A note on the location of zeros of entire functions of finite order in a certain domain

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Abstract

In the paper we wish to establish the zero free region for some entire functions of finite order. A few examples with related figures are given here to justify the results obtained.

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1 Introduction, Definitions and Notations.

The study of the problems concerning the distribution of zeros of polynomials have a long history. Gauss, Cauchy and Enström-Kakeya were the first contributors in this field of study [5] and consequently a large numbers of papers devoted in this area of subject can be found in the literature {[1],[2],[6], [8] & [9]}. A function of one complex variable analytic in the finite complex plane $\mathbb{C}$ is called an entire function.If a function $f(z)$ is entire then it can be represented by an every where convergent power series like

$$f(z) = a_0 + a_1 z + ... + a_n z^n + ...$$

Thus the entire functions form natural generalization of polynomials.

The prime concern of this paper is to derive the zero free region for some entire functions of finite order under various conditions using the coefficients a_n's. We do not explain the standard theories, notations and definitions of entire functions as those are available in [10] & [12].

Some well known definitions are as follows:

Definition 1. *The order ρ of an entire function $f(z)$ is defined as*

$$\rho = \limsup_{r \longrightarrow \infty} \frac{\log^{[2]} M(r, f)}{\log r}$$

and the type σ of an entire function $f(z)$ of order ρ as

$$\sigma = \limsup_{r \longrightarrow \infty} \frac{\log M(r,f)}{r^{\rho}}$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ *for* $k = 1, 2, 3, ...$ *and* $\log^{[0]} x = x$.

If $\rho < \infty$ then $f(z)$ is said to be of finite order. Also $\rho = 0$ means that $f(z)$ is of order zero. In this connection Datta and Biswas [4] gave the following definition :

Definition 2. *[4] Let $f(z)$ be an entire function of order zero. Then the quantities ρ^* is defined by*

$$\rho^* = \limsup_{r \to \infty} \frac{\log M(r,f)}{\log r}.$$

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. *[11] If a_j is any complex number with $\mid \arg a_j - \beta \mid \leq \alpha \leq \frac{\pi}{2}$ for some real numbers α and β, then*

$$\mid a_j - a_{j-1} \mid \leq \mid\mid a_j \mid - \mid a_{j-1} \mid\mid \cos\alpha + (\mid a_j \mid + \mid a_{j-1} \mid) \sin\alpha.$$

The following lemma is due to Schwarz .

Lemma 2. *[3] If $g(z)$ is analytic in $\mid z \mid \leq R$, $g(0) = 0$ and $\mid g(z) \mid \leq M$ for $\mid z \mid = R$, then*

$$|g(z)| \leq \frac{M|z|}{R}.$$

Lemma 3. *[7] Let $f(z)$ be analytic for $\mid z \mid < R$. Suppose $f(0) \neq 0$ and let $r_1, r_2, ..., r_n, ...$ be the moduli of the zeros of $f(z)$ in $\mid z \mid < R$ arranged as a non decreasing sequence. If $r_n \leq r \leq r_{n+1}$, then*

$$\log \frac{r^n \mid f(0) \mid}{r_1 r_2 ... r_n} = \frac{1}{2\pi} \int_0^{2\pi} \log \mid f(re^{i\theta}) \mid d\theta,$$

where a zero of order p is counted p times.

Remark 1. *Lemma 3 is known as Jensen's Theorem .*

Lemma 4. *Let $f(z)$ be an entire function of non zero finite order ρ and type σ with $f(0) \neq 0$. Then for every $\epsilon > 0$ and for all sufficiently large values of r , the inequality*

$$N(r) \leq \frac{1}{\log 2} \{(\sigma + \epsilon)(2r)^{\rho} - \log \mid f(0) \mid\}$$

holds where $N(r)$ is the number of zeros of $f(z)$ in $\mid z \mid < r$.

Proof. Since $f(z)$ is an entire function of non zero finite order ρ and type σ, for sufficiently large values of r and for every $\epsilon > 0$,

$$\max_{|z|=r} \mid f(z) \mid = M_f(r) \leq e^{(\sigma+\epsilon)r^\rho}.$$

Let $z_1, z_2, z_3, ..., z_n$ be n zeros of $f(z)$ with $\mid z_1 \mid \leq \mid z_2 \mid \leq ... \leq \mid z_n \mid < R = 2r$. Then by Lemma 3, it follows that

$$\sum_{i=1}^{N(2r)} \log \frac{2r}{\mid z_i \mid} = \frac{1}{2\pi} \int_0^{2\pi} \log \mid f(2re^{i\theta}) \mid d\theta - \log \mid f(0) \mid$$

$$\leq e^{(\sigma+\epsilon)(2r)^\rho} - \log \mid f(0) \mid .$$

Therefore,

$$N(r) \log 2 \leq \sum_{i=1}^{N(r)} \log \frac{2r}{\mid z_i \mid} \leq \sum_{i=1}^{N(2r)} \log \frac{2r}{\mid z_i \mid} \leq e^{(\sigma+\epsilon)(2r)^\rho} - \log \mid f(0) \mid$$

i.e, $N(r) \leq \frac{1}{\log 2}\{(\sigma+\epsilon)(2r)^\rho - \log \mid f(0) \mid\}$.
This completes the proof of the lemma.

□

Lemma 5. *Let $f(z)$ be an entire function of order zero with $f(0) \neq 0$. Then for every $\epsilon > 0$ and for all sufficiently large values of r , the inequality*

$$N(r) \leq \frac{1}{\log 2}\{(\rho^* + \epsilon) \log 2r - \log \mid f(0) \mid\}$$

holds where $N(r)$ is the number of zeros of $f(z)$ in $\mid z \mid < r$.

Proof. If $f(z)$ is an entire function of order zero, by Definition 2 it follows that for sufficiently large values of r and for every $\epsilon > 0$,

$$\max_{|z|=r} \mid f(z) \mid = M_f(r) \leq r^{\rho^*+\epsilon}.$$

Therefore in a like manner as in Lemma 4, we have

$$N(r) \leq \frac{1}{\log 2}\{(\rho^* + \epsilon) \log 2r - \log \mid f(0) \mid\}.$$

This proves the lemma.

□

3 Theorems.

In this section we present the main results of the paper.

Theorem 3.1. *Let* $f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + ... + a_{n-1} z^{n-1} + a_n z^n + ...$ *be an entire function of non zero finite order* $\rho(\geq 1)$ *and type* σ *with* $f(0) \neq 0$. *Also let* N *be the greatest positive integer less than or equal to* $N(r)$ *in* $| z |< r$ *such that* $a_N \neq 0$. *If for some real numbers* α *and* β,

$$| \arg a_j - \beta |\leq \alpha \leq \frac{\pi}{2}, j = 0, 1, 2, .., N$$

and

$$\rho \mid a_0 \mid\geq r \mid a_1 \mid\geq r^2 \mid a_2 \mid\geq ... \geq r^N \mid a_N \mid$$

then $f(z)$ *does not vanish in* $| z |< t_0$ *within the open disk* $| z |< r$ *where* t_0 *is the least positive root of the equation*

$$g(t) \equiv\mid a_0 \mid r^{N+1} - (\rho \mid a_0 \mid +B)r^N t + ((\rho - 1) \mid a_0 \mid +B)r^{N-1} t^2 - 2Mt^{N+1} = 0$$

with $B = (\rho \mid a_0 \mid -r^N \mid a_N \mid) \cos\alpha + (\rho \mid a_0 \mid +r^N \mid a_N \mid) \sin\alpha +$ $2\sin\alpha \sum_{j=1}^{N-1} \mid a_j \mid r^j$ *and* $\max_{|z|=r} \mid f(z) \mid\leq M$.

Proof. Clearly, $\lim_{n\to\infty} a_n r^n = 0$.
Let $F(z) = (z - r)f(z)$.
So,

$$\begin{aligned} F(z) &= (z - r)(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + ... + a_{n-1} z^{n-1} + a_n z^n + ...) \\ \text{i.e,} F(z) &= -ra_0 + (a_0 - ra_1)z + (a_1 - ra_2)z^2 + ... + (a_{N-1} - ra_N)z^N + \\ &\quad \sum_{j=N+1}^{\infty} (a_{j-1} - ra_j)z^j \\ \text{i.e,} F(z) &= -ra_0 + (a_0 - \rho a_0 + \rho a_0 - ra_1)z + (a_1 - ra_2)z^2 + ... + (a_{N-1} - ra_N)z^N + \\ &\quad \sum_{j=N+1}^{\infty} (a_{j-1} - ra_j)z^j \\ \text{i.e,} F(z) &= -ra_0 + (1 - \rho)a_0 z + (\rho a_0 - ra_1)z + (a_1 - ra_2)z^2 + ... + (a_{N-1} - ra_N)z^N + \\ &\quad \sum_{j=N+1}^{\infty} (a_{j-1} - ra_j)z^j \\ \text{i.e,} F(z) &= -ra_0 + (1 - \rho)a_0 z + G(z) + H(z). \qquad (1) \end{aligned}$$

Now for $\mid z \mid= r$, applying Lemma 1 it follows that

$$\begin{aligned}
\mid G(z) \mid &\leq\mid \rho a_0 - ra_1 \mid\mid z \mid + \mid a_1 - ra_2 \mid\mid z \mid^2 +...+ \mid a_{N-1} - ra_N \mid\mid z \mid^N \\
&\leq\mid \rho a_0 - ra_1 \mid r+ \mid a_1 - ra_2 \mid r^2 + ...+ \mid a_{N-1} - ra_N \mid r^N \\
&\leq \{\mid \rho \mid a_0 \mid -r \mid a_1 \mid\mid \cos\alpha + (\rho \mid a_0 \mid +r \mid a_1 \mid) \sin\alpha\}r+ \\
&\{\mid\mid a_1 \mid -r \mid a_2 \mid\mid \cos\alpha + (\mid a_1 \mid +r \mid a_2 \mid) \sin\alpha\}r^2+ \\
&\{\mid\mid a_{N-1} \mid -r \mid a_N \mid\mid \cos\alpha + (\mid a_{N-1} \mid +r \mid a_N \mid) \sin\alpha\}r^N \\
&= (\rho \mid a_0 \mid -r \mid a_1 \mid)r\cos\alpha + (\rho \mid a_0 \mid +r \mid a_1 \mid)r\sin\alpha + (\mid a_1 \mid -r \mid a_2 \mid)r^2\cos\alpha \\
&+ (\mid a_1 \mid +r \mid a_2 \mid)r^2\sin\alpha + (\mid a_{N-1} \mid -r \mid a_N \mid r^N\cos\alpha + (\mid a_{N-1} \mid +r \mid a_N \mid)r^N\sin\alpha \\
&= (\rho \mid a_0 \mid -r^N \mid a_N \mid)r\cos\alpha + (\rho \mid a_0 \mid +r^N \mid a_N \mid)r\sin\alpha + 2r\sin\alpha \sum_{j=1}^{N-1} \mid a_j \mid r^j \\
&= Br
\end{aligned}$$

where $B = (\rho \mid a_0 \mid -r^N \mid a_N \mid)\cos\alpha + (\rho \mid a_0 \mid +r^N \mid a_N \mid)\sin\alpha + 2\sin\alpha \sum_{j=1}^{N-1} \mid a_j \mid r^j$.

Since $G(z)$ is analytic in $\mid z \mid\leq r$, $G(0) = 0$ and $\mid G(z) \mid\leq Br$ for $\mid z \mid= r$, we obtain by Lemma 2 that

$$\mid G(z) \mid\leq \frac{Br \mid z \mid}{r} = B \mid z \mid \qquad (2)$$

Also, as for the coefficients of the power series $\sum_{n=0}^{\infty} a_n z^n$ in $\mid z \mid\leq r$, $\mid a_n \mid\leq \frac{M}{r^n}$ and $\mid a_{j-1} - a_j \mid\leq \frac{2M}{r^{j-1}}$, we get for $\mid z \mid< r$ that

$$\mid H(z) \mid\leq \frac{2M}{r^{N-1}}.\frac{\mid z \mid^{N+1}}{r- \mid z \mid}. \qquad (3)$$

Hence by using (2) and (3), for $\mid z \mid< r$ it follows from (1) that

$$\begin{aligned}
\mid F(z) \mid &\geq\mid a_0 \mid r - (\rho - 1) \mid a_0 \mid\mid z \mid - \mid G(z) \mid - \mid H(z) \mid \\
&\geq\mid a_0 \mid r - (\rho - 1) \mid a_0 \mid\mid z \mid -B \mid z \mid - \frac{2M}{r^{N-1}}.\frac{\mid z \mid^{N+1}}{r- \mid z \mid} \\
&= \frac{\mid a_0 \mid r^{N+1} - (\rho \mid a_0 \mid +B)r^N \mid z \mid +((\rho - 1) \mid a_0 \mid +B)r^{N-1} \mid z \mid^2 -2M \mid z \mid^{N+1}}{r^{N-1}(r- \mid z \mid)}.
\end{aligned}$$

Let $g(t) \equiv\mid a_0 \mid r^{N+1} - (\rho \mid a_0 \mid +B)r^N t + ((\rho - 1) \mid a_0 \mid +B)r^{N-1}t^2 - 2Mt^{N+1}$.

We see that the number of changes in sign in the coefficients of $g_1(t)$ is 3. Hence by Descarte's rule of sign, the number of positive real roots of $g_1(t) = 0$ will be either 3 or 1.

Let t_0 be the least positive root of the equation $g(t) = 0$.

Since $g(0) = \mid a_0 \mid r^{N+1} > 0$ and $g(\infty) = -\infty < 0$, $g(t) > 0$ if $t < t_0$, otherwise there will be another positive root in $(0, t_0)$, which makes a contradiction.

Hence for $\mid z \mid < r$,

$$\mid F(z) \mid > 0 \text{ if } \mid z \mid < t_0.$$

Therefore $F(z)$ does not vanish in $\mid z \mid < t_0$.
Consequently, no zeros of $f(z)$ lie in $\mid z \mid < t_0$ contained in $\mid z \mid < r$.

This proves the theorem. □

Remark 2. *The following example with related figure ensures the validity of Theorem 3.1.*

Let $f(z) = \cos 2z + 2z + 3 = 4 + 2z - 2z^2 + \frac{2}{3}z^4 - ....$

Here, $\rho = 1$ *and* $\sigma = 2$.

Since $\frac{\pi}{2} = \mid \arg a_j - \frac{\pi}{2} \mid \leq \alpha \leq \frac{\pi}{2}$ *for* $j = 0, 1, 2, ...,$ *we may take* $\alpha = \frac{\pi}{2}$.

Now considering $r = 1$ *and* $\epsilon = .0001$, *by Lemma 4 we get* $N(r) \leq 3.77$. *Since* $2 =$ *highest positive integer* $\leq N(r)$ *and* $a_2 \neq 0$, *we may take* $N = 2$.

Now, $B = (\rho \mid a_0 \mid - r^N \mid a_N \mid) \cos\alpha + (\rho \mid a_0 \mid + r^N \mid a_N \mid) \sin\alpha + 2\sin\alpha \sum_{j=1}^{N-1} \mid a_j \mid r^j = 10$ *and* $M = \frac{e^2 + e^{-2}}{2} + 2 + 3 \approx 8.76$.

Thus
$$g(t) = \mid a_0 \mid r^{N+1} - (\rho \mid a_0 \mid + B) r^N t + ((\rho - 1) \mid a_0 \mid + B) r^{N-1} t^2 - 2Mt^{N+1}$$
$$= 4 - 14t + 10t^2 - 17.52t^3.$$

Since $g(.31) > 0$, $g(.32) < 0$ *and also* $g(t) > 0$ *for* $0 \leq t \leq .31$, *the least positive root of* $g(t)$ *lies between* .31 *and* .32.

Hence by Theorem 3.1, $f(z)$ *does not vanish in* $\mid z \mid \leq .31$ *contained in* $\mid z \mid < 1$.

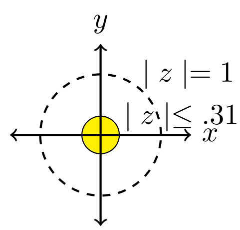

Fig. 1: Zero free region of $f(z) = \cos 2z + 2z + 3$ in $\mid z \mid < 1$

Remark 3. *Let us consider a polynomial* $P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + ... + a_{n-1} z^{n-1} + a_n z^n, a_n \neq 0$.
Now for sufficiently large values of r, $M_P(r) = \mid a_n \mid r^n$. *Thus we see by Definition 1 and Definition 2 respectively that* $\rho = 0$ *and* $\rho^* = n \geq 1$.

In view of above, we may state the following theorem for entire functions of order zero. The proof of which can be carried out in the line of Theorem 3.1 and therefore proof is omitted.

Theorem 3.2. *Let $f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + ... + a_{n-1} z^{n-1} + a_n z^n + ...$ be an entire function of non zero finite order $\rho^*(\geq 1)$ with $f(0) \neq 0$. Also let N be the greatest positive integer less than or equal to $N(r)$ in $\mid z \mid< r$ such that $a_N \neq 0$. If for some real numbers α and β,*

$$\mid \arg a_j - \beta \mid \leq \alpha \leq \frac{\pi}{2}, j = 0, 1, 2, ..., N$$

and

$$\rho^* \mid a_0 \mid \geq r \mid a_1 \mid \geq r^2 \mid a_2 \mid \geq ... \geq r^N \mid a_N \mid$$

then $f(z)$ does not vanish in $\mid z \mid< t_0$ within the open disk $\mid z \mid< r$ where t_0 is the least positive root of the equation

$$g(t) \equiv \mid a_0 \mid r^{N+1} - (\rho^* \mid a_0 \mid +B) r^N t + ((\rho^* - 1) \mid a_0 \mid +B) r^{N-1} t^2 - 2M t^{N+1} = 0$$

with $B = (\rho^* \mid a_0 \mid -r^N \mid a_N \mid) \cos\alpha + (\rho^* \mid a_0 \mid +r^N \mid a_N \mid) \sin\alpha + 2 \sin\alpha \sum_{j=1}^{N-1} \mid a_j \mid r^j$ and $\max\limits_{|z|=r} \mid f(z) \mid \leq M$.

Theorem 3.3. *Let $f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + ... + a_{n-1} z^{n-1} + a_n z^n + ...$ be an entire function of non zero finite order $\rho(\geq 1)$ and type $\sigma(\geq 1)$ with $f(0) \neq 0$. Also let $a_j = \alpha_j + i\beta_j, j = 0, 1, 2, ...$ and N be the greatest positive integer less than or equal to $N(r)$ in $\mid z \mid< r$ such that $a_N \neq 0$.If*

$$\rho\alpha_0 \geq r\alpha_1 \geq r^2\alpha_2 \geq ... \geq r^N \alpha_N$$
and
$$\sigma\beta_0 \geq r\beta_1 \geq r^2\beta_2 \geq ... \geq r^N \beta_N$$

then no zeros of $f(z)$ lie in $\mid z \mid< t_0'$ contained in $\mid z \mid< r$ where t_0' is the least positive root of the equation

$$h(t) \equiv \mid a_0 \mid r^{N+1} - (C + D + \mid a_0 \mid) r^N t + (C + D) r^{N-1} t^2 - 2M t^{N+1} = 0$$

with $C = (\rho\alpha_0 - r^N \alpha_N) + (\sigma\beta_0 - r^N \beta_N)$, $D = (\rho - 1) \mid \alpha_0 \mid +(\sigma - 1) \mid \beta_0 \mid$ and $\max\limits_{|z|=r} \mid f(z) \mid \leq M$.

Proof. Clearly, $\lim\limits_{n\to\infty} a_n r^n = 0$.
Let $F(z) = (z - r) f(z)$.

So,

$$
\begin{aligned}
F(z) &= (z-r)(a_0 + a_1 z + a_2 z^2 + a_3 z^3 + ... + a_{n-1}z^{n-1} + a_n z^n + ...)\\
\text{i.e,}F(z) &= -ra_0 + (a_0 - ra_1)z + (a_1 - ra_2)z^2 + ... + (a_{N-1} - ra_N)z^N +\\
&\quad \sum_{j=N+1}^{\infty}(a_{j-1} - ra_j)z^j\\
\text{i.e,}F(z) &= -ra_0 + \{(\alpha_0 - r\alpha_1) + i(\beta_0 - r\beta_1)\}z + \{(\alpha_1 - r\alpha_2) + i(\beta_1 - r\beta_2)\}z^2 + ...\\
&\quad + \{(\alpha_{N-1} - r\alpha_N) + i(\beta_{N-1} - r\beta_N)\}z^N + \sum_{j=N+1}^{\infty}(a_{j-1} - ra_j)z^j\\
\text{i.e,}F(z) &= -ra_0 + \{(\alpha_0 - \rho\alpha_0 + \rho\alpha_0 - r\alpha_1) + i(\beta_0 - \sigma\beta_0 + \sigma\beta_0 - r\beta_1)\}z +\\
&\quad \{(\alpha_1 - r\alpha_2) + i(\beta_1 - r\beta_2)\}z^2 + ... + \{(\alpha_{N-1} - r\alpha_N) + i(\beta_{N-1} - r\beta_N)\}z^N +\\
&\quad \sum_{j=N+1}^{\infty}(a_{j-1} - ra_j)z^j\\
\text{i.e,}F(z) &= -ra_0 + \{(1-\rho)\alpha_0 + i(1-\sigma)\beta_0\}z + \{(\rho\alpha_0 - r\alpha_1) + i(\sigma\beta_0 - r\beta_1)\}z +\\
&\quad \{(\alpha_1 - r\alpha_2) + i(\beta_1 - r\beta_2)\}z^2 + ... + \{(\alpha_{N-1} - r\alpha_N) + i(\beta_{N-1} - r\beta_N)\}z^N +\\
&\quad \sum_{j=N+1}^{\infty}(a_{j-1} - ra_j)z^j\\
\text{i.e,}F(z) &= -ra_0 + \{(1-\rho)\alpha_0 + i(1-\sigma)\beta_0\}z + G(z) + H(z) \qquad (1)
\end{aligned}
$$

where $G(z) = \{(\rho\alpha_0 - r\alpha_1) + i(\sigma\beta_0 - r\beta_1)\}z + \{(\alpha_1 - r\alpha_2) + i(\beta_1 - r\beta_2)\}z^2 + ...+ \{(\alpha_{N-1} - r\alpha_N) + i(\beta_{N-1} - r\beta_N)\}z^N$ and $H(z) = \sum_{j=N+1}^{\infty}(a_{j-1} - ra_j)z^j$.
Now for $| z |= r$, we get that

$$
\begin{aligned}
| G(z) | &\leq \{| \rho\alpha_0 - r\alpha_1 | + | \sigma\beta_0 - r\beta_1 |\}r + \{| \alpha_1 - r\alpha_2 | + | \beta_1 - r\beta_2 |\}r^2 + ...\\
&\quad + \{| \alpha_{N-1} - r\alpha_N | + | \beta_{N-1} - r\beta_N |\}r^N\\
&\leq \{(\rho\alpha_0 - r^N\alpha_N) + (\sigma\beta_0 - r^N\beta_N)\}r\\
&= rC \text{ where } C = (\rho\alpha_0 - r^N\alpha_N) + (\sigma\beta_0 - r^N\beta_N).
\end{aligned}
$$

Since $G(z)$ is analytic in $| z |\leq r$, $G(0) = 0$ and $| G(z) |\leq rC$ for $| z |= r$, we obtain by Lemma 2 that

$$
| G(z) |\leq \frac{rC \mid z \mid}{r} = C \mid z \mid . \qquad (2)
$$

Also from (3) of Theorem 3.1, it follows for $| z |< r$ that

$$
| H(z) |\leq \frac{2M}{r^{N-1}}.\frac{\mid z \mid^{N+1}}{r- \mid z \mid}. \qquad (3)
$$

Hence by using (2), (3), we get from (1) for $\mid z \mid< r$ that

$$\begin{aligned}
\mid F(z) \mid &=\mid -ra_0 + \{(1-\rho)\alpha_0 + i(1-\sigma)\beta_0\}z + G(z) + H(z) \mid \\
&\geq\mid -ra_0 \mid - \mid \{(1-\rho)\alpha_0 + i(1-\sigma)\beta_0\}z + G(z) + H(z) \mid \\
&\geq r \mid a_0 \mid -\{(\rho-1) \mid \alpha_0 \mid +(\sigma-1) \mid \beta_0 \mid\} \mid z \mid - \mid G(z) \mid - \mid H(z) \mid \\
&\geq r \mid a_0 \mid -\{(\rho-1) \mid \alpha_0 \mid +(\sigma-1) \mid \beta_0 \mid\} \mid z \mid -C \mid z \mid - \frac{2M}{r^{N-1}}.\frac{\mid z \mid^{N+1}}{r- \mid z \mid} \\
&= \frac{\begin{bmatrix} \mid a_0 \mid r^{N+1} - \{(\rho-1) \mid \alpha_0 \mid +(\sigma-1) \mid \beta_0 \mid +C+ \mid a_0 \mid\}r^N \mid z \mid \\ + \{(\rho-1) \mid \alpha_0 \mid +(\sigma-1) \mid \beta_0 \mid +C\}r^{N-1} \mid z \mid^2 -2M \mid z \mid^{N+1} \end{bmatrix}}{r^{N-1}(r- \mid z \mid)} \\
&= \frac{\mid a_0 \mid r^{N+1} - (C+D+ \mid a_0 \mid)r^N \mid z \mid +(C+D)r^{N-1} \mid z \mid^2 -2M \mid z \mid^{N+1}}{r^{N-1}(r- \mid z \mid)}
\end{aligned}$$

where $D = (\rho-1) \mid \alpha_0 \mid +(\sigma-1) \mid \beta_0 \mid$.

Let $h(t) \equiv\mid a_0 \mid r^{N+1} - (C+D+ \mid a_0 \mid)r^N t + (C+D)r^{N-1}t^2 - 2Mt^{N+1}$.

We see that the number of changes in sign in the coefficients of $h(t)$ is 3. So by Descarte's rule of sign the number of positive roots of $h(t) = 0$ is either 3 or 1.

Let t_0' be the least positive root of $h(t) = 0$.

Clearly $h(0) > 0$ and $h(\infty) < 0$. Therefore $h(t) > 0$ for $t < t_0'$, otherwise there will be another positive root in $(0, t_0')$ which is a contradiction.

Hence for $\mid z \mid< r$,

$$\mid F(z) \mid> 0 \text{ if } \mid z \mid< t_0' .$$

From above it follows that $F(z)$ has no zeros in $\mid z \mid< t_0'$.

Consequently, no zeros of $f(z)$ lie in $\mid z \mid< t_0'$.

Thus the theorem is established. □

Remark 4. *The following example with related figure justifies the validity of Theorem 3.3.*

Let

$$f(z) = (1+i)\cos 2z + (2+3i) = (3+4i) - (2+2i)z^2 + (\frac{2}{3} + \frac{2}{3}i)z^4 - ... .$$

Here, $\rho = 1$ *and* $\sigma = 2$.

Let us take $r = 1, \epsilon = .0001$. *Then by Lemma 4,* $N(r) \leq 3.45$. *Since* 2 *is the highest positive integer* $\leq N(r)$ *and* $a_2 \neq 0$, *we may take* $N = 2$.

Here, $\alpha_0 = 3, \alpha_1 = 0, \alpha_2 = -2, \beta_0 = 4, \beta_1 = 0$ *and* $\beta_2 = -2$.

Also, $C = (\rho\alpha_0 - r^N\alpha_N) + (\sigma\beta_0 - r^N\beta_N) = 15$, $D = (\rho-1) \mid \alpha_0 \mid +(\sigma-1) \mid \beta_0 \mid= 4$, *and* $M = \sqrt{2}.\frac{e^2+e^{-2}}{2} + \sqrt{13} \approx 8.93$.

Now, $h(t) \equiv \mid a_0 \mid r^{N+1} - (C + D + \mid a_0 \mid) r^N t + (C + D) r^{N-1} t^2 - 2Mt^{N+1} = 5 - 24t + 19t^2 - 17.86t^3$.

As $h(.24) > 0$, $h(.25) < 0$ *and* $g(t) > 0$ *for* $0 \leq t \leq .24$, *the least positive root of* $h(t)$ *lies between* $.24$ *and* $.25$.

Hence by Theorem 3.3, $f(z)$ *does not vanish in* $\mid z \mid \leq .24$.

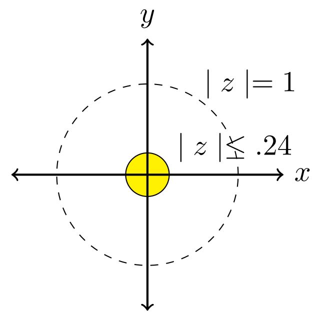

Fig. 2: Zero free region of $f(z) = (1 + i)\cos 2z + (2 + 3i)$ in $\mid z \mid < 1$

Future prospect. In the line of the works as carried out in the paper one may think of proving analogous results for entire functions of infinite order.

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