k-Number of Complete Muti-Partite Graphs

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Abstract

An $L(3,2,1)$-labeling of a graph G is an assignment f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x)-f(y)| \geq 3$ if x and y are adjacent, $|f(x)-f(y)| \geq 2$ if x and y are at distance 2, and $|f(x)-f(y)| \geq 1$ if x and y are at distance 3, for all x and y in $V(G)$. The $L(3,2,1)$-labeling number $k(G)$ of G is the smallest positive integer k such that G has an $L(3,2,1)$-labeling with k as the maximum label. In this paper, we consider complete 3-Partite, complete 4-Partite, complete 5-partite graphs and generalize the result to complete n-Partite graph and find the k-numbers of them.

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1 Introduction

For standard terminology and notation, we follow Bondy and Murty [1] or Murugan [2]. Here, graph means only a simple, finite, connected, undirected graph.

Graph Theory is one of the important branch of mathematics in which graph labeling is an active research area due to its practical applications. It has applications in many fields like communication network, circuit design, television / radio frequency assignment, radar, coding theory etc. Now, due to installation of more television / radio stations assignment of frequencies become a very critical problem. When we assign frequencies to different stations, we have to consider the possibility of interference. Here we face two type of collisions, namely, direct collision and hidden collision.

If two neighbouring stations receive same frequency, they perform direct collision and stations nearby the neighbouring stations perform hidden collision.

In frequency assignment problem, we assign frequencies to a given set of Television / Radio transmitters so that transmitters are assigned frequency with a minimum allowed separation. Closer stations have a stronger interference and so there should be a greater difference between their assigned channels.

Hale introduced a graph model of the channel assignment problem in 1980 [3]. Robert modified this with stations which are "close" and "very close" which correspond to stations at distance two and stations at distance one in graph theoretic terms [4]. The mathematical abstraction of this problem was introduced by Griggs and Yeh as $L(2,1)$ problem [5]. An $L(2,1)$-labeling of a graph G is an assignment f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x)-f(y)| \geq 2$ if x and y are adjacent and $|f(x)-f(y)| \geq 1$ if x and y are at distance 2, for all x and y in $V(G)$. The generalization of this concept is as below.

For positive integers k, d_1, d_2, a k-$L(d_1, d_2)$-labeling of a graph G is a function $f: V(G) \to \{0, 1, 2, \dots, k\}$ such that $|f(u) - f(v)| \geq d_i$ whenever the distance between u and v in G, $d_G(u,v) = i$, for i = 1, 2. The $L(d_1, d_2)$-number of G, $\lambda_{d_1,d_2}(G)$, is the smallest k such that there exists a k-$L(d_1, d_2)$-labeling of G.

But, practically, interference among channels may go beyond two levels. Liu and Shao modified the above problem, by considering stations at distance 1, 2 and 3 and it was called as $L(3,2,1)$ problem [6].

An $L(3,2,1)$-labeling of a graph G is an assignment f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x)-f(y)| \geq 3$ if x and y are adjacent, $|f(x)-f(y)| \geq 2$ if x and y are at distance 2, and $|f(x)-f(y)| \geq 1$ if x and y are at distance 3, for all x and y in $V(G)$. The $L(3,2,1)$-labeling number $k(G)$ of G is the smallest positive integer k such that G has an $L(3,2,1)$-labeling with k as the maximum label. This $k(G)$ is called the k-number of the graph G.

2 Some Existing Results

- Jean Clipperton et al., [7] determined the $L(3,2,1)$-labeling number for complete graphs, complete bipartite graphs paths, cycles, caterpillars, and n-array trees and they have proved that the k-number of $K_{a,b}$ is $2(a+b)$.
- Murugan and Suriya [8] determined the $L(3,2,1)$-labeling number for Fan, Double Fan, Wheel, Friendship graph in terms of the maximum degree of the graphs.
- Shao [9] determined bounds for the $L(3,2,1)$-labeling numbers for Kneser graphs, extremely irregular graphs, Halin graphs.
- Ma-Lian Chia et al., [10] determined the $L(3,2,1)$-labeling number for Cartesian product of paths and cycles, and the power of paths. Also, they presented upper bounds for the $L(3,2,1)$-labeling numbers of general graphs and trees.
- Liu and Shao [6] proved that for a planar graph G, $k(G) \leq 15(\Delta^2 - \Delta + 1)$, where Δ is the maximum degree of G.

3 Multi-Partite Graphs

A bipartite graph or bigraph is one whose vertex set can be partitioned into two subsets X and Y such that each edge has one end in X and the other end in Y; such a partition (X, Y) is called a bipartition of the graph.

A complete bipartite graph is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y.

For $n \geq 2$, a graph G is an n-partite graph if $V(G)$ can be partitioned into n non-empty subsets $V_1, V_2, \dots, V_n$ such that no edge of G joins vertices in the same set. The sets $V_1, V_2, \dots, V_n$ are called partite sets of G.

If G is a simple n-partite graph having partite sets $V_1, V_2, \ldots, V_n$ such that every vertex of V_i is joined to every vertex of V_j, where $1 \leq i < j \leq n$, then G is called a complete n-partite graph. It is also called a complete multi-partite graph and it is denoted by $K_{a_1,a_2,\ldots,a_n}$ where $a_i = |V_i|$.

In this paper, we consider complete 3-Partite, complete 4-Partite, complete 5-partite graphs and generalize the result to complete n-Partite graph and find the k-numbers of them.

4 Results

Theorem 4.1. *The k-number of $K_{a,\alpha,b}$ is $2(a+b+\alpha)$.*

Proof. Consider the tri-partite graph $K_{a,\alpha,b}$. Let the vertices of the 1^{st} partition be $u_1, u_2, \ldots, u_a$, the vertices of the 2^{nd} partition be $v_1, v_2, \ldots, v_\alpha$ and the vertices of the 3^{rd} partition be $w_1, w_2, \ldots, w_b$. Now we define a labeling $f : V(K_{a,\alpha,b}) \to \mathbb{N} \cup \{0\}$ such that f is an $L(3,2,1)$-labeling.

$$\begin{array}{rcll} f(v_i) & = & 2i-2 & \text{if} \quad i = 1,2,\ldots,\alpha \\ f(u_i) & = & f(v_\alpha) + 2i + 1 & \text{if} \quad i = 1,2,\ldots,a \\ f(w_i) & = & f(v_\alpha) + 2a + 1 + 2i + 1 & \text{if} \quad i = 1,2,\ldots,b \end{array}$$

We note that each partition is an independent set and any two vertices in different partitions are adjacent. First we consider adjacent vertices.

For $i = 1,2,\ldots,a$ and $j = 1,2,\ldots,\alpha$,

$$|f(u_i) - f(v_j)| \geq |f(u_1) - f(v_\alpha)| = |f(v_\alpha) + 3 - f(v_\alpha)| \geq 3.$$

For $i = 1,2,\ldots,b$ and $j = 1,2,\ldots,\alpha$,

$$|f(w_i) - f(v_j)| \geq |f(w_1) - f(v_\alpha)| = |f(v_\alpha) + 2a + 4 - f(v_\alpha)| = 2a + 4 \geq 3.$$

For $i = 1,2,\ldots,b$ and $j = 1,2,\ldots,a$,

$$|f(w_i) - f(u_j)| \geq |f(w_1) - f(u_a)| = |f(v_\alpha) + 2a + 4 - f(v_\alpha)| - 2a - 1| \geq 3.$$

In each partition, any two vertices are at distance 2 and since their labels are increasing by 2, the absolute value of their label difference is always greater than or equal to 2. We note that there are no two vertices at distance 3. Hence f is an $L(3,2,1)$-labeling. Also since $f(w_b) = 2(a+b+\alpha)$, we have $k(K_{a,\alpha,b}) \leq 2(a+b+\alpha)$.

Since each partition is an independent set, in an optimal labeling we need at least $2\alpha - 2$ labels to label the vertices $v_1, v_2, \ldots, v_\alpha$ of the second partition. Since each vertex of the second partition is adjacent to each vertex of the first partition, the minimum label needed to label the vertices $u_1, u_2, \ldots, u_a$ is $2\alpha - 2 + 2a + 1$. Since each vertex of the second partition is adjacent to each vertex of the third partition, the minimum label needed to label the vertices $w_1, w_2, \ldots, w_b$ is $2\alpha - 2 + 2a + 1 + 2b + 1 = 2(a+b+\alpha)$. That is, $k(K_{a,\alpha,b}) \geq 2(a+b+\alpha)$. Hence, $k(K_{a,\alpha,b}) = 2(a+b+\alpha)$. □

Theorem 4.2. *The k-number of $K_{a,\alpha,\beta,b}$ is $2(a+b+\alpha+\beta)+1$.*

Proof. Consider the 4-partite graph $K_{a,\alpha,\beta,b}$. Let the vertices of the 1^{st} partition be $u_1, u_2, \ldots, u_a$, the vertices of the 2^{nd} partition be $v_1, v_2, \ldots, v_\alpha$, the vertices of the 3^{rd} partition be $w_1, w_2, \ldots, w_\beta$ and the vertices of the 4^{th} partition be $x_1, x_2, \ldots, x_b$. Now we define $f : V(K_{a,\alpha,\beta,b}) \to \mathbb{N} \cup \{0\}$ such that f is an $L(3, 2, 1)$-labeling.

$$\begin{array}{rcll} f(v_i) & = & 2i-2 & \text{if} \quad i = 1, 2, \ldots, \alpha \\ f(u_i) & = & f(v_\alpha) + 2i + 1 & \text{if} \quad i = 1, 2, \ldots, a \\ f(w_i) & = & f(v_\alpha) + 2a + 2i + 2 & \text{if} \quad i = 1, 2, \ldots, \beta \\ f(x_i) & = & f(v_\alpha) + 2a + 2\beta + 2i + 3 & \text{if} \quad i = 1, 2, \ldots, b \end{array}$$

We note that each partition is an independent set and any two vertices in different partitions are adjacent. First we consider adjacent vertices.

For $i = 1, 2, \ldots, a$ and $j = 1, 2, \ldots, \alpha$,

$$|f(u_i) - f(v_j)| \geq |f(u_1) - f(v_\alpha)| = |f(v_\alpha) + 3 - f(v_\alpha)| \geq 3.$$

For $i = 1, 2, \ldots, \beta$ and $j = 1, 2, \ldots, \alpha$,

$$|f(w_i) - f(v_j)| \geq |f(w_1) - f(v_\alpha)| = |f(v_\alpha) + 2a + 4 - f(v_\alpha)| = 2a + 4 \geq 3.$$

For $i = 1, 2, \ldots, \beta$ and $j = 1, 2, \ldots, a$,

$$|f(w_i) - f(u_j)| \geq |f(w_1) - f(u_a)| = |f(v_\alpha) + 2a + 4 - f(v_\alpha) - 2a - 1| \geq 3.$$

For $i = 1, 2, \ldots, b$ and $j = 1, 2, \ldots, \beta$,

$$|f(x_i) - f(w_j)| \geq |f(x_1) - f(w_\beta)| = |f(v_\alpha) + 2a + 2\beta + 5 - f(v_\alpha) - 2a - 2\beta - 2| \geq 3.$$

For $i = 1, 2, \ldots, b$ and $j = 1, 2, \ldots, \alpha$,

$$|f(x_i) - f(v_j)| \geq |f(x_1) - f(v_\alpha)| = |f(v_\alpha) + 2a + 2\beta + 5 - f(v_\alpha)| = |2a + 2\beta + 5| \geq 3.$$

For $i = 1, 2, \ldots, b$ and $j = 1, 2, \ldots, a$,

$$|f(x_i) - f(u_j)| \geq |f(x_1) - f(u_a)| = |f(v_\alpha) + 2a + 2\beta + 5 - f(v_\alpha) - 2a - 1| = |2\beta + 4| \geq 3.$$

In each partition, any two vertices are at distance 2 and since their labels are increasing by 2, the absolute value of their label difference is always greater than or equal to 2. We note that there are no two vertices at distance 3. Hence f is an $L(3, 2, 1)$-labeling. Also since $f(w_b) = 2(a + b + \alpha + \beta) + 1$, we have $k(K_{a,\alpha,\beta,b}) \leq 2(a + b + \alpha + \beta) + 1$.

Since each partition is an independent set, in an optimal labeling we need at least $2\alpha - 2$ labels to label the vertices $v_1, v_2, \ldots, v_\alpha$ of the second partition. Since each vertex of the second partition is adjacent to each vertex of the first partition, the minimum label needed to label the vertices $u_1, u_2, \ldots, u_a$ is $2\alpha - 2 + 2a + 1$. Since each vertex of the second partition is adjacent to each vertex of the third partition, the minimum label needed to label the vertices $w_1, w_2, \ldots, w_\beta$ is $2\alpha - 2 + 2a + 1 + 2\beta + 1 = 2(a + \alpha + \beta)$. Since each vertex of the third partition is adjacent to each vertex of the fourth partition, the minimum label needed to label the vertices $x_1, x_2, \ldots, x_b$ is $2(a + \alpha + \beta) + 2b + 1 = 2(a + b + \alpha + \beta) + 1$. That is, $k(K_{a,\alpha,\beta,b}) \geq 2(a + b + \alpha + \beta) + 1$. Hence, $k(K_{a,\alpha,\beta,b}) = 2(a + b + \alpha + \beta) + 1$. □

Theorem 4.3. *The k-number of $K_{a,\alpha,\beta,\gamma,b}$ is* $2(a+b+\alpha+\beta+\gamma)+2$.

Proof. Consider the 5-partite graph $K_{a,\alpha,\beta,\gamma,b}$. Let the vertices of the 1^{st} partition be $u_1, u_2, \ldots, u_a$, the vertices of the 2^{nd} partition be $v_1, v_2, \ldots, v_\alpha$, the vertices of the 3^{rd} partition be $w_1, w_2, \ldots, w_\beta$, the vertices of the 4^{th} partition be $x_1, x_2, \ldots, x_\gamma$ and the vertices of the 5^{th} partition be $y_1, y_2, \ldots, y_b$. Now we define $f : V(K_{a,\alpha,\beta,\gamma,b}) \to \mathbb{N} \cup \{0\}$ such that f is an $L(3,2,1)$-labeling.

$$\begin{array}{lcll} f(v_i) & = & 2i-2 & \text{if} \quad i = 1,2,\ldots,\alpha \\ f(u_i) & = & f(v_\alpha)+2i+1 & \text{if} \quad i = 1,2,\ldots,a \\ f(w_i) & = & f(v_\alpha)+2a+2i+2 & \text{if} \quad i = 1,2,\ldots,\beta \\ f(x_i) & = & f(v_\alpha)+2a+2\beta+2i+3 & \text{if} \quad i = 1,2,\ldots,\gamma \\ f(y_i) & = & f(v_\alpha)+2a+2\beta+2\gamma+2i+4 & \text{if} \quad i = 1,2,\ldots,b \end{array}$$

We note that each partition is an independent set and any two vertices in different partitions are adjacent. First we consider adjacent vertices.

For $i = 1,2,\ldots,a$ and $j = 1,2,\ldots,\alpha$,

$$|f(u_i) - f(v_j)| \geq |f(u_1) - f(v_\alpha)| = |f(v_\alpha) + 3 - f(v_\alpha)| \geq 3.$$

For $i = 1,2,\ldots,\beta$ and $j = 1,2,\ldots,\alpha$,

$$|f(w_i) - f(v_j)| \geq |f(w_1) - f(v_\alpha)| = |f(v_\alpha) + 2a + 4 - f(v_\alpha)| = 2a + 4 \geq 3.$$

For $i = 1,2,\ldots,\beta$ and $j = 1,2,\ldots,a$,

$$|f(w_i) - f(u_j)| \geq |f(w_1) - f(u_a)| = |f(v_\alpha) + 2a + 4 - f(v_\alpha) - 2a - 1| \geq 3.$$

For $i = 1,2,\ldots,\gamma$ and $j = 1,2,\ldots,\beta$,

$$|f(x_i) - f(w_j)| \geq |f(x_1) - f(w_\beta)| = |f(v_\alpha) + 2a + 2\beta + 5 - f(v_\alpha) - 2a - 2\beta - 2| \geq 3.$$

For $i = 1,2,\ldots,\gamma$ and $j = 1,2,\ldots,\alpha$,

$$|f(x_i) - f(v_j)| \geq |f(x_1) - f(v_\alpha)| = |f(v_\alpha) + 2a + 2\beta + 5 - f(v_\alpha)| = |2a + 2\beta + 5| \geq 3.$$

For $i = 1,2,\ldots,\gamma$ and $j = 1,2,\ldots,a$,

$$|f(x_i) - f(u_j)| \geq |f(x_1) - f(u_a)| = |f(v_\alpha) + 2a + 2\beta + 5 - f(v_\alpha) - 2a - 1| = |2\beta + 4| \geq 3.$$

For $i = 1,2,\ldots,b$ and $j = 1,2,\ldots,\gamma$,

$$|f(y_i) - f(x_j)| \geq |f(y_1) - f(x_\gamma)| = |f(v_\alpha) + 2a + 2\beta + 2\gamma + 6 - f(v_\alpha) - 2a - 2\beta - 2\gamma - 3| \geq 3.$$

For $i = 1,2,\ldots,b$ and $j = 1,2,\ldots,\beta$,

$$|f(y_i) - f(w_j)| \geq |f(y_1) - f(w_\beta)| = |f(v_\alpha) + 2a + 2\beta + 2\gamma + 6 - f(v_\alpha) - 2a - 2\beta - 2|$$

$$= |2\gamma + 4| \geq 3.$$

For $i = 1, 2, \ldots, b$ and $j = 1, 2, \ldots, \alpha$,

$$|f(y_i) - f(v_j)| \geq |f(y_1) - f(v_\alpha)| = |f(v_\alpha) + 2a + 2\beta + 2\gamma + 6 - f(v_\alpha)|$$
$$= |2\alpha + 2\beta + 2\gamma + 6| \geq 3.$$

For $i = 1, 2, \ldots, b$ and $j = 1, 2, \ldots, a$,

$$|f(y_i) - f(u_j)| \geq |f(y_1) - f(u_a)| = |f(v_\alpha) + 2a + 2\beta + 2\gamma + 6 - f(v_\alpha) - 2a - 1|$$
$$= |2\beta + 2\gamma + 5| \geq 3.$$

In each partition, any two vertices are at distance 2 and since their labels are increasing by 2, the absolute value of their label difference is always greater than or equal to 2. We note that there are no two vertices at distance 3. Hence f is an $L(3, 2, 1)$-labeling. Also since $f(y_b) = 2(a + b + \alpha + \beta + \gamma) + 2$, we have $k(K_{a,\alpha,\beta,\gamma,b}) \leq 2(a + b + \alpha + \beta + \gamma) + 2$.

Since each partition is an independent set, in an optimal labeling we need at least $2\alpha - 2$ labels to label the vertices $v_1, v_2, \ldots, v_\alpha$ of the second partition. Since each vertex of the second partition is adjacent to each vertex of the first partition, the minimum label needed to label the vertices $u_1, u_2, \ldots, u_a$ is $2\alpha - 2 + 2a + 1$. Since each vertex of the second partition is adjacent to each vertex of the third partition, the minimum label needed to label the vertices $w_1, w_2, \ldots, w_\beta$ is $2\alpha - 2 + 2a + 1 + 2\beta + 1 = 2(a + \alpha + \beta)$. Since each vertex of the third partition is adjacent to each vertex of the fourth partition, the minimum label needed to label the vertices $x_1, x_2, \ldots, x_\gamma$ is $2(a + \alpha + \beta) + 2\gamma + 1 = 2(a + \alpha + \beta + \gamma) + 1$.

Since each vertex of the fourth partition is adjacent to each vertex of the fifth partition, the minimum label needed to label the vertices $y_1, y_2, \ldots, y_b$ is $2(a + \alpha + \beta + \gamma) + 2b + 2 = 2(a + b + \alpha + \beta + \gamma) + 2$. Hence $k(K_{a,\alpha,\beta,\gamma,b}) \geq 2(a + b + \alpha + \beta + \gamma) + 2$. Therefore, $k(K_{a,\alpha,\beta,\gamma,b}) = 2(a + b + \alpha + \beta + \gamma) + 2$. □

Theorem 4.4. *The k-number of $K_{a_1,a_2,\ldots,a_n}$ is $2(a_1 + a_2 + \cdots + a_n) + n - 3$, $n \geq 3$.*

Proof. We prove this theorem by induction on the number of partitions of $K_{a_1,a_2,\ldots,a_n}$, that is, n. For small values of n, the result is true by Theorems 4.1,4.2, and 4.3. Now the assume the result for n. That is, let the k-number of $K_{a_1,a_2,\ldots,a_n}$ is $2(a_1 + a_2 + \cdots + a_n) + n - 3$. Now we prove the result for $n + 1$. Let G be a multi-partite graph with $n + 1$ partitions. Now remove the vertices of $(n+1)^{th}$ partition which are $v_1, v_2, \ldots, v_{a_{n+1}}$ (say) and call the resulting graph as G_1. Clearly G_1 is an n-partite graph. By our assumption, the theorem is true for G_1. That is, the k-number of G_1 is $2(a_1 + a_2 + \cdots + a_n) + n - 3$.

Now consider G. In an optimal labeling of G, the minimum label of v_1 should be $2(a_1 + a_2 + \cdots + a_n) + n - 3 + 3$ and so the minimum label of $v_{a_{n+1}}$ should be $2(a_1 + a_2 + \cdots + a_n) + n - 3 + 3 + 2(a_{n+1} - 1) = 2(a_1 + a_2 + \cdots + a_n + a_{n+1}) + n - 2$, which is $2(a_1 + a_2 + \cdots + a_n + a_{n+1}) + (n + 1) - 3$. Hence the theorem. □

5 Conclusion

We have determined k-numbers of complete 3-Partite, complete 4-Partite, complete 5-partite graphs and generalized the result to complete n-Partite graph. We believe that this work will create an interest on researchers about $L(3, 2, 1)$-labeling.

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