On Conformal and Almost Conformal Ricci Solitons Concerning Lorentzian Trans-Sasakian Manifolds

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Abstract

In this paper, we have shown that if a 3-dimensional Lorentzian trans-Sasakian manifold M admits conformal Ricci soliton (g, V, λ) and if the vector field V is pointwise collinear with the unit vector field ξ, then V is a constant multiple of ξ. Similarly, we have proved that an almost conformal Ricci soliton becomes conformal Ricci soliton. Next, we have studied conformal curvature tensor admitting conformal Ricci soliton and found that Lorentzian trans-Sasakian manifold admitting conformal Ricci soliton with the condition $R(\xi, X)C = 0$ is $\eta-$ Einstein manifold. We also discuss the example of a 3-dimensional Lorentzian trans-Sasakian manifold. .

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1 Introduction

In 1982, the concept of Ricci flow, the notion of Ricci soliton and its existence introduced by Hamilton [9]. We need this concept to answer Thurston's geometric conjecture. Thurstons geometric conjecture says that a 3-dimensional manifold admits a geometric decomposition if it is closed. All compact manifolds of dimensional four with positive curvature also classified by Hamilton. the equation of Ricci flow is as follows

$$\frac{\delta g}{\delta t} = -2S, \tag{1.1}$$

Ricci soliton emerges as the limit of the soliton of Ricci flow, a soliton to the Ricci flow [9, 14] to be a Ricci soliton [8] if it has only one moving parameter group of diffeomorphism and scaling. the Ricci soliton equation is expressed by

$$\ell_X g + 2S - 2\lambda g = 0, \tag{1.2}$$

In the above equation, S denotes the Ricci tensor, g is the Riemannian metric, ℓ_X is the Lie derivative, X is a vector field, and λ is a scalar.
In the above equation if
(a) $\lambda > 0$ (positive) then it defines as Shrinking Ricci Soliton.

(b) $\lambda < 0$ (negative) then it is defined as Expanding Ricci soliton.
(c) $\lambda = 0$ then it is defined as Steady Ricci Soliton.

Next, A. E. Fischler developed the new concept of conformal Ricci flow [7] during 2003-04, which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The new equations define the conformal Ricci flow on M, where M is considered as a smooth closed connected oriented n-manifold, and the equation is as follows [7]

$$\frac{\delta g}{\delta t} + 2(S + \frac{g}{n}) = -pg, \tag{1.3}$$

Where $r(g)$ the scalar curvature of the manifold is defined by $r(g) = -1$, p is a scalar non-dynamical field, and the dimension of the manifold is n.

Similarly, N. Basu and A. Bhattacharya [2] in 2015 introduce the notion of conformal Ricci soliton equation as follows

$$\ell_X g + 2S = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g, \tag{1.4}$$

The concept of Ricci almost soliton was firstly introduced by S. Pigola, M. Rigoli, M. Rimoldi, A. G. Setti in 2010 [11]. R. Sharma also started the study of Ricci soliton and done excellent work [12].
In this paper, we use the notion of almost conformal Ricci soliton by

$$(\ell_X g)(X, \Upsilon) + 2S(X, \Upsilon) = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X, \Upsilon) \tag{1.5}$$

Where $\lambda : M^n \to R$ is a amooth function.

2 Preliminaries :

Let us define almost contact metric structure (φ, ξ, η, g) on M and it is said to be Trans-Sasakian [10] if $(M \times R, J, G)$ belonging to the class W_4, with the almost complex structure J on $M \times R$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\varphi X - f\xi, \eta(X)\frac{d}{dt}\right), \tag{2.1}$$

For all vectors fields, X on M. f is a smooth function on $M \times R$ and G is the product metric on $M \times R$ it can be expressed by [4].

$$(\triangledown_X \varphi)\Upsilon = \alpha\left(g(X, \Upsilon)\xi - \eta(\Upsilon)X\right) + \beta\left(g(\varphi X, \Upsilon)\xi - \eta(\Upsilon)\varphi X\right), \tag{2.2}$$

α and β are smooth functions on M.
In view of (2.2) the Trans Sasakian structures hold given results:

$$\triangledown_X \xi = -\alpha(\varphi X) + \beta\left(X - \eta(X)\xi\right) \tag{2.3}$$

$$(\nabla_X \eta)\Upsilon = -\alpha g(\varphi X, \Upsilon) + \beta g(\varphi X, \varphi \Upsilon), \tag{2.4}$$

3-Dimensional Lorentzian Trans Sasakian Manifold

Now, let us take a 3-dimensional differentiable manifold M, and it is said to be a Lorentzian Trans Sasakian manifold if it admits a (1,1) tensor field φ a contravariant vector field ξ a covariant vector field η and the Lorentzian metric g holds the conditions [6]

$$\varphi^2 X = X + \eta(X)\xi, \eta(\xi) = -1, \eta(\varphi X) = 0, \varphi(\xi) = 0 \tag{2.5}$$

$$g(\varphi X, \varphi \Upsilon) = g(X, \Upsilon) + \eta(X)\eta(\Upsilon) \tag{2.6}$$

$$g(X, \varphi \Upsilon) = g(\varphi X, \Upsilon) \tag{2.7}$$

$$g(X, \xi) = \eta(X), \tag{2.8}$$

For all vector fields $X, Y \in \chi(M)$.
For Lorentzian Trans-Sasakian manifold following relations are apparent:

$$\nabla_X \xi = -\alpha(\varphi X) - \beta\left(X + \eta(X)\xi\right) \tag{2.9}$$

$$(\nabla_X \eta)\Upsilon = \alpha g(\varphi X, \Upsilon) + \beta g(\varphi X, \varphi \Upsilon), \tag{2.10}$$

where ∇ denotes the operator of covariant differentiation concerning the Lorentzian metric g.
In a 3-dimensional Lorentzian Trans Sasakian manifold, Ricci tensor given by

$$\begin{aligned}(2.11) \S(X,Y) \;=\; & (\frac{r}{2} + \xi\beta - (\alpha^2 + \beta^2) + \psi(\xi\alpha - 2\alpha\beta))g(X,\Upsilon) \\ & +(\frac{r}{2} + \xi\beta - 3(\alpha^2 + \beta^2) - 4\alpha\beta\psi)\eta(X)\eta(\Upsilon) + \eta(X)[-\Upsilon B + (\varphi\Upsilon)\alpha \\ & -\psi(\Upsilon\alpha)] - \eta(\Upsilon)(X\beta - (\varphi X)\alpha + \psi(X\alpha)) + (2\alpha\beta - \xi\alpha)g(\varphi X, \Upsilon),\end{aligned}$$

(2.12) $S(X, \xi) = (2(\alpha^2 + \beta^2) - \xi\beta)\eta(X) + (X\beta) - (\varphi X)\alpha + \psi(2\alpha\beta\eta(X) + X\alpha)$

Where α and β are smooth functions on M and r it's scalar curvature.
For α, β=constant, the following relations holds:

$$S(X, \Upsilon) = \left(\frac{r}{2} - (\alpha^2 + \beta^2)\right) g(X, \Upsilon) + \left(\frac{r}{2} - 3(\alpha^2 + \beta^2)\right) \eta(X)\eta(\Upsilon) \tag{2.13}$$

$$S(X, \xi) = 2(\alpha^2 + \beta^2)\eta(X) \tag{2.14}$$

$$R(X,\Upsilon)\xi = (\alpha^2+\beta^2)(X\eta(\Upsilon) - \eta(X)\Upsilon) \tag{2.15}$$

$$QX = \left(\frac{r}{2} - (\alpha^2+\beta^2)\right)X + \left(\frac{r}{2} - 3(\alpha^2+\beta^2)\right)\eta(X)\xi \tag{2.16}$$

$$R(\xi,X)\Upsilon = (\alpha^2+\beta^2)\left[g(X,\Upsilon)\xi - \eta(\Upsilon)X\right] \tag{2.17}$$

Here Q is the Ricci operator and defined by $S(X,\Upsilon) = g(QX,\Upsilon)$.
Now next, we solve and obtained

$$\begin{aligned}(\ell_\xi g)(X,\Upsilon) &= (\nabla_\xi g)(X,\Upsilon) + \alpha g(\varphi X,\Upsilon) + 2\beta\eta(X)\eta(\Upsilon) - \alpha g(X,\varphi\Upsilon)\\ &= 2\beta g(X,\Upsilon) - 2\beta\eta(X)\eta(\Upsilon)\end{aligned} \tag{2.18}$$

Putting above in (1.5) the conformal Ricci soliton equation and take the dimension of manifold equal to three, we have

$$2\beta g(X,\Upsilon) - 2\beta\eta(X)\eta(\Upsilon) + 2S(X,\Upsilon) = \left[2\lambda - \left(p + \frac{2}{3}\right)\right] g(X,\Upsilon) \tag{2.19}$$

Solving for S, we get

$$\begin{aligned}S(X,\Upsilon) &= \frac{1}{2}\left[2\lambda - \left(p+\frac{2}{3}\right)\right] g(X,\Upsilon) - \frac{1}{2}[2\beta g(X,\Upsilon) - 2\beta\eta(X)\eta(\Upsilon)]\\ &= (A-\beta)g(X,\Upsilon) + \beta\eta(X)\eta(\Upsilon)\end{aligned} \tag{2.20}$$

where $A = \frac{1}{2}\left[2\lambda - \left(p+\frac{2}{3}\right)\right]$. Hence we concluded the following.

Preposition 2.1: If a 3-dimensional Lorentzian trans-Sasakian manifold M admits conformal Ricci soliton (g,ξ,λ) , the manifold M becomes an $\eta-$ Einstein manifold.

Comparing equation (2.20) with (2.16) we have also

$$QX = AX - \beta X - \beta\eta(X)\xi \tag{2.21}$$

Let us use this for almost conformal Ricci soliton, so we have

$$\begin{aligned}S(X,\Upsilon) &= \lambda g(X,\Upsilon) - \frac{1}{2}\left(p+\frac{2}{3}\right) g(X,\Upsilon) - \beta\eta(X)\eta(\Upsilon)\\ &= (\lambda - B - \beta)g(X,\Upsilon) - \beta\eta(X)\eta(\Upsilon)\end{aligned} \tag{2.22}$$

In the above expression, B defines as $B = \frac{1}{2}\left(p + \frac{2}{d3}\right)$.

This leads to the result as

Preposition 2.2: A 3-dimensional Lorentzian trans-Sasakian manifold M admitting almost conformal Ricci soliton (g,ξ,λ) is an $\eta-$Einstein manifold.

3 Some results for Conformal and almost conformal Ricci soliton on 3dimensional Lorentzian Trans- Sasakian manifold :

A conformal Ricci soliton equation on a Riemannian manifold M is defined by

$$\ell_V g + 2S - \left[2\lambda - \left(p + \frac{2}{3}\right)\right] g = 0 \tag{3.1}$$

Taking $V = \gamma\xi$, i.e. V be pointwise co-linear vector field with ξ and in which γ is a function on 3-dimensional Lorentzian trans-Sasakian manifold. so

$$(\ell_{\gamma\xi})g(X,\Upsilon) + 2S(X,\Upsilon) - \left[2\lambda - \left(p + \frac{2}{3}\right)\right] g(X,\Upsilon) = 0, \tag{3.2}$$

By the use of the Levi-Civita connection property and Lie derivative, the above equation becomes

$$\begin{aligned} \gamma g(\nabla_X \xi, \Upsilon) + (X\gamma)g(\xi,\Upsilon) + (\Upsilon\gamma)g(\xi,X) + \gamma g(\nabla_{\Upsilon\xi,X}) \\ +2S(X,\Upsilon) - \left[2\lambda - \left(p + \frac{2}{3}\right)\right] g(X,\Upsilon) = 0 \end{aligned} \tag{3.3}$$

Using (2.7) and (2.9) in the above equation, we obtain

$$\begin{aligned} -2\beta\gamma g(X,\Upsilon) - 2\beta\gamma\eta(X)\eta(\Upsilon) + (X\gamma)\eta(\Upsilon) + (\Upsilon\gamma)\eta(X) + 2S(X,\Upsilon) \\ - \left[2\lambda - \left(p + \frac{2}{3}\right)\right] g(X,\Upsilon) = 0 \end{aligned} \tag{3.4}$$

Replacing Υ by ξ

$$\begin{aligned} -2\beta\gamma g(X,\xi) - 2\beta\gamma\eta(X)\eta(\xi) + (X\gamma)\eta(X) + (\xi\gamma)\eta(X) + 2S(X,\xi) \\ - \left[2\lambda - \left(p + \frac{2}{3}\right)\right] g(X,\xi) = 0 \end{aligned} \tag{3.5}$$

Now using (2.5), (2.8) and (2.14) in (3.5), we get

$$-X\gamma + (\xi\Upsilon)\eta(X) + 2[2(\alpha^2 + \beta^2)\eta(X)] - \left[2\lambda - \left(p + \frac{2}{3}\right)\right] \eta(X) = 0 \tag{3.6}$$

Now put $X = \xi$ in the given equation, then we have

$$\xi\gamma = \frac{1}{2}\left[2\lambda - \left(p + \frac{2}{3}\right)\right] - 2(\alpha^2 + \beta^2) \tag{3.7}$$

Using the above equation in (3.6) we have

$$\begin{aligned} -X\gamma + \left[\frac{1}{2}\{2\lambda - \left(p + \frac{2}{3}\right)\} - 2(\alpha^2 + \beta^2)\right] \eta(X) + 2[2(\alpha^2 + \beta^2)\eta(X)] \\ - \left[2\lambda - \left(p + \frac{2}{3}\right)\right] \eta(X) = 0 \end{aligned} \tag{3.8}$$

Which gives us

$$X\gamma = -\frac{1}{2}\left[2\lambda - \left(p + \frac{2}{3}\right)\right]\eta(X) + 2(\alpha^2 + \beta^2)\eta(X) \tag{3.9}$$

Applying exterior differentiation in (3.7) and consider λ as constant, we get

$$-\frac{1}{2}\left[2\lambda - \left(p + \frac{2}{3}\right)\right] + 2(\alpha^2 + \beta^2) = 0 \tag{3.10}$$

Using (3.10) in (3.10)

$$X\gamma = 0 \tag{3.11}$$

Implies γ is constant. Hence because of (3.5), we have

$$-2\beta\gamma g(X,\Upsilon) - 2\beta\gamma\eta(X)\eta(\Upsilon) + 2S(X,\Upsilon) - \left[2\lambda - \left(p + \frac{2}{3}\right)\right] g(X,\Upsilon) = 0 \tag{3.12}$$

Solving for Ricci tensor S we get the expression

$$S(X,\Upsilon) = \frac{1}{2}\left[2\lambda - \left(p + \frac{2}{3}\right)\right] g(X,\Upsilon) + \beta\gamma g(X,\Upsilon) + \beta\gamma\eta(X)\eta(\Upsilon) \tag{3.13}$$

Let us take e_i is an orthonormal basis of the tangent space TM and taking $X = \Upsilon = e_i$ and summing over i, we obtain

$$\lambda = \frac{1}{2}p - \frac{4}{3}\beta\gamma + \frac{1}{3} \tag{3.14}$$

So by the above result, we can state the theorem as follows

Theorem 3.1. *A 3-dimensional Lorentzian trans-Sasakian manifold admitting conformal Ricci soliton and if V is pointwise collinear with ξ then V is a constant multiple of ξ also the value of $\lambda = \frac{1}{2}p + \frac{4}{3}\beta\gamma + \frac{1}{3}$, provided α, β are constant.*
Again for almost conformal Ricci soliton, we know that λ behave like a smooth function. Now applying the exterior derivative in (3.9), we get

$$-\frac{1}{2}\left[2\lambda - \left(p + \frac{2}{3}\right)\right] + 2(\alpha^2 + \beta^2) = 0, and \tag{3.15}$$

$$d\lambda = 0, \tag{3.16}$$

So λ is a constant function and from (3.9) and (3.15), we get γ is constant. So we have the following

Theorem 3.2. *If a 3-dimensional Lorentzian trans-Sasakian manifold admits almost conformal Ricci soliton. If V is pointwise collinear with ξ then almost conformal Ricci soliton becomes conformal Ricci soliton i.e., V is a constant multiple of ξ as well as λ becomes a constant function.*

Now from the conformal Ricci solution equation, we have

$$(\ell_\xi g)(X,\Upsilon) = 2\beta[g(X,\Upsilon) - \eta(X)\eta(\Upsilon)] \tag{3.17}$$

Using (2.13) in the above equation and from (1.4), we have

$$2\beta[g(X,\Upsilon) + \eta(X)\eta(\Upsilon)] + 2[\left(\frac{r}{2} - (\alpha^2+\beta^2)\right) g(X,\Upsilon) + \left(\frac{r}{2} - 3(\alpha^2+\beta^2)\right)\eta(X)\eta(\Upsilon)] - \left[2\lambda - \left(p+\frac{2}{3}\right)\right] g(X,\Upsilon) = 0 \tag{3.18}$$

Since $r=-1$ in conformal Ricci soliton, so the above equation becomes

$$\left[2\beta - 1 - 2(\alpha^2+\beta^2) - 2\lambda + \left(p+\frac{2}{3}\right)\right] g(X,\Upsilon) + [2\beta - 1 \tag{3.19}$$

$$-6(\alpha^2+\beta^2)]\eta(X)\eta(\Upsilon) = 0 \tag{3.20}$$

Now taking $X=\Upsilon=\xi$ in (3.20), we get

$$2\lambda + 2(\alpha^2+\beta^2) - \left(p+\frac{2}{3}\right) - 6(\alpha^2+\beta^2) = 0 \tag{3.21}$$

which gives

$$\lambda = \frac{1}{2}\left[4(\alpha^2+\beta^2) + \left(p+\frac{2}{3}\right)\right] \tag{3.22}$$

Since $\alpha^2+\beta^2$ is always positive. So values of λ depending on $\left(p+\frac{2}{3}\right)$ i.e. on p, we concluded that

Theorem 3.3. *A 3-dimensional Lorentzian trans-Sasakian manifold concerning a conformal Ricci soliton (g,ξ,λ) satisfies the following relations:*
(i) if $\alpha^2+\beta^2+\frac{p}{4} > \left(-\frac{1}{6}\right)$ which implies $\lambda > 0$, and the conformal Ricci solution becomes shrinking.
(ii) if $\alpha^2+\beta^2+\frac{p}{4} < \left(-\frac{1}{6}\right)$ which implies $\lambda < 0$, and the conformal Ricci soliton becomes expanding.
(iii) if $\alpha^2+\beta^2+\frac{p}{4} = \left(-\frac{1}{6}\right)$ which implies $\lambda = 0$, and the conformal Ricci soliton becomes steady.

4 Lorentzian trans-Sasakian manifold with conformal Ricci soliton and satisfy the condition $R(\xi, X).C = 0$:

The conformal curvature tensor C on a Lorentzian trans-Sasakian manifoldM, given by

$$C(X,\Upsilon)Z = R(X,\Upsilon)Z - [S(\Upsilon,Z)X - S(X,Z)\Upsilon + g(\Upsilon,Z)QX - g(X,Z)\Upsilon] + \frac{r}{2}[g(\Upsilon,Z)X - g(X,Z)\Upsilon] \tag{4.1}$$

Where r is the scalar curvature.
Now M admitting a conformal Ricci solution (g, V, λ) and We know that scalar curvature $r = -1$ in conformal Ricci soliton, using this and putting $Z = \xi$ in equation (4.2), we have

$$C(X,\Upsilon)\xi = R(X,\Upsilon)\xi - [S(\Upsilon,\xi)X - S(X,\xi)\Upsilon + g(\Upsilon,\xi)QX - g(X,\xi)\Upsilon] - \frac{1}{2}[g(\Upsilon,\xi)X - g(X,\xi)\Upsilon] \tag{4.2}$$

Using equation (2.13)-(2.16), we get

$$(\alpha^2+\beta^2)(X\eta(\Upsilon) - \eta(X)\Upsilon) - [2(\alpha^2+\beta^2)\eta(\Upsilon)X - 2(\alpha^2+\beta^2)\eta(X)\Upsilon + \eta(\Upsilon)(AX - \beta X - \beta\eta(X)\xi) - \eta(X)(A\Upsilon - \beta\Upsilon - \beta(\Upsilon)\xi)] - \frac{1}{2}(\eta(\Upsilon)X - \eta(X)\Upsilon) \tag{4.3}$$

After a brief implication, we obtain

$$C(X,\Upsilon)\xi = \left[A - (\alpha^2+\beta^2) - \beta - \frac{1}{2}\right](X\eta(\Upsilon) - \eta(X)\Upsilon) \tag{4.4}$$

Considering $B = A - (\alpha^2+\beta^2) - \beta - \frac{1}{2}$. then (4.1)d becomes

$$C(X,\Upsilon)\xi = B[X\eta(\Upsilon) - \Upsilon\eta(X)], and \quad g(C(X,\Upsilon)\xi, Z) = B[\eta(\Upsilon)g(X,Z) - \eta(X)g(\Upsilon,Z)] \tag{4.5}$$

Implies that

$$-\eta(C(X,\Upsilon)Z) = B\left[\eta(\Upsilon)g(X,Z) - \eta(X)g(\Upsilon,Z)\right] \tag{4.6}$$

Now using the condition of Weyl conformally semi symmetric in equation (4.6) with conformal Ricci soliton on M then we get

$$R(\xi,X)(C(\Upsilon,Z)W) - C(R(\xi,X)\Upsilon,Z)W - C(\Upsilon,R(\xi,X)Z)W - C(\Upsilon,Z)R(\xi,X)W = 0 \tag{4.7}$$

Using equation (2.17) in (4.8) and after putting $W = \xi$, we have

$$g(X,C(\Upsilon,Z)\xi)\xi - \eta(C(\Upsilon,Z)\xi)X - g(X,\Upsilon)C(\xi,Z)\xi + \eta(\Upsilon)C(X,Z)\xi - g(X,Z)C(\Upsilon,\xi)\xi + \eta(Z)C(\Upsilon,X)\xi - g(X,\xi)C(\Upsilon,Z)\xi + \eta(\xi)C(\Upsilon,Z)X = 0 \tag{4.8}$$

In equation (4.9) let us take the inner product of ξ and using (2.1), we have

$$(4.9) g(X,C(\Upsilon,Z)\xi) - g(X,\Upsilon)\eta(C(\xi,Z)\xi) + \eta(\Upsilon)\eta(C(X,Z)\xi) - g(X,Z)\eta(C(\Upsilon,\xi)\xi) + \eta(Z)\eta(C(\Upsilon,X)\xi) - \eta(X)\eta(C(\Upsilon,Z)\xi) - \eta(C(\Upsilon,Z)X) = 0$$

Using (4.6) in (4.10) and putting $Z=\xi$, we have

$$Bg(X,\Upsilon) + B\eta(X)\eta(\Upsilon) - \eta(C(\Upsilon,\xi)X) = 0 \tag{4.10}$$

Now from (3.5) , we can write

$$C(\Upsilon,\xi)X = R(\Upsilon,\xi)X - [S(\xi,X)\Upsilon - S(\Upsilon,X)\xi + g(\xi,X)Q\Upsilon - g(\Upsilon,X)Q\xi] - \frac{1}{2}[g(\xi,X - g(\Upsilon,X)\xi] \tag{4.11}$$

In the above equation, let us take the inner product of ξ and using equations (2.1), (2.14) and (2.17), we have

$$\eta(C(\Upsilon,\xi)X) = (\alpha^2+\beta^2)[\eta(\Upsilon)\eta(X) - g(X,\Upsilon)] - [2(\alpha^2+\beta^2)\eta(X)\eta(\Upsilon) + S(X,\Upsilon) + A\eta(X)\eta(\Upsilon) + g(X,\Upsilon)A] - \frac{1}{2}[\eta(X)\eta(\Upsilon) + g(X,\Upsilon)] = 0 \tag{4.12}$$

After putting (4.13) in (4.10), the equation reduces to

$$Bg(X,\Upsilon) + B\eta(X)\eta(\Upsilon) + (\alpha^2+\beta^2)\eta(X)\eta(\Upsilon) + (\alpha^2+\beta^2)g(X,\Upsilon) + S(X,\Upsilon)\eta(X)\eta(\Upsilon)A + g(X,\Upsilon)A + \frac{1}{2}\eta(X)\eta(\Upsilon) + \frac{1}{2}g(X,\Upsilon) = 0 \tag{4.13}$$

On simplifying, we get

$$g(X,\Upsilon)\left[B + (\alpha^2+\beta^2) + A + \frac{1}{2}\right] + \eta(X)\eta(\Upsilon)\left[B - (\alpha^2+\beta^2) + A + \frac{1}{2}\right] + S(X,\Upsilon) = 0 \tag{4.14}$$

Can be written in the form

$$S(X,\Upsilon) = \nu g(X,\Upsilon) + \omega\eta(X)\eta(\Upsilon) \tag{4.15}$$

Where $\nu = \left[-B - (\alpha^2+\beta^2 - A - \frac{1}{2})\right]$ and $\omega = \left[-B + (\alpha^2+\beta^2) - A - \frac{1}{2}\right]$
From (4.15) it is apparent that it is an Einstein manifold. So we have concluded that

Theorem 4.1. *If a Lorentzian trans-Sasakian manifold admits conformal Ricci soliton and is Weyl conformally semi symmetric, then the manifold is $\eta-$ Einstein manifold.*

Example :

For construct example, we consider a 3-dimensional manifold $M = \{(x, y, z) \in {}^3, Z \neq 0\}$, where (x, y, z) are the coordinates of ^{3}
Take the linearly independent vector fields on M as

$$e_1 = z\frac{\partial}{\partial x},\ e_2 = z\frac{\partial}{\partial y},\ e_3 = z\frac{\partial}{\partial z}$$

Now introduce the Riemannian metric g according as follow

$$g(e_1, e_3) = 0,\ g(e_1, e_2) = 0,\ g(e_2, e_3) = 0$$

The Riemannian metric g becomes

$$g = \frac{dx^2}{z^2} + \frac{dy^2}{z^2} - \frac{dz^2}{z^2}$$

Let η be the 1-form satisfy the condition $\eta(e_3) = -1$ and φ be the (1,1) tensor field defined by $\varphi(e_1) = -e_2$, $\varphi(e_2) = -e_1$, $\varphi(e_3) = 0$.Then using the identity of φ and g, we have

$$\varphi^2(Z) = Z + \eta(Z)e_3$$
$$g(\varphi Z, \varphi W) = g(Z, W) + \eta(Z)\eta(W)$$

For any $Z, W \in \chi(M^3)$. Thus $e_3 = \xi$, the structure (φ, ξ, η, g) defines a Lorentzian structure on M. solving all these, we get

$$[e_1, e_3] = -e_1,\ [e_1, e_2] = 0,\ [e_2, e_3] = -e_2$$

Again using Koszuls formula, solving all of these, we have

$$\nabla_{e_1} e_3 = -e_1,\ \nabla_{e_1} e_2 = 0,\ \nabla_{e_1} e_1 = -e_3$$
$$\nabla_{e_2} e_3 = -e_2,\ \nabla_{e_2} e_2 = -e_3,\ \nabla_{e_2} e_1 = 0$$
$$\nabla_{e_3} e_3 = 0,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_1 = 0$$

From the above, we have found that $\alpha = 0, \beta = 1$ Which is a Lorentzian trans-Sasakian manifold.

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