gx_{n+1}), \tag{2.30}$$

we conclude that $\{fx_n\}$ also converges to same u in K. Now $fK \subseteq gK \subseteq K$ (see, (2.25)) implies $\{gx_n\}$ and $\{fx_n\}$ converges to same u in K. Therefore, there exists a point z in K such that $gz = u$. For $x = z$ and $y = x_n$, (2.26) becomes

$$d(u,fz) + d(u,u) \leq qd(u,u)$$

and we get $d(u,fz) \leq 0$, which implies $u = fz = gz$ and the proof of the theorem is complete. □

Now we present an example to support Theorem 2.7.

Let $K = [0, 1]$ and a metric defined on K is given by $d(x, y) = |x - y|$. Also for all $x \in K$, functions $f, g : K \to K$ are defined by $f(x) = x^2$ and $g(x) = x$.
For f and g defined above, we have

$$g(K) \supseteq (1 - h)g(K) + hf(K) \text{ for } h \in (0, 1).$$

Now we investigate contraction condition (2.26) satisfies for all $x, y \in K$.
If $x > y$, then $x^2 > y^2$ and we have two cases: either $y^2 < x^2 < y < x$ or $y^2 < y < x^2 < x$. In both cases, we get

$$\begin{aligned} d(gx, fx) + d(gy, fy) = d(x, x^2) + d(y, y^2) &= x - x^2 + y - y^2, \text{ (as, } x > x^2 \text{ and } y > y^2) \\ &\leq x - y^2 + x - y^2, \text{ (as, } -x^2 < -y^2 \text{ and } y < x) \\ &= 2(x - y^2 + y - y), \text{ (adding and subtracting } y) \\ &\leq 2[(x - y) + (y - y^2)(x - y)], \text{ (as, } x - y > 0) \\ &= 2(1 + y - y^2)d(x, y) \\ &= qd(x, y), \text{ where } q = 2(1 + y - y^2). \end{aligned}$$

Similarly, if $x < y$, we have

$$d(gx, fx) + d(gy, fy) \leq qd(x, y), \text{ where } q = 2(1 + x - x^2).$$

Hence contraction condition (2.26) is satisfied. For the value $x \in [0, 1]$, the maximum value of $1 + x - x^2$ is 1.25. Therefore minimum value of q will be 2.5 and we choose $h = 0.4$, so that, $2.5 \leq q < 5$ by (2.27). Hence all the conditions of Theorem 2.7 are satisfied and $x = 0, 1$ are two coincidence points of f and g.

Corollary 5. *Let K be as in Theorem 2.7 and let $f, g : K \to K$ be two mappings satisfying the condition (2.25) and*

$$d(gx, fy) + d(gy, fx) \leq pd(gx, gy), \tag{2.31}$$

for all x, y in K, where

$$(1/h) - 2 \leq p < (2/h) - 2.$$

Then f and g have at least one coincidence point.

Proof. Using the triangular inequality, we have

$$\begin{aligned} d(gx, fx) + d(gy, fy) &\leq d(gx, fy) + 2d(gx, gy) + d(gy, fx) \\ &\leq (p + 2)d(gx, gy). \end{aligned}$$

So, f and g satisfy (2.26) with $q = p + 2$ and further proof follows from Theorem 2.7. □

Theorem 2.8. *Let f and g be two self mappings defined on a closed and convex subset K of a Banach space with the norm $||x|| = d(x,0)$ and satisfying the (2.25) and*

$$d(fx,fy) + d(gx,fx) + d(gy,fy) \leq qd(gx,gy), \tag{2.32}$$

for all x, y in K, where

$$1 \leq q < 1 + (2/h). \tag{2.33}$$

Then f and g have at least one coincidence point.

Proof. Consider a sequence $\{gx_n\}$ in K be defined by (2.28). For this sequence, the inequality (2.29) and

$$\begin{aligned} d(fx_{n-1}, gx_n) &= ||(fx_{n-1} - (1-h)gx_{n-1} - hfx_{n-1})|| \\ &= (1-h)d(fx_{n-1}, gx_{n-1}) = \{(1-h)/h\}d(gx_{n-1}, gx_n) \end{aligned} \tag{2.34}$$

hold. Then by triangle inequality

$$d(gx_n, fx_n) - d(fx_{n-1}, gx_n) \leq d(fx_{n-1}, fx_n), \tag{2.35}$$

we get

$$(1/h)d(gx_n, gx_{n+1}) - \{(1-h)/h\}d(gx_{n-1}, gx_n) \leq d(fx_{n-1}, fx_n), \tag{2.36}$$

with help of (2.29) and (2.34).
Now, if we put $x = x_{n-1}, y = x_n$ in (2.32), using (2.29) and (2.36), we get

$$\begin{aligned} &(1/h)d(gx_n, gx_{n+1}) - \{(1-h)/h\}d(gx_{n-1}, gx_n) \\ &\quad + (1/h)d(gx_{n-1}, gx_n) + (1/h)d(gx_n, gx_{n+1}) \leq qd(gx_{n-1}, gx_n) \end{aligned} \tag{2.37}$$

and hence

$$d(gx_n, gx_{n+1}) \leq cd(gx_{n-1}, gx_n), \text{ where } c = \{(q-1)h/2\},$$

since $1 \leq q < (2/h)$, it follows that $0 \leq c < 1$. Therefore, $\{gx_n\}$ is a Cauchy sequence in K and hence converges to some u in K. Since (2.30) holds, we conclude that $\{fx_n\}$ also converges to same u in K. Now $fK \subseteq gK \subseteq K$ (see, (2.25)) implies $\{gx_n\}$ and $\{fx_n\}$ converges to same u in K, therefore there exist a point z in K such that $gz = u$. For $x = z$ and $y = x_n$ in (2.32), we get $2d(u, fz) \leq 0$, which gives $u = fz = gz$ and the proof of the theorem is complete. □

Now we give a general form of inequality (2.32) which includes (2.26) and (2.32).

Theorem 2.9. *Let f and g be two self mappings defined on a closed and convex subset K of a Banach space with the norm $||x|| = d(x,0)$. If there exist real numbers a, b and q such that*

$$g(K) \supseteq (1-h)g(K) + hf(K), \text{ where } h \in (0,1), \tag{2.38}$$

$$0 \leq q + (1/h)\{(1-h)|a| - b\} < (1/h)(a+b), \tag{2.39}$$

$$ad(fx,fy) + b\{d(gx,fx) + d(gy,fy)\} \leq qd(gx,gy), \tag{2.40}$$

for all x, y in K, then f and g have at least one coincidence point.

Proof. For the sequence defined by (2.28), we have (2.29) and (2.34). If $a \geq 0$, (2.35) takes the form

$$(a/h)d(gx_n, gx_{n+1}) - \{(1-h)/h\}|a|d(gx_{n-1}, gx_n) \leq d(fx_{n-1}, fx_n) \tag{2.41}$$

and hence with $x = x_{n-1}, y = x_n$, (2.40) becomes

$$(a/h)d(gx_n, gx_{n+1}) - \{(1-h)/h\}|a|d(gx_{n-1}, gx_n) + (b/h)d(gx_{n-1}, gx_n) + (b/h)d(gx_n, gx_{n+1}) \leq qd(gx_{n-1}, gx_n). \tag{2.42}$$

If $a < 0$, we use the inequality

$$d(gx_n, fx_n) + d(fx_{n-1}, gx_n) \geq d(fx_{n-1}, fx_n), \tag{2.43}$$

instead of (2.35). Therefore, (2.41) holds for all a, b and q. From (2.41) we get

$$d(gx_n, gx_{n+1}) \leq cd(gx_{n-1}, gx_n),$$

where $c = [q + (1/h)\{(1-h)|a| - b\}]/(1/h)(a+b)$.
From (2.39), we have $0 \leq c < 1$, it follows that $\{gx_n\}$ is a Cauchy sequence in K and hence converges to some u in K. Since (2.30) holds, we conclude that $\{fx_n\}$ also converges to same u in K. Now $fK \subseteq gK \subseteq K$ (see, (2.38)) implies sequence $\{gx_n\}$ and $\{fx_n\}$ converges to same u in K, therefore there exist a point z in K such that $gz = u$. For $x = z$ and $y = x_n$ in (2.40), we get $(a+b)d(u, fz) \leq 0$, which gives $u = fz = gz$ as $(a+b) > 0$ and the proof of the theorem is complete. □

By setting g to be the identity mapping in Theorem 2.7, Theorem 2.8 and Theorem 2.9, we have the following corollaries.

Corollary 6. *Let K be a closed and convex subset of a Banach space with the norm $||x|| = d(x, 0)$ and $f : K \to K$ a mapping which satisfies the condition*

$$d(x, fx) + d(y, fy) \leq qd(x, y), \tag{2.44}$$

$$K \supseteq (1-h)K + hf(K), \text{ where } h \in (0, 1), \tag{2.45}$$

for all x, y in K, where

$$1/h \leq q < 2/h. \tag{2.46}$$

Then f has at least one fixed point.

Remark 1. *Identity mapping satisfies (2.44), therefore mapping which satisfy (2.44) may have many fixed points.*

Remark 2. *It is interesting to note that when $1/h$(say r) $\in \mathbb{R}(h \in (0, 1))$, then $r \leq q < 2r$ in (2.46) and Corollary 6 can be stated as:*

Let K be a closed and convex subset of a Banach space with the norm $||x|| = d(x, 0)$ and $f : K \to K$ be a mapping which satisfies the condition (2.44) and (2.45) where $1/h$(say r) $\in \mathbb{R}$, for some fixed $h \in (0, 1)$ and $r \leq q < 2r$, then f has at least one fixed point.

Corollary 7. *Let K be a closed and convex subset of a Banach space with the norm $||x|| = d(x,0)$ and $f : K \to K$ a mapping which satisfies (2.45) and the condition*

$$d(fx, fy) + d(x, fx) + d(y, fy) \leq qd(x, y), \tag{2.47}$$

for all x, y in K, where

$$1 \leq q < 1 + (2/h). \tag{2.48}$$

Then f has at least one fixed point.

Corollary 8. *Let f be a self mapping on K, where K be a closed and convex subset of a Banach space with the norm $||x|| = d(x,0)$. If there exist real numbers a, b and q such that*

$$ad(fx, fy) + b\{d(x, fx) + d(y, fy)\} \leq qd(x, y) \tag{2.49}$$

and (2.45) holds for all x, y in K, where

$$0 \leq q + \{(1-h)/h\}|a| - b/h < (1/h)(a + b). \tag{2.50}$$

Then f has at least one fixed point.

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