A note on some growth properties of composite entire function on the basis of their generalized type (α, β) and generalized weak type (α, β)

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Abstract

The main aim of this paper is to prove some results related to the growth rates of composite entire functions on the basis of their generalized type (α, β) and generalized weak type (α, β), where α, β are continuous non-negative functions on $(-\infty, +\infty)$.

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Entire function, growth, generalized order (α, β), generalized type (α, β), generalized weak type (α, β)

1. Introduction,Definitions and Notations.

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$. The maximum modulus function $M_f(r)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined as $M_f = \max_{|z|=r} |f(z)|$. Moreover, if f is non-constant entire then $M_f(r)$ is also strictly increasing and continuous functions of r. Therefore its inverse $M_f^{-1} : (M_f(0), \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to+\infty} M_f^{-1}(s) = \infty$. We use the standard notations and definitions of the theory of entire functions which are available in [8] and [9], and therefore we do not explain those in details.

For $x \in [0, \infty)$ and $k \in N$ where $\mathbb{N}$ be the set of all positive integers, we define iterations of the exponential and logarithmic functions as $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ and $\log^{[k]} x = \log(\log^{[k-1]} x)$, with convention that $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$, and $\exp^{[-1]} x = \log x$. Further we assume that p and q always denote positive integers. Now considering this, let us recall that Juneja et al. [5] defined the (p, q)-th order and (p, q)-th lower order of an entire function, respectively, as follows:

Definition 1.1.[5] Let $p \geq q$. The (p, q)-th order $\varrho^{(p,q)}(f)$ and (p, q)-th lower order $\lambda^{(p,q)}(f)$ of an entire function f are defined as:

$$\varrho^{(p,q)}(f) = \limsup_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \liminf_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}.$$

The function f is said to be of regular (p,q) growth when (p,q)-th order and (p,q)-th lower order of f are the same. Functions which are not of regular (p,q) growth are said to be of irregular (p,q) growth.

Extending the notion (p,q)-th order, recently Shen et al. [4] introduced the new concept of $[p,q]$-φ order of entire function where $p \geq q$. Later on, combining the definition of (p,q)-order and $[p,q]$-φ order, Biswas (see, e.g., [1]) redefined the (p,q)-order of an entire function without restriction $p \geq q$.

However the above definition is very useful for measuring the growth of entire functions. If $p = l$ and $q = 1$ then we write $\varrho^{(l,1)}(f) = \varrho^{(l)}(f)$ and $\lambda^{(l,1)}(f) = \lambda^{(l)}(f)$ where $\varrho^{(l)}(f)$ and $\lambda^{(l)}(f)$ are respectively known as generalized order and generalized lower order of function f (see, e.g., [7]). Also for $p = 2$ and $q = 1$, we respectively denote $\varrho^{(2,1)}(f)$ and $\lambda^{(2,1)}(f)$ by $\varrho(f)$ and $\lambda(f)$ which are classical growth indicators such as order and lower order of entire function f.

Now let L be a class of continuous non-negative on $(-\infty,+\infty)$ function α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$ and $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$. We say that $\alpha \in L^0$, if $\alpha \in L$ and $\alpha(cx) = (1+o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0,+\infty)$, i.e., α is slowly increasing function. Clearly $L^0 \subset L$.

Further we assume that throughout the present paper α, α_1, α_2, β, β_1 and β_2 always denote the functions belonging to L^0. The value

$$\varrho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log M_f(r))}{\beta(\log r)} (\alpha \in L, \beta \in L)$$

is called [6] generalized order (α,β) of an entire function f .

In order to keep accordance with Definition 1.1, here we rewrite the definition of the generalized order (α,β) and generalized lower order (α,β) of an entire function after giving a minor modification to the original definition of generalized order (α,β) of an entire function (e.g. see, [6]).

Definition 1.2. The generalized order (α,β) denoted by $\varrho_{(\alpha,\beta)}[f]$ and generalized lower order (α,β) denoted by $\lambda_{(\alpha,\beta)}[f]$ of an entire function f are defined as:

$$\varrho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)} \text{ and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)}$$

Definition 1.1 is a special case of Definition 1.2 for $\alpha(r) = \log^{[p]} r$ and $\beta(r) = \log^{[q]} r$.

The function f is said to be of regular generalized (α,β) growth when generalized order (α,β) and generalized lower order (α,β) of f are the same. Functions which are not of regular generalized (α,β) growth are said to be of irregular generalized (α,β) growth.

Now in order to refine the growth scale namely the generalized order (α,β) of an entire function, we introduce the definitions of another growth indicators, called generalized type (α,β) and generalized lower type (α,β) respectively of an entire function which are as follows:

Definition 1.3. The generalized type (α,β) denoted by $\sigma_{(\alpha,\beta)}[f]$ and generalized lower type (α,β) denoted by $\overline{\sigma}_{(\alpha,\beta)}[f]$ of an entire function f having finite positive generalized order (α,β) $(0 < \varrho_{(\alpha,\beta)}[f] < \infty)$ are defined as :

$$\sigma_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r))^{\varrho_{(\alpha,\beta)}[f]}} \text{ and } \overline{\sigma}_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r))^{\varrho_{(\alpha,\beta)}[f]}}$$

It is obvious that $0 \leq \overline{\sigma}_{(\alpha,\beta)}[f] \leq \sigma_{(\alpha,\beta)}[f] \leq \infty$.

Analogously, to determine the relative growth of two entire functions having same non zero finite generalized lower order (α,β), one can introduced the definition of generalized weak type

(α, β) and generalized upper weak type (α, β) of an entire function f of finite positive generalized lower order (α, β), $\lambda_{(\alpha,\beta)}[f]$ in the following way:

Definition 1.4. The generalized upper weak type (α, β) denoted by $\overline{\tau}_{(\alpha,\beta)}[f]$ and generalized weak type (α, β) denoted by $\tau_{(\alpha,\beta)}[f]$ of an entire function f having finite positive generalized lower order (α, β) $(0 < \lambda_{(\alpha,\beta)}[f] < \infty)$ are defined as :

$$\overline{\tau}_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]}} \text{ and } \tau_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]}}$$

It is obvious that $0 \le \tau_{(\alpha,\beta)}[f] \le \overline{\tau}_{(\alpha,\beta)}[f] \le \infty$.

In this paper we wish to prove some results related to the growth rates of composite entire functions on the basis of their generalized order (α, β), generalized type (α, β) and generalized weak type (α, β). In fact some works in this direction have already been explored in [2, 3].

2. Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1.[3] Let f and g are any two entire functions with $g(0) = 0$. Also let B satisfy $0 < B < 1$ and $c(B) = \frac{(1-B)^2}{4B}$. Then for all sufficiently large values of r,

$$M_f(c(B)M_g(Br)) \le M_{f(g)}(r) \le M_f(M_g(r)).$$

In addition if $B = \frac{1}{2}$, then for all sufficiently large values of r,

$$M_{f(g)}(r) \ge M_f(\frac{1}{8}M_g(\frac{r}{2})).$$

Lemma 2.2.[2] Suppose f is an entire function and $a > 1$, $0 < b < a$. Then for all sufficiently large r,

$$M_f(ar) \ge bM_f(r).$$

3. Main Results

In this section we present the main results of the paper.

Theorem 3.1. Let f and g be any two entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f] \le \varrho_{(\alpha_1,\beta_1)}[f] < \infty$ and $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$ where $\beta_1(r) \le \exp(\alpha_2(r))$. Then

$$\limsup_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(r))}{\alpha_1(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]}))} \le \frac{\sigma_{(\alpha_2,\beta_2)}[g] \cdot \varrho_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_1,\beta_1)}[f]}.$$

Proof. In view of Lemma 2.1, it follows for all sufficiently large values of r that

$$\alpha_1(M_{f(g)}(r)) \le \alpha_1(M_f(M_g(r)))$$

$$i.e., \ \alpha_1(M_{f(g)}(r)) \le (\varrho_{(\alpha_1,\beta_1)}[f] + \varepsilon)\beta_1(M_g(r)).$$

Since $\beta_1(r) \le \exp(\alpha_2(r))$, we get from above for all sufficiently large values of r that

$$\alpha_1(M_{f(g)}(r)) \le (\varrho_{(\alpha_1,\beta_1)}[f] + \varepsilon)\exp(\alpha_2(M_g(r)))$$

$$\text{(3.1)} \qquad i.e., \ \alpha_1(M_{f(g)}(r)) \le (\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\sigma_{(\alpha_2,\beta_2)}[g]+\varepsilon)(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]}.$$

Now from the definition of $\lambda_{(\alpha_1,\beta_1)}[f]$, we obtain for all sufficiently large values of r that

$$\text{(3.2)} \qquad \alpha_1(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]})) \ge (\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]}.$$

Therefore from (3.1) and (3.2), it follows for all sufficiently large values of r that

$$\frac{\alpha_1(M_{f(g)}(r))}{\alpha_1(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]}))} \le \frac{(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\sigma_{(\alpha_2,\beta_2)}[g]+\varepsilon)(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]}}{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]}}$$

$$i.e., \limsup_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(r))}{\alpha_1(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]}))} \le \frac{\sigma_{(\alpha_2,\beta_2)}[g]\cdot\varrho_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_1,\beta_1)}[f]}.$$

Thus the theorem is established.

Remark 3.1. In Theorem 3.1, if we will replace "$\sigma_{(\alpha_2,\beta_2)}[g]$" by "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$", then Theorem 3.1 remains valid with " limit inferior" replaced by " limit superior".

Now we state the following theorem without its proof as it can easily be carried out in the line of Theorem 3.1.

Theorem 3.2. Let f and g be any two entire functions such that $\varrho_{(\alpha_1,\beta_1)}[f] < \infty$, $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ and $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$ where $\beta_1(r) \le \exp(\alpha_2(r))$. Then

$$\limsup_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(r))}{\alpha_2(M_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]}))} \le \frac{\sigma_{(\alpha_2,\beta_2)}[g]\cdot\varrho_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_2,\beta_2)}[g]}.$$

Remark 3.2. In Theorem 3.2, if we will replace "$\sigma_{(\alpha_2,\beta_2)}[g]$" by "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$", then Theorem 3.2 remains valid with " limit inferior" replaced by " limit superior".

Remark 3.3. We remark that in Theorem 3.2, if we will replace the condition " $\varrho_{(\alpha_1,\beta_1)}[f] < \infty$" by " $\lambda_{(\alpha_1,\beta_1)}[f] < \infty$" , then

$$\text{(3.3)} \qquad \liminf_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(r))}{\alpha_2(M_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]}))} \le \frac{\sigma_{(\alpha_2,\beta_2)}[g]\cdot\lambda_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_2,\beta_2)}[g]}.$$

Remark 3.4. In Remark 3.3, if we will replace the conditions " $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ and $\lambda_{(\alpha_1,\beta_1)}[f] < \infty$" by "$\varrho_{(\alpha_2,\beta_2)}[g] > 0$ and $\varrho_{(\alpha_1,\beta_1)}[f] < \infty$" respectively, then is need to go the same replacement in right part of (3.3).

Using the concept of the growth indicator $\overline{\tau}_{(\alpha_2,\beta_2)}[g]$ of an entire function g, we may state the subsequent two theorems without their proofs since those can be carried out in the line of Theorem 3.1 and Theorem 3.2 respectively.

Theorem 3.3. Let f and g be any two entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f] \le \varrho_{(\alpha_1,\beta_1)}[f] < \infty$ and $\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$ where $\beta_1(r) \le \exp(\alpha_2(r))$. Then

$$\limsup_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(r))}{\alpha_1(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]}))} \le \frac{\overline{\tau}_{(\alpha_2,\beta_2)}[g]\cdot\varrho_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_1,\beta_1)}[f]}.$$

Remark 3.5. We remark that in Theorem 3.3, if we will replace the condition " $0 < \lambda_{(\alpha_1,\beta_1)}[f] \le \varrho_{(\alpha_1,\beta_1)}[f] < \infty$ and $\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$" by " $0 < \lambda_{(\alpha_1,\beta_1)}[f] < \infty$ or $0 < \varrho_{(\alpha_1,\beta_1)}[f] < \infty$ and $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$", then

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(r))}{\alpha_1(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]}))} \le \sigma_{(\alpha_2,\beta_2)}[g].$$

Theorem 3.4. Let f and g be any two entire functions such that $\varrho_{(\alpha_1,\beta_1)}[f] < \infty$, $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ and $\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$ where $\beta_1(r) \le \exp(\alpha_2(r))$. Then

$$\limsup_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(r))}{\alpha_2(M_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]}))} \le \frac{\overline{\tau}_{(\alpha_2,\beta_2)}[g] \cdot \varrho_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_2,\beta_2)}[g]}.$$

Further using the notion of generalized weak type (α, β) we may also state the following two theorems without proof because it can be carried out in the line of Theorem 3.3 and Theorem 3.4 respectively.

Theorem 3.5. Let f and g be any two entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f] \le \varrho_{(\alpha_1,\beta_1)}[f] < \infty$ and $\tau_{(\alpha_2,\beta_2)}[g] < \infty$ where $\beta_1(r) \le \exp(\alpha_2(r))$. Then

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(r))}{\alpha_1(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]}))} \le \frac{\tau_{(\alpha_2,\beta_2)}[g] \cdot \varrho_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_1,\beta_1)}[f]}.$$

Remark 3.6. We remark that in Theorem 3.5, if we will replace the condition " $0 < \lambda_{(\alpha_1,\beta_1)}[f] \le \varrho_{(\alpha_1,\beta_1)}[f] < \infty$ and $\tau_{(\alpha_2,\beta_2)}[g] < \infty$" by " $0 < \lambda_{(\alpha_1,\beta_1)}[f] < \infty$ or $0 < \varrho_{(\alpha_1,\beta_1)}[f] < \infty$ and $\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$", then

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(r))}{\alpha_1(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]}))} \le \overline{\tau}_{(\alpha_2,\beta_2)}[g].$$

Theorem 3.6. Let f and g be any two entire functions such that $\varrho_{(\alpha_1,\beta_1)}[f] < \infty$, $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ and $\tau_{(\alpha_2,\beta_2)}[g] < \infty$ where $\beta_1(r) \le \exp(\alpha_2(r))$. Then

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(r))}{\alpha_2(M_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]}))} \le \frac{\tau_{(\alpha_2,\beta_2)}[g] \cdot \varrho_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_2,\beta_2)}[g]}.$$

Remark 3.7. We remark that in Theorem 3.6, if we will replace the condition " $\varrho_{(\alpha_1,\beta_1)}[f] < \infty$ and $\tau_{(\alpha_2,\beta_2)}[g] < \infty$" by " $\lambda_{(\alpha_1,\beta_1)}[f] < \infty$ and $\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$" , then

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(r))}{\alpha_2(M_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]}))} \le \frac{\overline{\tau}_{(\alpha_2,\beta_2)}[g] \cdot \lambda_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_2,\beta_2)}[g]}. \tag{3.4}$$

Remark 3.8. In Remark 3.7, if we will replace the conditions " $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ and $\lambda_{(\alpha_1,\beta_1)}[f] < \infty$" by "$\varrho_{(\alpha_2,\beta_2)}[g] > 0$ and $\varrho_{(\alpha_1,\beta_1)}[f] < \infty$" respectively, then is need to go the same replacement in right part of (3.4).

Theorem 3.7. Let f and g be any two entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f] \le \varrho_{(\alpha_1,\beta_1)}[f] < \infty$ and $\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$ where $\beta_1(r) \ge \exp(\alpha_2(r))$. Then for any $\eta > 16$

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(\eta r))}{\alpha_1(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]}))} \ge \frac{\overline{\sigma}_{(\alpha_2,\beta_2)}[g] \cdot \lambda_{(\alpha_1,\beta_1)}[f]}{\varrho_{(\alpha_1,\beta_1)}[f]}.$$

Proof. In view of Lemma 2.1 and Lemma 2.2, we get for any $\eta > 16$ and all sufficiently large values of r that

$$\alpha_1(M_{f(g)}(\eta r)) \geq \alpha_1(M_f(M_g(r)))$$
$$i.e.,\ \alpha_1(M_{f(g)}(\eta r)) \geq (\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)\beta_1(M_g(r)).$$

Since $\beta_1(r) \geq \exp(\alpha_2(r))$, we get from above for all sufficiently large values of r that

$$\alpha_1(M_{f(g)}(\eta r)) \geq (\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)\exp(\alpha_2(M_g(r)))$$
$$i.e.,\ \alpha_1(M_{f(g)}(\eta r)) \geq$$

$$(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)(\overline{\sigma}_{(\alpha_2,\beta_2)}[g]-\varepsilon)(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]}. \tag{3.5}$$

Now from the definition of $\lambda_{(\alpha_1,\beta_1)}[f]$, we obtain for all sufficiently large values of r that

$$\alpha_1(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]})) \leq (\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]}. \tag{3.6}$$

Therefore from (3.5) and (3.6), it follows for all sufficiently large values of r that

$$\frac{\alpha_1(M_{f(g)}(\eta r))}{\alpha_1(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]}))} \geq$$
$$\frac{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)(\overline{\sigma}_{(\alpha_2,\beta_2)}[g]-\varepsilon)(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]}}{(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]}}$$
$$i.e., \liminf_{r\to+\infty}\frac{\alpha_1(M_{f(g)}(\eta r))}{\alpha_1(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]}))} \geq \frac{\overline{\sigma}_{(\alpha_2,\beta_2)}[g]\cdot\lambda_{(\alpha_1,\beta_1)}[f]}{\varrho_{(\alpha_1,\beta_1)}[f]}.$$

Thus the theorem is established.

Remark 3.9. In Theorem 3.7, if we will replace "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$" by "$\sigma_{(\alpha_2,\beta_2)}[g]$", then Theorem 3.7 remains valid with " limit superior" replaced by " limit inferior".

Remark 3.10. We remark that in Theorem 3.7, if we will replace the condition " $0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \varrho_{(\alpha_1,\beta_1)}[f] < \infty$" by " $0 < \lambda_{(\alpha_1,\beta_1)}[f] < \infty$ or $0 < \varrho_{(\alpha_1,\beta_1)}[f] < \infty$" , then

$$\limsup_{r\to+\infty}\frac{\alpha_1(M_{f(g)}(\eta r))}{\alpha_1(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]}))} \geq \overline{\sigma}_{(\alpha_2,\beta_2)}[g].$$

Now we state the following theorem without its proof as it can easily be carried out in the line of Theorem 3.7.

Theorem 3.8. Let f and g be any two entire functions such that $\lambda_{(\alpha_1,\beta_1)}[f] > 0$, $\varrho_{(\alpha_2,\beta_2)}[g] < \infty$ and $\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$ where $\beta_1(r) \geq \exp(\alpha_2(r))$. Then for any $\eta > 16$

$$\liminf_{r\to+\infty}\frac{\alpha_1(M_{f(g)}(\eta r))}{\alpha_2(M_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]}))} \geq \frac{\overline{\sigma}_{(\alpha_2,\beta_2)}[g]\cdot\lambda_{(\alpha_1,\beta_1)}[f]}{\varrho_{(\alpha_2,\beta_2)}[g]}.$$

Remark 3.11. In Theorem 3.8, if we will replace "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$" by "$\sigma_{(\alpha_2,\beta_2)}[g]$", then Theorem 3.8 remains valid with " limit inferior" replaced by " limit superior".

Remark 3.12. We remark that in Theorem 3.8, if we will replace the condition " $\varrho_{(\alpha_2,\beta_2)}[g] < \infty$" by " $\lambda_{(\alpha_2,\beta_2)}[g] < \infty$" , then

$$\limsup_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(\eta r))}{\alpha_2(M_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\varrho_{(\alpha_2,\beta_2)}[g]}))} \geq \frac{\overline{\sigma}_{(\alpha_2,\beta_2)}[g] \cdot \lambda_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_2,\beta_2)}[g]}. \tag{3.7}$$

Remark 3.13. In Remark 3.12, if we will replace the conditions " $0 < \lambda_{(\alpha_1,\beta_1)}[f]$ and $\lambda_{(\alpha_2,\beta_2)}[g] < \infty$" by "$0 < \varrho_{(\alpha_1,\beta_1)}[f]$ and $\varrho_{(\alpha_2,\beta_2)}[g] < \infty$" respectively, then is need to go the same replacement in right part of (3.7).

Using the concept of generalized weak type (α, β) of an entire function, we may state the subsequent two theorems without their proofs since those can be carried out in the line of Theorem 3.7 and Theorem 3.8 respectively.

Theorem 3.9. Let f and g be any two entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \varrho_{(\alpha_1,\beta_1)}[f] < \infty$ and $\tau_{(\alpha_2,\beta_2)}[g] < \infty$ where $\beta_1(r) \geq \exp(\alpha_2(r))$. Then for any $\eta > 16$

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(\eta r))}{\alpha_1(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]}))} \geq \frac{\tau_{(\alpha_2,\beta_2)}[g] \cdot \lambda_{(\alpha_1,\beta_1)}[f]}{\varrho_{(\alpha_1,\beta_1)}[f]}.$$

Remark 3.14. In Theorem 3.9, if we will replace "$\tau_{(\alpha_2,\beta_2)}[g]$" by "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$", then Theorem 3.9 remains valid with " limit superior" replaced by " limit inferior".

Remark 3.15. We remark that in Theorem 3.9, if we will replace the condition " $0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \varrho_{(\alpha_1,\beta_1)}[f] < \infty$" by " $0 < \lambda_{(\alpha_1,\beta_1)}[f] < \infty$ or $0 < \varrho_{(\alpha_1,\beta_1)}[f] < \infty$" , then

$$\limsup_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(\eta r))}{\alpha_1(M_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]}))} \geq \tau_{(\alpha_2,\beta_2)}[g].$$

Theorem 3.10. Let f and g be any two entire functions such that $\lambda_{(\alpha_1,\beta_1)}[f] > 0$, $\varrho_{(\alpha_2,\beta_2)}[g] < \infty$ and $\tau_{(\alpha_2,\beta_2)}[g] < \infty$ where $\beta_1(r) \geq \exp(\alpha_2(r))$. Then for any $\eta > 16$

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(\eta r))}{\alpha_2(M_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]}))} \geq \frac{\tau_{(\alpha_2,\beta_2)}[g] \cdot \lambda_{(\alpha_1,\beta_1)}[f]}{\varrho_{(\alpha_2,\beta_2)}[g]}.$$

Remark 3.16. In Theorem 3.10, if we will replace "$\tau_{(\alpha_2,\beta_2)}[g]$" by "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$", then Theorem 3.10 remains valid with " limit superior" replaced by " limit inferior".

Remark 3.17. We remark that in Theorem 3.10, if we will replace the condition " $0 < \lambda_{(\alpha_1,\beta_1)}[f]$" by " $0 < \varrho_{(\alpha_1,\beta_1)}[f]$", then

$$\limsup_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(\eta r))}{\alpha_2(M_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]}))} \geq \frac{\tau_{(\alpha_2,\beta_2)}[g] \cdot \varrho_{(\alpha_1,\beta_1)}[f]}{\varrho_{(\alpha_2,\beta_2)}[g]}. \tag{3.8}$$

Remark 3.18. In Remark 3.17, if we will replace the conditions " $0 < \varrho_{(\alpha_1,\beta_1)}[f]$ and $0 < \varrho_{(\alpha_2,\beta_2)}[g]$" by "$0 < \lambda_{(\alpha_1,\beta_1)}[f]$ and $0 < \lambda_{(\alpha_2,\beta_2)}[g]$", then is need to go the same replacement in right part of (3.8).

Theorem 3.11. Let f and g be any two entire functions such that $0 < \varrho_{(\alpha_1,\beta_1)}[f] < \infty$, $\varrho_{(\alpha_1,\beta_1)}[f] = \varrho_{(\alpha_2,\beta_2)}[g]$, $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$ and $0 < \sigma_{(\alpha_1,\beta_1)}[f] < \infty$ where $\beta_1(r) \leq \exp(\alpha_2(r))$. Then

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(r))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))} \leq \frac{\varrho_{(\alpha_1,\beta_1)}[f] \cdot \sigma_{(\alpha_2,\beta_2)}[g]}{\sigma_{(\alpha_1,\beta_1)}[f]}. \tag{3.9}$$

Proof. In view of the condition $\varrho_{(\alpha_1,\beta_1)}[f]=\varrho_{(\alpha_2,\beta_2)}[g]$,we obtain from (3.1) for all sufficiently large values of r that

$$\alpha_1(M_{f(g)}(r))\leq(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\sigma_{(\alpha_2,\beta_2)}[g]+\varepsilon)(\exp(\beta_2(r)))^{\varrho_{(\alpha_1,\beta_1)}[f]}. \tag{3.10}$$

Now using the definition of $\sigma_{(\alpha_2,\beta_2)}[g]$, we get from above for a sequence of values of r tending to infinity that

$$\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))\geq(\sigma_{(\alpha_1,\beta_1)}[f]-\varepsilon)(\exp(\beta_2(r)))^{\varrho_{(\alpha_1,\beta_1)}[f]}. \tag{3.11}$$

Now from (3.10) and (3.11), it follows for a sequence of values of r tending to infinity that

$$\frac{\alpha_1(M_{f(g)}(r))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))}\leq\frac{(\varrho_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\sigma_{(\alpha_2,\beta_2)}[g]+\varepsilon)(\exp(\beta_2(r)))^{\varrho_{(\alpha_1,\beta_1)}[f]}}{(\sigma_{(\alpha_1,\beta_1)}[f]-\varepsilon)(\exp(\beta_2(r)))^{\varrho_{(\alpha_1,\beta_1)}[f]}}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to+\infty}\frac{\alpha_1(M_{f(g)}(r))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))}\leq\frac{\varrho_{(\alpha_1,\beta_1)}[f]\cdot\sigma_{(\alpha_2,\beta_2)}[g]}{\sigma_{(\alpha_1,\beta_1)}[f]}.$$

Remark 3.19. In Theorem 3.11, if we will replace the conditions " $\sigma_{(\alpha_2,\beta_2)}[g]<\infty$" and "$0<\sigma_{(\alpha_1,\beta_1)}[f]<\infty$" by "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]<\infty$" and "$0<\overline{\sigma}_{(\alpha_1,\beta_1)}[f]<\infty$", then is need to go the same replacement in right part of (3.9). Also if we replace the conditions $0<\varrho_{(\alpha_1,\beta_1)}[f]<\infty$ and $0<\sigma_{(\alpha_1,\beta_1)}[f]<\infty$ of Theorem 3.11 by $0<\lambda_{(\alpha_1,\beta_1)}[f]\leq\varrho_{(\alpha_1,\beta_1)}[f]<\infty$ and $0<\overline{\sigma}_{(\alpha_1,\beta_1)}[f]<\infty$ respectively, then

$$\liminf_{r\to+\infty}\frac{\alpha_1(M_{f(g)}(r))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))}\leq\frac{\lambda_{(\alpha_1,\beta_1)}[f]\cdot\sigma_{(\alpha_2,\beta_2)}[g]}{\overline{\sigma}_{(\alpha_1,\beta_1)}[f]}.$$

Further if we replace the condition $0<\overline{\sigma}_{(\alpha_1,\beta_1)}[f]<\infty$ of Theorem 3.11 by $0<\sigma_{(\alpha_1,\beta_1)}[f]<\infty$, then Theorem 3.11 remains valid with " limit superior" replaced by " limit inferior".

Now we state the following three theorems without their proofs as those can easily be carried out in the line of Theorem 3.11.

Theorem 3.12. Let f and g be any two entire functions such that $0<\varrho_{(\alpha_1,\beta_1)}[f]<\infty$, $\lambda_{(\alpha_1,\beta_1)}[f]=\lambda_{(\alpha_2,\beta_2)}[g]$, $\overline{\tau}_{(\alpha_2,\beta_2)}[g]<\infty$ and $0<\overline{\tau}_{(\alpha_1,\beta_1)}[f]<\infty$ where $\beta_1(r)\leq\exp(\alpha_2(r))$. Then

$$\liminf_{r\to+\infty}\frac{\alpha_1(M_{f(g)}(r))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))}\leq\frac{\varrho_{(\alpha_1,\beta_1)}[f]\cdot\overline{\tau}_{(\alpha_2,\beta_2)}[g]}{\overline{\tau}_{(\alpha_1,\beta_1)}[f]}. \tag{3.12}$$

Remark 3.20. In Theorem 3.12, if we will replace the conditions " $\overline{\tau}_{(\alpha_2,\beta_2)}[g]<\infty$" and "$0<\overline{\tau}_{(\alpha_1,\beta_1)}[f]<\infty$" by "$\tau_{(\alpha_2,\beta_2)}[g]<\infty$" and "$0<\tau_{(\alpha_1,\beta_1)}[f]<\infty$", then is need to go the same replacement in right part of (3.12). Also if we replace the conditions $0<\lambda_{(\alpha_1,\beta_1)}[f]\leq\varrho_{(\alpha_1,\beta_1)}[f]<\infty$ and $0<\overline{\tau}_{(\alpha_1,\beta_1)}[f]<\infty$ of Theorem 3.12 by $0<\lambda_{(\alpha_1,\beta_1)}[f]<\infty$ and $0<\tau_{(\alpha_1,\beta_1)}[f]<\infty$ respectively, then

$$\liminf_{r\to+\infty}\frac{\alpha_1(M_{f(g)}(r))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))}\leq\frac{\lambda_{(\alpha_1,\beta_1)}[f]\cdot\overline{\tau}_{(\alpha_2,\beta_2)}[g]}{\tau_{(\alpha_1,\beta_1)}[f]}.$$

Further if we replace the condition $0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f] < \infty$ of Theorem 3.12 by $0 < \tau_{(\alpha_1,\beta_1)}[f] < \infty$, then Theorem 3.12 remains valid with " limit superior" replaced by " limit inferior".

Theorem 3.13. Let f and g be any two entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \varrho_{(\alpha_1,\beta_1)}[f] < \infty$, $\lambda_{(\alpha_1,\beta_1)}[f] = \varrho_{(\alpha_2,\beta_2)}[g]$, $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$ and $0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f] < \infty$ where $\beta_1(r) \leq \exp(\alpha_2(r))$. Then

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(r))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))} \leq \frac{\varrho_{(\alpha_1,\beta_1)}[f] \cdot \sigma_{(\alpha_2,\beta_2)}[g]}{\overline{\tau}_{(\alpha_1,\beta_1)}[f]}. \tag{3.13}$$

Remark 3.21. In Theorem 3.13, if we will replace the conditions " $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f] < \infty$" by "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \tau_{(\alpha_1,\beta_1)}[f] < \infty$", then is need to go the same replacement in right part of (3.13). Also if we replace the conditions $0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \varrho_{(\alpha_1,\beta_1)}[f] < \infty$ and $0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f] < \infty$ of Theorem 3.13 by $0 < \lambda_{(\alpha_1,\beta_1)}[f] < \infty$ and $0 < \tau_{(\alpha_1,\beta_1)}[f] < \infty$ respectively, then

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(r))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))} \leq \frac{\lambda_{(\alpha_1,\beta_1)}[f] \cdot \sigma_{(\alpha_2,\beta_2)}[g]}{\tau_{(\alpha_1,\beta_1)}[f]}.$$

Further if we replace the condition $0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f] < \infty$ of Theorem 3.13 by $0 < \tau_{(\alpha_1,\beta_1)}[f] < \infty$, then Theorem 3.13 remains valid with " limit superior" replaced by " limit inferior".

Theorem 3.14. Let f and g be any two entire functions such that $0 < \varrho_{(\alpha_1,\beta_1)}[f] < \infty$, $\varrho_{(\alpha_1,\beta_1)}[f] = \lambda_{(\alpha_2,\beta_2)}[g]$, $\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$ and $0 < \sigma_{(\alpha_1,\beta_1)}[f] < \infty$ where $\beta_1(r) \leq \exp(\alpha_2(r))$. Then

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(r))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))} \leq \frac{\varrho_{(\alpha_1,\beta_1)}[f] \cdot \overline{\tau}_{(\alpha_2,\beta_2)}[g]}{\sigma_{(\alpha_1,\beta_1)}[f]}. \tag{3.14}$$

Remark 3.22. In Theorem 3.14, if we will replace the conditions " $\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \sigma_{(\alpha_1,\beta_1)}[f] < \infty$" by "$\tau_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] < \infty$", then is need to go the same replacement in right part of (3.14). Also if we replace the conditions $0 < \varrho_{(\alpha_1,\beta_1)}[f] < \infty$ and $0 < \sigma_{(\alpha_1,\beta_1)}[f] < \infty$ of Theorem 3.14 by $0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \varrho_{(\alpha_1,\beta_1)}[f] < \infty$ and $0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] < \infty$ respectively, then

$$\liminf_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(r))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))} \leq \frac{\lambda_{(\alpha_1,\beta_1)}[f] \cdot \overline{\tau}_{(\alpha_2,\beta_2)}[g]}{\overline{\sigma}_{(\alpha_1,\beta_1)}[f]}.$$

Further if we replace the condition $0 < \sigma_{(\alpha_1,\beta_1)}[f] < \infty$ of Theorem 3.14 by $0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] < \infty$, then Theorem 3.14 remains valid with " limit superior" replaced by " limit inferior".

Theorem 3.15. Let f and g be any two entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \varrho_{(\alpha_1,\beta_1)}[f] < \infty$, $\varrho_{(\alpha_1,\beta_1)}[f] = \varrho_{(\alpha_2,\beta_2)}[g]$, $\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$ and $0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] < \infty$ where $\beta_1(r) \geq \exp(\alpha_2(r))$. Then for any $\eta > 16$

$$\limsup_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(\eta r))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))} \geq \frac{\lambda_{(\alpha_1,\beta_1)}[f] \cdot \overline{\sigma}_{(\alpha_2,\beta_2)}[g]}{\overline{\sigma}_{(\alpha_1,\beta_1)}[f]}. \tag{3.15}$$

Proof. In view of the condition $\varrho_{(\alpha_1,\beta_1)}[f] = \varrho_{(\alpha_2,\beta_2)}[g]$, we obtain from (3.5) for all sufficiently large values of r that

$$\alpha_1(M_{f(g)}(\eta r)) \geq$$

$$(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)(\overline{\sigma}_{(\alpha_2,\beta_2)}[g]-\varepsilon)(\exp(\beta_2(r)))^{\varrho_{(\alpha_1,\beta_1)}[f]}. \tag{3.16}$$

Further in view of definition of $\overline{\sigma}_{(\alpha_1,\beta_1)}[f]$, we get for a sequence of values of r tending to infinity that

$$\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r))))) \geq (\overline{\sigma}_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\exp(\beta_2(r)))^{\varrho_{(\alpha_1,\beta_1)}[f]}. \tag{3.17}$$

Now from (3.16) and (3.17),it follows for a sequence of values of r tending to infinity that

$$\frac{\alpha_1(M_{f(g)}(\eta r))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))} \geq \frac{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)(\overline{\sigma}_{(\alpha_2,\beta_2)}[g]-\varepsilon)(\exp(\beta_2(r)))^{\varrho_{(\alpha_1,\beta_1)}[f]}}{(\overline{\sigma}_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\exp(\beta_2(r)))^{\varrho_{(\alpha_1,\beta_1)}[f]}}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\limsup_{r\to+\infty}\frac{\alpha_1(M_{f(g)}(\eta r))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))} \geq \frac{\lambda_{(\alpha_1,\beta_1)}[f]\cdot\overline{\sigma}_{(\alpha_2,\beta_2)}[g]}{\overline{\sigma}_{(\alpha_1,\beta_1)}[f]}.$$

Remark 3.23. In Theorem 3.15, if we will replace the conditions " $\overline{\sigma}_{(\alpha_2,\beta_2)}[g]<\infty$" and "$0<\overline{\sigma}_{(\alpha_1,\beta_1)}[f]<\infty$" by "$\sigma_{(\alpha_2,\beta_2)}[g]<\infty$" and "$0<\sigma_{(\alpha_1,\beta_1)}[f]<\infty$", then is need to go the same replacement in right part of (3.15). Also if we replace the conditions $0<\lambda_{(\alpha_1,\beta_1)}[f]\leq\varrho_{(\alpha_1,\beta_1)}[f]<\infty$ and $0<\overline{\sigma}_{(\alpha_1,\beta_1)}[f]<\infty$ of Theorem 3.15 by $0<\varrho_{(\alpha_1,\beta_1)}[f]<\infty$ and $0<\sigma_{(\alpha_1,\beta_1)}[f]<\infty$ respectively, then

$$\limsup_{r\to+\infty}\frac{\alpha_1(M_{f(g)}(\eta r))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))} \geq \frac{\varrho_{(\alpha_1,\beta_1)}[f]\cdot\overline{\sigma}_{(\alpha_2,\beta_2)}[g]}{\sigma_{(\alpha_1,\beta_1)}[f]}.$$

Further if we replace the condition $0<\overline{\sigma}_{(\alpha_1,\beta_1)}[f]<\infty$ of Theorem 3.15 by $0<\sigma_{(\alpha_1,\beta_1)}[f]<\infty$, then Theorem 3.15 remains valid with " limit inferior" replaced by " limit superior".

Now we state the following three theorems without their proofs as those can easily be carried out in the line of Theorem 3.15.

Theorem 3.16. Let f and g be any two entire functions such that $0<\lambda_{(\alpha_1,\beta_1)}[f]<\infty$, $\lambda_{(\alpha_1,\beta_1)}[f]=\lambda_{(\alpha_2,\beta_2)}[g]$, $\tau_{(\alpha_2,\beta_2)}[g]<\infty$ and $0<\tau_{(\alpha_1,\beta_1)}[f]<\infty$ where $\beta_1(r)\geq\exp(\alpha_2(r))$. Then for any $\eta>16$

$$\limsup_{r\to+\infty}\frac{\alpha_1(M_{f(g)}(\eta r))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))} \geq \frac{\lambda_{(\alpha_1,\beta_1)}[f]\cdot\tau_{(\alpha_2,\beta_2)}[g]}{\tau_{(\alpha_1,\beta_1)}[f]}. \tag{3.18}$$

Remark 3.24. In Theorem 3.16, if we will replace the conditions " $\tau_{(\alpha_2,\beta_2)}[g]<\infty$" and "$0<\tau_{(\alpha_1,\beta_1)}[f]<\infty$" by "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]<\infty$" and "$0<\overline{\tau}_{(\alpha_1,\beta_1)}[f]<\infty$", then is need to go the same replacement in right part of (3.18). Also if we replace the conditions $0<\lambda_{(\alpha_1,\beta_1)}[f]<\infty$ and $0<\tau_{(\alpha_1,\beta_1)}[f]<\infty$ of Theorem 3.16 by $0<\lambda_{(\alpha_1,\beta_1)}[f]\leq\varrho_{(\alpha_1,\beta_1)}[f]<\infty$ and $0<\overline{\tau}_{(\alpha_1,\beta_1)}[f]<\infty$ respectively, then

$$\limsup_{r\to+\infty}\frac{\alpha_1(M_{f(g)}(\eta r))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))} \geq \frac{\varrho_{(\alpha_1,\beta_1)}[f]\cdot\tau_{(\alpha_2,\beta_2)}[g]}{\overline{\tau}_{(\alpha_1,\beta_1)}[f]}.$$

Further if we replace the condition $0<\tau_{(\alpha_1,\beta_1)}[f]<\infty$ of Theorem 3.16 by $0<\overline{\tau}_{(\alpha_1,\beta_1)}[f]<\infty$, then Theorem 3.16 remains valid with " limit inferior" replaced by " limit superior".

Theorem 3.17. Let f and g be any two entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f] < \infty$, $\lambda_{(\alpha_1,\beta_1)}[f] = \varrho_{(\alpha_2,\beta_2)}[g]$, $\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$ and $0 < \tau_{(\alpha_1,\beta_1)}[f] < \infty$ where $\beta_1(r) \geq \exp(\alpha_2(r))$. Then for any $\eta > 16$

$$\limsup_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(\eta r))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))} \geq \frac{\lambda_{(\alpha_1,\beta_1)}[f] \cdot \overline{\sigma}_{(\alpha_2,\beta_2)}[g]}{\tau_{(\alpha_1,\beta_1)}[f]}. \tag{3.19}$$

Remark 3.25. In Theorem 3.17, if we will replace the conditions " $\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \tau_{(\alpha_1,\beta_1)}[f] < \infty$" by "$\sigma_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f] < \infty$", then is need to go the same replacement in right part of (3.19). Also if we replace the conditions $0 < \lambda_{(\alpha_1,\beta_1)}[f] < \infty$ and $0 < \tau_{(\alpha_1,\beta_1)}[f] < \infty$ of Theorem 3.17 by $0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \varrho_{(\alpha_1,\beta_1)}[f] < \infty$ and $0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f] < \infty$ respectively, then

$$\limsup_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(\eta r))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))} \geq \frac{\varrho_{(\alpha_1,\beta_1)}[f] \cdot \overline{\sigma}_{(\alpha_2,\beta_2)}[g]}{\overline{\tau}_{(\alpha_1,\beta_1)}[f]}.$$

Further if we replace the condition $0 < \tau_{(\alpha_1,\beta_1)}[f] < \infty$ of Theorem 3.17 by $0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f] < \infty$, then Theorem 3.17 remains valid with " limit inferior" replaced by " limit superior".

Theorem 3.18. Let f and g be any two entire functions such that $0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \varrho_{(\alpha_1,\beta_1)}[f] < \infty$, $\varrho_{(\alpha_1,\beta_1)}[f] = \lambda_{(\alpha_2,\beta_2)}[g]$, $\tau_{(\alpha_2,\beta_2)}[g] < \infty$ and $0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] < \infty$ where $\beta_1(r) \geq \exp(\alpha_2(r))$. Then for any $\eta > 16$

$$\limsup_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(\eta r))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))} \geq \frac{\lambda_{(\alpha_1,\beta_1)}[f] \cdot \tau_{(\alpha_2,\beta_2)}[g]}{\overline{\sigma}_{(\alpha_1,\beta_1)}[f]}. \tag{3.20}$$

Remark 3.26. In Theorem 3.18, if we will replace the conditions " $\tau_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] < \infty$" by "$\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \sigma_{(\alpha_1,\beta_1)}[f] < \infty$", then is need to go the same replacement in right part of (3.20). Also if we replace the conditions $0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \varrho_{(\alpha_1,\beta_1)}[f] < \infty$ and $0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] < \infty$ of Theorem 3.18 by $0 < \varrho_{(\alpha_1,\beta_1)}[f] < \infty$ and $0 < \sigma_{(\alpha_1,\beta_1)}[f] < \infty$ respectively, then

$$\limsup_{r\to+\infty} \frac{\alpha_1(M_{f(g)}(\eta r))}{\exp(\alpha_1(M_f(\beta_1^{-1}(\beta_2(r)))))} \geq \frac{\varrho_{(\alpha_1,\beta_1)}[f] \cdot \tau_{(\alpha_2,\beta_2)}[g]}{\sigma_{(\alpha_1,\beta_1)}[f]}.$$

Further if we replace the condition $0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] < \infty$ of Theorem 3.18 by $0 < \sigma_{(\alpha_1,\beta_1)}[f] < \infty$, then Theorem 3.18 remains valid with " limit inferior" replaced by " limit superior".

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