Evaluation of Some Definite Integrals $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos\theta \pm \sin\theta)^{\alpha} d\theta$

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Abstract

In this article, we provide the analytical solutions of two definite integrals $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos\theta \pm \sin\theta)^{\alpha} d\theta$ (with suitable convergence conditions) by using the approaches of gamma and hypergeometric functions. Further we also obtain a summation formula for Kampé de Fériet's double hypergeometric function having the arguments ± 1.

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1 Introduction and Preliminaries

In our investigations, we shall use the following standard notations:
$\mathbb{N} := \{1, 2, 3, \cdots\}; \mathbb{N}_0 := \mathbb{N} \bigcup \{0\}; \mathbb{Z}_0^- := \mathbb{Z}^- \bigcup \{0\} = \{0, -1, -2, -3, \cdots\}$.
The symbols $\mathbb{C}, \mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers, respectively.

Pochhammer symbol

The Pochhammer symbol $(\alpha)_p (\alpha, p \in \mathbb{C})$[7, p.22, Eq.(1), p.32, Q.N.(8) and Q.N.(9)], see also [11, p.23, Eq.(22) and Eq.(23)] is defined by

$$(\alpha)_p := \frac{\Gamma(\alpha+p)}{\Gamma(\alpha)} = \begin{cases} 1 & ;(p=0; \alpha \in \mathbb{C}\setminus\{0\}), \\ \alpha(\alpha+1)\cdots(\alpha+n-1) & ;(p=n \in \mathbb{N}; \alpha \in \mathbb{C}), \\ \frac{(-1)^k n!}{(n-k)!} & ;(\alpha=-n; p=k; k, n \in \mathbb{N}_0; 0 \leqq k \leqq n), \\ 0 & ;(\alpha=-n; p=k; k, n \in \mathbb{N}_0; k > n), \\ \frac{(-1)^n}{(1-\alpha)_n} & ;(p=-n; n \in \mathbb{N}; \alpha \in \mathbb{C}\setminus\mathbb{Z}). \end{cases}$$

It being understood conventionally that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

Beta function[11, p.25, Eq.(43) and p.26, Eq.(48)]

(1.1)

$$B(\alpha,\beta) = \begin{cases} \int_0^1 u^{\alpha-1}(1-u)^{\beta-1} du = \int_0^{\infty} \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}} du, & ; Re(\alpha) > 0, Re(\beta) > 0, \\ \frac{\Gamma(\alpha)\ \Gamma(\beta)}{\Gamma(\alpha+\beta)}, & ; Re(\alpha) < 0, Re(\beta) < 0, \end{cases} \qquad (\alpha, \beta \in \mathbb{C}\setminus\mathbb{Z}_0^-).$$

Recurrence relation[11, p.20, Eq.(5)]

$$\Gamma(z+1) = z\Gamma(z). \tag{1.2}$$

Decomposition of unilateral series

$$\sum_{r=0}^{\infty} \Phi(r) = \sum_{r=0}^{\infty} \Phi(2r) + \sum_{r=0}^{\infty} \Phi(2r+1), \tag{1.3}$$

provided that each infinite series of both sides is absolutely convergent.

$$\sum_{r=0}^{2p} \Psi(r) = \sum_{r=0}^{p} \Psi(2r) + \sum_{r=0}^{p-1} \Psi(2r+1). \tag{1.4}$$

$$\sum_{r=0}^{2p+1} \Theta(r) = \sum_{r=0}^{p} \Theta(2r) + \sum_{r=0}^{p} \Theta(2r+1). \tag{1.5}$$

Binomial theorem[11, p.44, Eq.(8)]
When $a \in \mathbb{C}$ and $|z| < 1$, then

$$(1-z)^{-a} = {}_1F_0\left[\begin{matrix} a \ ; \\ - \ ; \end{matrix} \ z\right]. \tag{1.6}$$

Integral formula[11, p.26, Eq.(49)]

$$\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}, \tag{1.7}$$

where $Re(p) > -1$ and $Re(q) > -1$.
Legendre's duplication formula[11, p.23, Eq.(25)]

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right), \tag{1.8}$$

where $2z \in \mathbb{C}\backslash\mathbb{Z}_0^-$.
Reflection formula[11, p.20, Eq.(8)]

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}; \qquad (z \neq 0, \pm1, \pm2, \pm3, ...). \tag{1.9}$$

Property of definite integral

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx. \tag{1.10}$$

Double hypergeometric function of Kampé de Fériet
Just as the Gaussian ${}_2F_1$ function was generalized to ${}_pF_q$ by increasing the number of the numerator and denominator parameters, the four Appell functions, seven Humbert functions were unified and generalized by Kampé de Fériet [5] who defined a general hypergeometric function of two variables.
The notation introduced by Kampé de Fériet for his double hypergeometric function [1, p.150, Eq.(26)] of superior order was subsequently abbreviated by Burchnall and Chaundy [2, p.112]. We recall here the definition of a more general double hypergeometric function (than the one defined by Kampé de Fériet) in a slightly modified notation [12, p.423, Eq.(26)]:

$$(1.11)\quad F^{p:q;k}_{\ell:m;n}\left[\begin{array}{ccccc}(a_p) & : & (b_q) & ; & (c_k) ; \\ (\alpha_\ell) & : & (\beta_m) & ; & (\gamma_n) ;\end{array}\; x,\ y\right] = \sum_{r,s=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{\ell}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\frac{x^r}{r!}\frac{y^s}{s!},$$

where for convergence,

$$(i)\quad p+q<\ell+m+1,\quad p+k<\ell+n+1,\quad |x|<\infty,\quad |y|<\infty,$$

$$(ii)\quad p+q=\ell+m+1,\quad p+k=\ell+n+1 \text{ and}$$

$$\begin{cases}|x|^{1/(p-\ell)}+|y|^{1/(p-\ell)}<1; & \text{if } p>\ell,\\ \max\{|x|,|y|\}<1; & \text{if } p\le\ell.\end{cases}$$

For absolutely and conditionally convergence of double series (1.11), the readers can refer the paper of Hái *et al.* [4].
Motivated by the work of Srivastava *et al.* [8, 10] and Qureshi *et al.* [3, 6], we obtain the analytical solution of certain definite integrals in terms of gamma and hypergeometric functions.
The present article is organized as follows: In section 2, we provide the values of the definite integrals $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos\theta\pm\sin\theta)^\alpha d\theta$. In section 3, we have derived the definite integrals, by using some suitable substitutions. In section 4, we obtain a summation formula for Kampé de Fériet's double hypergeometric function having the arguments ± 1.

2 Main Integrals:

Any values of parameters and arguments leading to the results, which do not make sense, are tacitly excluded.

$$(2.1)\quad \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos\theta\pm\sin\theta)^\alpha d\theta = \frac{1}{\alpha}2^{\frac{\alpha}{2}}\frac{\sqrt{\pi}\,\Gamma\left(\frac{\alpha}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)},$$

where $Re\,(\alpha)>-1$.

$$(2.2)\quad \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos\theta\pm\sin\theta)^\alpha d\theta = 2\ F^{1:\ 2;\ 1}_{1:\ 1;\ 0}\left[\begin{array}{ccccc}\frac{1}{2} & : & \frac{-\alpha}{2},\frac{-\alpha+1}{2}; & \frac{\alpha+2}{2} & ; \\ \frac{3}{2} & : & \frac{1}{2} ; & - & ;\end{array}\; 1,\ -1\right],$$

where $Re\,(\alpha)>-1$.

3 Analytical Solutions of the Definite Integrals

Solution of the integral (2.1), in terms of Gamma functions:

$$\text{Let} \quad I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos\theta \pm \sin\theta)^{\alpha} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos^2\theta + \sin^2\theta \pm 2\sin\theta\cos\theta)^{\frac{\alpha}{2}} d\theta$$

$$(3.1) \qquad = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \{1 \pm \sin 2\theta\}^{\frac{\alpha}{2}} \, d\theta.$$

If $\theta = \left(\frac{\pi}{4} - \phi\right)$ or $\phi = \left(\frac{\pi}{4} - \theta\right)$, $d\theta = -d\phi$.
When $\theta = -\frac{\pi}{4}$, then $\phi = \frac{\pi}{2}$ and when $\theta = \frac{\pi}{4}$, then $\phi = 0$.
Put these values in equation (3.1), we get

$$I = -\int_{\frac{\pi}{2}}^{0} \left\{1 \pm \sin 2\left(\frac{\pi}{4} - \phi\right)\right\}^{\frac{\alpha}{2}} d\phi$$

$$= \int_{0}^{\frac{\pi}{2}} \left\{1 \pm \sin\left(\frac{\pi}{2} - 2\phi\right)\right\}^{\frac{\alpha}{2}} d\phi$$

$$= \int_{0}^{\frac{\pi}{2}} \{1 \pm \cos 2\phi\}^{\frac{\alpha}{2}} \, d\phi$$

$$= 2^{\frac{\alpha}{2}} \int_{0}^{\frac{\pi}{2}} \{\cos\phi\}^{\alpha} \, d\phi \quad or \quad 2^{\frac{\alpha}{2}} \int_{0}^{\frac{\pi}{2}} \{\sin\phi\}^{\alpha} \, d\phi$$

$$(3.2) \qquad I = 2^{\frac{\alpha}{2}} \int_{0}^{\frac{\pi}{2}} (\sin\phi)^{0} (\cos\phi)^{\alpha} d\phi$$

Further using integral formula (1.7) on the right hand side of equation (3.2), we have

$$I = 2^{\frac{\alpha}{2}} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1+\alpha}{2}\right)}{2\Gamma\left(\frac{\alpha+2}{2}\right)}$$

$$= 2^{\frac{\alpha}{2}-1} \frac{\sqrt{\pi}\ \Gamma\left(\frac{1+\alpha}{2}\right)}{\frac{\alpha}{2}\Gamma\left(\frac{\alpha}{2}\right)} \qquad ; Re\,(\alpha) > -1,$$

which, after simplification, yields the required result (2.1).

Another solution of the same integral in terms of Kampé de Fériet's double hypergeometric function:

$$\bullet\text{Let} \quad I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos\theta + \sin\theta)^{\alpha} d\theta$$

$$= \int_{-\frac{\pi}{4}}^{0} (\cos\theta + \sin\theta)^{\alpha} d\theta + \int_{0}^{\frac{\pi}{4}} (\cos\theta + \sin\theta)^{\alpha} d\theta. \tag{3.3}$$

If $\theta = -t,\ d\theta = -dt$, when $\theta = -\frac{\pi}{4},\ t = \frac{\pi}{4}$ and when $\theta = 0$, then $t = 0$.
Put these values in equation (3.3), we get

$$I = -\int_{\frac{\pi}{4}}^{0} (\cos t - \sin t)^{\alpha} dt + \int_{0}^{\frac{\pi}{4}} (\cos\theta + \sin\theta)^{\alpha} d\theta$$

$$= \int_{0}^{\frac{\pi}{4}} (\cos\theta - \sin\theta)^{\alpha} d\theta + \int_{0}^{\frac{\pi}{4}} (\cos\theta + \sin\theta)^{\alpha} d\theta$$

$$= \int_{0}^{\frac{\pi}{4}} (\cos\theta)^{\alpha} \{1 - \tan\theta\}^{\alpha} d\theta + \int_{0}^{\frac{\pi}{4}} (\cos\theta)^{\alpha} \{1 + \tan\theta\}^{\alpha} d\theta\ ; \tag{3.4}$$

$(0 < \tan\theta < 1$; Since, $0 < \theta < \frac{\pi}{4})$.
Using binomial theorem (1.6) on right hand side of equation (3.4), we get
(3.5)

$$I = \int_{0}^{\frac{\pi}{4}} (\cos\theta)^{\alpha} {}_1F_0 \left[\begin{array}{lll} -\alpha & ; & \\ & & \tan\theta \\ - & ; & \end{array} \right] d\theta + \int_{0}^{\frac{\pi}{4}} (\cos\theta)^{\alpha} {}_1F_0 \left[\begin{array}{lll} -\alpha & ; & \\ & & -\tan\theta \\ - & ; & \end{array} \right] d\theta.$$

Similarly, we can also obtain
(3.6)

$$\bullet\bullet \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos\theta - \sin\theta)^{\alpha} d\theta = \int_{0}^{\frac{\pi}{4}} (\cos\theta)^{\alpha} {}_1F_0 \left[\begin{array}{ll} -\alpha; & \\ & -\tan\theta \\ -; & \end{array} \right] d\theta + \int_{0}^{\frac{\pi}{4}} (\cos\theta)^{\alpha} {}_1F_0 \left[\begin{array}{ll} -\alpha; & \\ & \tan\theta \\ -; & \end{array} \right] d\theta.$$

by proceeding on the same way as in equation (3.3).

Remark: We conclude that the equations (3.5) and (3.6) are same. For the sake of convenience, we shall consider only the equation (3.5) in all the following cases for derivation purpose.

Case(i)

$*$ Let α is not a positive integer in equation (3.5), we get

$$I = \int_{0}^{\frac{\pi}{4}} (\cos\theta)^{\alpha} \sum_{r=0}^{\infty} \frac{(-\alpha)_r (\tan\theta)^r}{r!} d\theta + \int_{0}^{\frac{\pi}{4}} (\cos\theta)^{\alpha} \sum_{r=0}^{\infty} \frac{(-\alpha)_r (-\tan\theta)^r}{r!} d\theta.$$

For $\alpha \neq 1, 2, 3, 4...$ we get

$$(3.7) \qquad I = \int_0^{\frac{\pi}{4}} (\cos\theta)^{\alpha} \sum_{r=0}^{\infty} \frac{(-\alpha)_r}{r!} \left\{(\tan\theta)^r + (-\tan\theta)^r\right\} d\theta,$$

Now applying the decomposition identity (1.3) on the right hand side of equation (3.7), we get

$$I = 2\int_0^{\frac{\pi}{4}} (\cos\theta)^{\alpha} \sum_{r=0}^{\infty} \frac{(-\alpha)_{2r}\,(\tan\theta)^{2r}}{2r!} d\theta$$

$$(3.8) \qquad = 2\sum_{r=0}^{\infty} \frac{(-\alpha)_{2r}}{(2r)!} \int_0^{\frac{\pi}{4}} (\cos\theta)^{\alpha} (\tan\theta)^{2r} d\theta.$$

If $\tan\theta = t$, $\sec^2\theta d\theta = dt$, $d\theta = \frac{dt}{\sec^2\theta} = \frac{dt}{1+\tan^2\theta} = \frac{dt}{1+t^2}$.
When $\theta = 0$, then $t = 0$ and when $\theta = \frac{\pi}{4}$, then $t = 1$.
Put these values in equation (3.8), we get

$$I = 2\sum_{r=0}^{\infty} \frac{(-\alpha)_{2r}}{(2r)!} \int_0^1 \left(\frac{1}{\sqrt{(1+t^2)}}\right)^{\alpha} t^{2r} \frac{dt}{1+t^2}$$

$$(3.9) \qquad = 2\sum_{r=0}^{\infty} \frac{(-\alpha)_{2r}}{(2r)!} \int_0^1 t^{2r} (1+t^2)^{-\left(\frac{\alpha}{2}+1\right)} dt.$$

Implementing the binomial theorem (1.6) on right hand side of equation (3.9), we get

$$I = 2\sum_{r=0}^{\infty} \frac{(-\alpha)_{2r}}{(2r)!} \int_0^1 t^{2r} {}_1F_0 \left[\begin{matrix} \frac{\alpha}{2}+1; \\ -\;\; ; \end{matrix} \; -t^2 \right] dt$$

$$= 2\sum_{r=0}^{\infty} \frac{(-\alpha)_{2r}}{(2r)!} \sum_{s=0}^{\infty} \frac{\left(\frac{\alpha}{2}+1\right)_s (-1)^s}{s!} \int_0^1 t^{2r+2s} dt$$

$$= 2\sum_{r=0}^{\infty} \frac{(-\alpha)_{2r}}{(2r)!} \sum_{s=0}^{\infty} \frac{\left(\frac{\alpha}{2}+1\right)_s (-1)^s}{\{1+2(r+s)\}\; s!}$$

$$= 2\sum_{r=0}^{\infty} \frac{(-\alpha)_{2r}}{(2r)!} \sum_{s=0}^{\infty} \frac{\left(\frac{\alpha}{2}+1\right)_s (-1)^s \left(\frac{1}{2}\right)_{r+s}}{s!\; \left(\frac{3}{2}\right)_{r+s}}$$

$$= 2\sum_{r=0}^{\infty} \frac{\left(\frac{-\alpha}{2}\right)_r \left(\frac{-\alpha+1}{2}\right)_r}{\left(\frac{1}{2}\right)_r r!} \sum_{s=0}^{\infty} \frac{\left(\frac{\alpha}{2}+1\right)_s (-1)^s \left(\frac{1}{2}\right)_{r+s}}{s!\; \left(\frac{3}{2}\right)_{r+s}}$$

$$(3.10)\quad I=2\sum_{r,s=0}^{\infty}\frac{\left(\frac{1}{2}\right)_{r+s}\left(-\frac{\alpha}{2}\right)_r\left(\frac{-\alpha+1}{2}\right)_r\left(\frac{\alpha+2}{2}\right)_s(-1)^s(1)^r}{\left(\frac{3}{2}\right)_{r+s}\left(\frac{1}{2}\right)_r r!\ s!}.$$

Recalling the definition of Kampé de Fériet's double hypergeometric function (1.11) on the right hand side of equation (3.10), we get the required result (2.2).

Case (ii)

**Let $\alpha = 2p$ (even positive integer) in equation (3.5), we get

$$I=\int_0^{\frac{\pi}{4}}(\cos\theta)^{2p}{}_1F_0\left[\begin{array}{ll}-2p\ ; & \\ & \tan\theta\\ -\ ; & \end{array}\right]d\theta+\int_0^{\frac{\pi}{4}}(\cos\theta)^{2p}{}_1F_0\left[\begin{array}{ll}-2p\ ; & \\ & -\tan\theta\\ -\ ; & \end{array}\right]d\theta$$

$$=\int_0^{\frac{\pi}{4}}(\cos\theta)^{2p}\sum_{r=0}^{2p}\frac{(-2p)_r\,(\tan\theta)^r}{r!}d\theta+\int_0^{\frac{\pi}{4}}(\cos\theta)^{2p}\sum_{r=0}^{2p}\frac{(-2p)_r\,(-\tan\theta)^r}{r!}d\theta.$$

For $\alpha = 2, 4, 6, ...$ or $p = 1, 2, 3, ...$

$$(3.11)\qquad I=\int_0^{\frac{\pi}{4}}(\cos\theta)^{2p}\sum_{r=0}^{2p}\frac{(-2p)_r}{r!}\left\{(\tan\theta)^r+(-\tan\theta)^r\right\}d\theta.$$

Further implementing the decomposition identity (1.4) on the right hand side of equation (3.11), we get

$$I=2\int_0^{\frac{\pi}{4}}(\cos\theta)^{2p}\sum_{r=0}^{p}\frac{(-2p)_{2r}\,(\tan\theta)^{2r}}{(2r)!}d\theta$$

$$(3.12)\qquad =2\sum_{r=0}^{p}\frac{(-2p)_{2r}}{(2r)!}\int_0^{\frac{\pi}{4}}(\cos\theta)^{2p}(\tan\theta)^{2r}d\theta.$$

If $\tan\theta = t,\ \sec^2\theta d\theta = dt,\ d\theta = \frac{dt}{\sec^2\theta} = \frac{dt}{1+\tan^2\theta} = \frac{dt}{1+t^2}$.
When $\theta = 0$, then $t = 0$ and when $\theta = \frac{\pi}{4}$, then $t = 1$.
Put these values in equation (3.12), we get

$$I=2\sum_{r=0}^{p}\frac{(-2p)_{2r}}{(2r)!}\int_0^{1}\left(\frac{1}{\sqrt{(1+t^2)}}\right)^{2p}t^{2r}\frac{dt}{1+t^2}$$

$$(3.13)\qquad =2\sum_{r=0}^{p}\frac{(-2p)_{2r}}{(2r)!}\int_0^{1}t^{2r}(1+t^2)^{-(p+1)}dt.$$

After using binomial theorem (1.6) on right hand side of equation (3.13), we have

$$I = 2\sum_{r=0}^{p}\frac{(-2p)_{2r}}{(2r)!}\int_0^1 t^{2r}\,{}_1F_0\left[\begin{matrix} p+1; \\ - ; \end{matrix} -t^2\right]dt$$

$$= 2\sum_{r=0}^{p}\frac{(-2p)_{2r}}{(2r)!}\sum_{s=0}^{\infty}\frac{(p+1)_s\,(-1)^s}{s!}\int_0^1 t^{2r+2s}dt$$

$$= 2\sum_{r=0}^{p}\frac{(-2p)_{2r}}{(2r)!}\sum_{s=0}^{\infty}\frac{(p+1)_s\,(-1)^s}{\{1+2(r+s)\}\;s!}$$

$$= 2\sum_{r=0}^{p}\frac{(-2p)_{2r}}{(2r)!}\sum_{s=0}^{\infty}\frac{(p+1)_s\,(-1)^s\left(\frac{1}{2}\right)_{r+s}}{s!\;\left(\frac{3}{2}\right)_{r+s}}$$

$$= 2\sum_{r=0}^{p}\frac{(-p)_r\left(\frac{-2p+1}{2}\right)_r}{\left(\frac{1}{2}\right)_r r!}\sum_{s=0}^{\infty}\frac{(p+1)_s\,(-1)^s\left(\frac{1}{2}\right)_{r+s}}{s!\;\left(\frac{3}{2}\right)_{r+s}}$$

$$I = 2\sum_{r=0}^{p}\sum_{s=0}^{\infty}\frac{\left(\frac{1}{2}\right)_{r+s}(-p)_r\left(\frac{-2p+1}{2}\right)_r(p+1)_s\,(-1)^s(1)^r}{\left(\frac{3}{2}\right)_{r+s}\left(\frac{1}{2}\right)_r r!\;s!}. \tag{3.14}$$

Finally, use the definition of Kampé de Fériet's double hypergeometric function (1.11) on the right hand side of equation (3.14), leads to the result

$$I = 2\;F^{1:\;2;\;1}_{1:\;1;\;0}\left[\begin{matrix} \frac{1}{2} : -p,\; \frac{-2p+1}{2};\; p+1\;; \\ \frac{3}{2} : \quad \frac{1}{2} \quad ; \quad - \quad ; \end{matrix}\quad 1,\; -1\right]$$

$$I = 2\;F^{1:\;2;\;1}_{1:\;1;\;0}\left[\begin{matrix} \frac{1}{2} : \frac{-\alpha}{2}, \frac{-\alpha+1}{2};\; \frac{\alpha+2}{2}\;; \\ \frac{3}{2} : \quad \frac{1}{2} \quad ; \quad - \quad ; \end{matrix}\quad 1,\; -1\right], \tag{3.15}$$

where $Re\,(\alpha) > -1$.

Case (iii)

$***$Let $\alpha = 2p+1$ (odd positive integer) in equation (3.5), we get

$$I = \int_0^{\frac{\pi}{4}}(\cos\theta)^{2p+1}{}_1F_0\left[\begin{matrix} -(2p+1)\;; \\ - \;; \end{matrix}\; \tan\theta\right]d\theta + \int_0^{\frac{\pi}{4}}(\cos\theta)^{2p+1}{}_1F_0\left[\begin{matrix} -(2p+1)\;; \\ - \;; \end{matrix}\; -\tan\theta\right]d\theta$$

$$= \int_0^{\frac{\pi}{4}}(\cos\theta)^{2p+1}\sum_{r=0}^{2p+1}\frac{-(2p+1)_r\,(\tan\theta)^r}{r!}d\theta + \int_0^{\frac{\pi}{4}}(\cos\theta)^{2p+1}\sum_{r=0}^{2p+1}\frac{-(2p+1)_r\,(-\tan\theta)^r}{r!}d\theta.$$

For $\alpha = 1, 3, 5, 7, ...$ or $p = 0, 1, 2, 3, ...$

$$(3.16)\quad I = \int_0^{\frac{\pi}{4}} (\cos\theta)^{2p+1} \sum_{r=0}^{2p+1} \frac{(-2p-1)_r}{r!} \left\{(\tan\theta)^r + (-\tan\theta)^r\right\} d\theta.$$

Using decomposition identity (1.5) on the right hand side of equation (3.16), we get

$$I = 2\int_0^{\frac{\pi}{4}} (\cos\theta)^{2p+1} \sum_{r=0}^{p} \frac{(-2p-1)_{2r} (\tan\theta)^{2r}}{(2r)!} d\theta$$

$$(3.17)\quad = 2\sum_{r=0}^{p} \frac{(-2p-1)_{2r}}{(2r)!} \int_0^{\frac{\pi}{4}} (\cos\theta)^{2p+1} (\tan\theta)^{2r} d\theta.$$

If $\tan\theta = t$, $\sec^2\theta d\theta = dt$, $d\theta = \frac{dt}{\sec^2\theta} = \frac{dt}{1+\tan^2\theta} = \frac{dt}{1+t^2}$.
When $\theta = 0$, then $t = 0$ and when $\theta = \frac{\pi}{4}$, then $t = 1$.
Put these values in equation (3.17), we get

$$I = 2\sum_{r=0}^{p} \frac{(-2p-1)_{2r}}{(2r)!} \int_0^1 \left(\frac{1}{\sqrt{(1+t^2)}}\right)^{2p+1} t^{2r} \frac{dt}{1+t^2}$$

$$(3.18)\quad = 2\sum_{r=0}^{p} \frac{(-2p-1)_{2r}}{(2r)!} \int_0^1 t^{2r} (1+t^2)^{\frac{-(2p+3)}{2}} dt.$$

Recalling binomial theorem (1.6) on right hand side of equation (3.18), we find

$$= 2\sum_{r=0}^{p} \frac{(-2p-1)_{2r}}{(2r)!} \int_0^1 t^{2r} {}_1F_0 \left[\begin{array}{ll} \frac{2p+3}{2}; & \\ -\;; & -t^2 \end{array}\right] dt$$

$$= 2\sum_{r=0}^{p} \frac{(-2p-1)_{2r}}{(2r)!} \sum_{s=0}^{\infty} \frac{\left(\frac{2p+3}{2}\right)_s (-1)^s}{s!} \int_0^1 t^{2r+2s} dt$$

$$= 2\sum_{r=0}^{p} \frac{(-2p-1)_{2r}}{(2r)!} \sum_{s=0}^{\infty} \frac{\left(\frac{2p+3}{2}\right)_s (-1)^s}{\{1+2(r+s)\}\ s!}$$

$$= 2\sum_{r=0}^{p} \frac{(-2p-1)_{2r}}{(2r)!} \sum_{s=0}^{\infty} \frac{\left(\frac{2p+3}{2}\right)_s (-1)^s \left(\frac{1}{2}\right)_{r+s}}{s!\ \left(\frac{3}{2}\right)_{r+s}}$$

$$= 2\sum_{r=0}^{p} \frac{(-p)_r \left(\frac{-2p-1}{2}\right)_r}{\left(\frac{1}{2}\right)_r r!} \sum_{s=0}^{\infty} \frac{\left(\frac{2p+3}{2}\right)_s (-1)^s \left(\frac{1}{2}\right)_{r+s}}{s! \left(\frac{3}{2}\right)_{r+s}}$$

$$I = 2\sum_{r=0}^{p}\sum_{s=0}^{\infty} \frac{\left(\frac{1}{2}\right)_{r+s} (-p)_r \left(\frac{-2p-1}{2}\right)_r \left(\frac{2p+3}{2}\right)_s (-1)^s (1)^r}{\left(\frac{3}{2}\right)_{r+s} \left(\frac{1}{2}\right)_r r!\ s!}. \quad (3.19)$$

Finally, use the definition of Kampé de Fériet's double hypergeometric function (1.11) on the right hand side of equation (3.19), leads to the result.

$$I = 2\ F^{1:\ 2;\ 1}_{1:\ 1;\ 0}\left[\begin{matrix} \frac{1}{2}: & -p, & \frac{-2p-1}{2}; & \frac{2p+3}{2} & ; \\ & & & & & 1, & -1 \\ \frac{3}{2}: & & \frac{1}{2} & ; & - & ; \end{matrix}\right]$$

$$I = 2\ F^{1:\ 2;\ 1}_{1:\ 1;\ 0}\left[\begin{matrix} \frac{1}{2} & : & \frac{-\alpha}{2}, \frac{-\alpha+1}{2}; & \frac{\alpha+2}{2} & ; \\ & & & & & 1, & -1 \\ \frac{3}{2} & : & \frac{1}{2} \quad ; & - & ; \end{matrix}\right], \quad (3.20)$$

where $Re\,(\alpha) > -1$.

4 A Summation Formula for Kampé de fériet's Function

On comparing the right hand sides of equations (2.1) and (2.2), we get a summation formula for Kampé de Fériet's double hypergeometric function having the arguments ± 1, in terms of Gamma functions.

$$F^{1:\ 2;\ 1}_{1:\ 1;\ 0}\left[\begin{matrix} \frac{1}{2} & : & \frac{-\alpha}{2}, \frac{-\alpha+1}{2}; & \frac{\alpha+2}{2} & ; \\ & & & & & 1, & -1 \\ \frac{3}{2} & : & \frac{1}{2} \quad ; & - & ; \end{matrix}\right] = \frac{1}{\alpha} 2^{\frac{\alpha-2}{2}} \frac{\sqrt{\pi}\ \Gamma\left(\frac{\alpha}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)}, \quad (4.1)$$

where $Re\,(\alpha) > -1$.

5 Special Case of Main Integrals

Put $\alpha = \frac{1}{3}$ in equation (2.1) and after using Legendre's duplication formula (1.8) and relations (1.2) and (1.9), we obtain

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos\theta \pm \sin\theta)^{\frac{1}{3}} d\theta = \frac{(\pi)\Gamma(\frac{1}{3})}{2^{\frac{1}{6}}\Gamma(\frac{1}{6})\Gamma(\frac{7}{6})}. \quad (5.1)$$

6 Concluding Remarks

We conclude the present investigation by noting that certain definite integrals can be solved by gamma and hypergeometric functions approach . We remark that the results in this paper are quite significant and accurate. Moreover, the presented results are expected to lead some potential applications in several fields of mathematical, physical, statistical and engineering sciences.

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