Existence and Uniqueness of Solutions for Nonlinear Difference Equations with Summation Boundary Conditions

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Abstract

In this article, we study the existence of solutions to boundary value problems for difference equations

$$\Delta\chi(v) = f(v, \chi(v)), \qquad for\ v \in [0, V]$$

with two-point and Summation boundary conditions

$$A\chi(0) + \sum_{u=0}^{V-1} \xi(u)\chi(u) + B\chi(V) = \sum_{u=0}^{V-1} g(u, \chi(u)).$$

Existence and uniqueness results are obtained by using well known fixed point theorems. Some illustrative examples are also discussed.

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1 Introduction

Due to wide application in many fields such as science, economics, neural network, ecology, the theory of difference equations has been widely studied since 1970. Agrawal [1, 2], Kelley and Peterson[15] had developed a theory of difference equation and their applications. Later some Existence, Uniqueness and comparison results on difference equation and summation equation are obtained by K.L. Bondar etal.[3, 4, 5, 6]. G. C. Done, K. L. Bondar and P. U. Chopade, investigated the existence and uniqueness results for summation-difference type equations and the study of non-homogeneous first order nonlinear difference equation with nonlocal condition in cone metric spaces can be found in [7, 8, 9, 10]. The purpose of this paper is study the existence and uniqueness of the solutions of nonlinear difference equations of the type

$$\Delta\chi(v) = f(v, \chi(v)), \qquad for\ v \in [0, V] \tag{1.1}$$

with two point and summation boundary condition

$$A\chi(0) + \sum_{u=0}^{V-1} \xi(u)\chi(u) + B\chi(V) = \sum_{u=0}^{V-1} g(u, \chi(u)), \tag{1.2}$$

where $A, B \in \mathbb{R}^{n\times n}$ are given matrices , det $(A+\sum_{u=0}^{V-1} \xi(u)+B) \neq 0$; $f, g : [0, V]\times\mathbb{R}^n \to \mathbb{R}^n$, are given functins. By $C([0, V], \mathbb{R}^n)$ we denote the Banach space of all continuous functins from $[0, V]$ into $\mathbb{R}^n$ with the norm

$$\|\chi\| = max\{|\chi(v)| : v \in [0, V]\}$$

where $|.|$ is the norm in space $\mathbb{R}^n$.

We prove some new existence and uniqueness results by using a variety of fixed point theorems. It is worth mention that the methods used in this paper are standard. Our impact is implementation of these methods to the solution of the problem (1.1)- (1.2).

2 Preliminaries

We define a solution of the problem (1.1)-(1.2) as follows:

Definition 1. *A function $\chi \in C([0, V], \mathbb{R}^n)$ is said to be solution of problem (1.1)-(1.2) if $\Delta\chi(v) = f(v, \chi(v))$, and for each $v \in [0, V]$, the boundary condition (1.2) is satisfied.*

Lemma 1. *Let $\psi, g \in C([0, V], \mathbb{R}^n)$. Then the unique solution of the difference equation*

$$\Delta\chi(v) = \psi(v), \qquad v \in [0, V] \tag{2.1}$$

with boundary condition

$$A\chi(0) + \sum_{u=0}^{V-1} \xi(u)\chi(u) + B\chi(V) = \sum_{u=0}^{V-1} g(u) \tag{2.2}$$

is given by

$$\chi(v) = C + \sum_{\vartheta=0}^{V-1} K(v, \vartheta)\psi(\vartheta) \tag{2.3}$$

where

$$K(v, \vartheta) = \begin{cases} \Gamma^{-1}(A + \sum_{\vartheta=0}^{v-1} \xi(\vartheta)), & 0 \leq \vartheta \leq v \\ -\Gamma^{-1}(\sum_{\vartheta=v}^{v-1} \xi(\vartheta) + B), & v \leq \vartheta \leq V \end{cases}$$

$$C = \Gamma^{-1} \sum_{u=0}^{V-1} g(u)$$

$$\Gamma = (A + \sum_{v=0}^{V-1} \xi(v) + B).$$

Proof. If $\chi = \chi(.)$ is a solution of the difference equation (2.1), then for $v \in (0, V)$,

$$\chi(v) = \chi(0) + \sum_{\vartheta=0}^{v-1} \psi(\vartheta), \tag{2.4}$$

where $\chi(0)$ is an arbitrary constant vector. In order to determine $\chi(0)$ we require that the function in equality (2.1) should satisfy condition (2.2), i.e.,

$$\Gamma\chi(0) = \sum_{u=0}^{V-1} g(u) - \sum_{v=0}^{V-1} \xi(v) \sum_{\vartheta=0}^{v-1} \psi(\vartheta) - B \sum_{v=0}^{V-1} \psi(v).$$

Since $\det \Gamma \neq 0$, we have

$$\chi(0) = C + \Gamma^{-1} \sum_{v=0}^{V-1} \sum_{\vartheta=v}^{V-1} \xi(\vartheta)\psi(v) - \Gamma^{-1} B \sum_{v=0}^{V-1} \psi(v). \tag{2.5}$$

Now in (2.4) we take into account the value $\chi(0)$ determined from the equality (2.5) and obtain

$$\chi(v) = C + \sum_{\vartheta=0}^{V-1} K(v, \vartheta)\psi(\vartheta)$$

Thus we have proved that one can write the boundary-value problem (2.1), (2.2) as the summation equation (2.3). One can immediately verify that a solution to the summation equation (2.3) also satisfies the boundary-value problem (2.1)-(2.2). □

Lemma 2. *Assume that $f, g \in C([0, V] \times \mathbb{R}^n, \mathbb{R}^n)$. Then the function $\chi(v)$ is a solution of the boundary-value problem (1.1)-(1.2) if and only if $\chi(v)$ is a solution of the summation equation*

$$\chi(v) = \sum_{u=0}^{V-1} K(v, u) f(u, \chi(u)) + \Gamma^{-1} \sum_{u=0}^{V-1} g(u, \chi(u)) \tag{2.6}$$

Proof. Let $\chi(v)$ be a solution of the boundary-value problem (1.1), (1.2). Then in the same way as in Lemma 2.1, we can prove that it is also a solution of the summation equation (2.6). By direct verification we can show that the solution of the summation equation (2.6) also satisfies equation (1.1) and non-local boundary condition (1.2). Lemma 2.2 is proved. □

3 Main Results

Define the operator $P : C([0,V],\mathbb{R}^n) \to P([0,V],\mathbb{R}^n)$ as

$$P\chi(v) = \Gamma^{-1}\sum_{u=0}^{V-1} g(u,\chi(u)) + \sum_{u=0}^{V-1} K(v,u)f(u,\chi(u)) \tag{3.1}$$

Obviously, the problem (1.1) – (1.2) is equivalent to the fixed point problem $\chi = P\chi$. In consequence, problem (1.1) – (1.2) has a solution if and only if the operator P has a fixed point.
Our first result is based on the Banach fixed point theorem. It uses the assumptions:

(H1) There exists a continuous function $N(v) > 0$ such that ,

$$|f(v,\chi) - f(v,\psi)| \le N(v)|\chi - \psi|,$$

for each $v \in [0,V]$ and all $\chi,\psi \in \mathbb{R}^n$;

(H2) There exists a continuous function $M(v) > 0$ such that ,

$$|g(v,\chi) - g(v,\psi)| \le M(v)|\chi - \psi|,$$

for each $v \in [0,V]$ and all $\chi,\psi \in \mathbb{R}^n$;

Theorem 3.1. *Assume (H1), (H2) hold, and*

$$L = V[SN + M]\|\Gamma^{-1}\| < 1. \tag{3.2}$$

Then the boundary-value problem (1.1)-(1.2) has a unique solution on $[0,V]$*, where*

$$N = \max_{[0,V]} N(v), \qquad M = \max_{[0,V]} M(v)$$

$$S = max\{\|(A + \sum_{\vartheta=0}^{v-1}\xi(\vartheta))\|, \|\sum_{\vartheta=v}^{V-1}\xi(\vartheta) + B\|\}$$

.

Proof. Let $\max_{[0,V]}|f(v,0)| = M_f$,$\max_{[0,V]}|g(v,0)| = M_g$ and choosing

$$\left[1 - \|\Gamma^{-1}\|V(SN+M)\right]^{-1}\|\Gamma^{-1}\|(M_f + M_g) \le r,$$

we show that $PB_r \subset B_r$, where

$$B_r = \{\chi \in C([0,V],\mathbb{R}^n) : \|\chi\| \le r\}.$$

For $\chi \in B_r$, we have

$$\|(P_\chi)(v)\| \le \max_{[0,V]}\Big[\sum_{u=0}^{V-1}|K(v,u)||f(u,\chi(u))|\Big] + \Big[\|\Gamma^{-1}\|\sum_{u=0}^{V-1}|g(u,\chi(u))|\Big]$$

$$\leq \max_{[0,V]} \Big[\sum_{u=0}^{V-1} |K(v,u)|(|f(u,\chi(u)) - f(u,0)| + |g(s,0)|) \Big]$$

$$+\|\Gamma^{-1}\| \sum_{u=0}^{V-1} (|g(u,\chi(u)) - g(u,0)| + |g(s,0)|)$$

$$\leq \|\Gamma^{-1}\| S(Nr + M_f)V + \|\Gamma^{-1}\|(M_r + M_g)V \leq r.$$

Now, for any $x, y \in B_r$ we have

$$|(Px)(v) - (Py)(v)|$$

$$\leq \|\Gamma^{-1}\| \sum_{v=0}^{V-1} |g(v,x(v)) - g(v,y(v))| + \sum_{\vartheta=0}^{V-1} |K(v,\vartheta)||f(v,x(\vartheta)) - f(v,y(\vartheta))|$$

$$\leq \|\Gamma^{-1}\| \sum_{v=0}^{V-1} M(v)|x(v) - y(v)| + \|\Gamma^{-1}\| S \sum_{v=0}^{V-1} N(v)|x(v) - y(v)|$$

$$\leq \|\Gamma^{-1}\| [M + NS]V \|x - y\|$$

or

$$\|Px - Py\| \leq L\|x - y\|. \tag{3.3}$$

From condition (3.2) it follows that $\|Px - Py\| \leq \|x - y\|$. Therefore, P is a contraction in B_r. Therefore, in view of the contraction principle the operator P defined by (3.1) has a unique fixed point in $C([0,V], \mathbb{R}^n)$. Consequently, the summation equation (2.6) (or the boundary-value problem (1.1)-(1.2) has a unique solution. □

The second result is based on Schaefer's fixed point theorem. It uses the assumptions:

(H3) The function $f : [0,V] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous;

(H4) There exists a constant $N_1 > 0$ such that $|f(v,\chi)| \leq N_1$ for each $v \in [0,V]$ and all $\chi \in \mathbb{R}^n$

(H5) The function $g : [0,V] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous;

(H6) There exists a constant $N_2 > 0$ such that $|g(v,\chi)| \leq N_2$ for each $v \in [0,V]$ and all $\chi \in \mathbb{R}^n$.

Theorem 3.2. *Assume (H3)-(H6). Then the boundary value problem (1.1)-(1.2) has at least one solution on* $[0,V]$.

Proof. We divide the proof into several main steps in which we show that under the assumptions of the theorem, the operator P has a fixed point.

Step 1. The operator P under the assumptions of the theorem is continuous. Let χ_n be a sequence such that $\chi_n \to \chi$ in $C([0,V],\mathbb{R}^n)$. Then for any $v \in (0,V)$,

$$\begin{aligned}
&|(P\chi_n)(v) - (P\chi)(v)| \\
&\leq \|\Gamma^{-1}\| \sum_{v=0}^{V-1} |g(v,\chi_n(v)) - g(v,\chi(v))| \\
&\quad + \sum_{\vartheta=0}^{V-1} |K(v,\vartheta)||f(v,\chi_n(\vartheta)) - f(v,\chi(\vartheta))| \\
&\leq VM\|\Gamma^{-1}\| \max_{[0,V]} |g(v,\chi_n(v)) - g(v,\chi(v))| \\
&\quad + VNS\|\Gamma^{-1}\| \max_{[0,V]} |f(v,\chi_n(\vartheta)) - f(v,\chi(\vartheta))|
\end{aligned}$$

Since f and g are continuous functions, we have

$$\|(P\chi_n)(v) - (P\chi)(v)\| \to 0, \quad \text{as } n \to \infty$$

Step 2. The operator P maps bounded sets into $C([0,V],\mathbb{R}^n)$. Indeed, it is sufficient to show that for any $\eta > 0$, there exists a positive constant l such that for each $\chi \in B_\eta = \{\chi \in C([0,V],\mathbb{R}^n) : \|\chi\| \leq \eta\}$, we have $\|P(\chi)\| \leq l$. By (H4) and (H6) we have for each $v \in [0,V]$,

$$|P(\chi)(v)| \leq \sum_{u=0}^{V-1} |K(v,u)||f(u,\chi(u))| + \|\Gamma^{-1}\| \sum_{u=0}^{V-1} |g(u,\chi(u))|$$

hence,

$$|P(\chi)(v)| \leq \|\Gamma^{-1}\|SVN_1 + \|\Gamma^{-1}\|VN_2$$

thus

$$\|P(\chi)(v)\| \leq \|\Gamma^{-1}\|SVN_1 + \|\Gamma^{-1}\|VN_2 = l$$

Step 3. The operator P maps bounded sets in to equi-continuous sets of $C([0,V],\mathbb{R}^n)$. Let $v_1, v_2 \in [0,V]$, $v_1 < v_2, B_\eta$ be a bounded set of $C([0,V],\mathbb{R}^n)$ as in Step 2, and let $x \in B_\eta$. Then

$$\begin{aligned}
&|P(\chi)(v_2) - P(\chi)(v_1)| \\
&= |\sum_{u=0}^{V-1} [K(v_2,u) - K(v_1,u)] f(u,\chi(u)| \\
&\leq |\Gamma^{-1} \sum_{v=v_1}^{v_2-1} [A + \sum_{\vartheta=0}^{v-1} \xi(\vartheta)] f(v,\chi(v)) + \Gamma^{-1} \sum_{v=v_1}^{v_2-1} [B + \sum_{\vartheta=0}^{V-1} \xi(\vartheta)] f(v,\chi(v))|
\end{aligned}$$

$$\leq 2S\|\Gamma^{-1}\| \sum_{v=v_1}^{v_2-1} |f(v,\chi(v))|$$

As $v_1 \to v_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzela-Ascoli theorem, we can conclude that $P : C([0,V],\mathbb{R}^n) \to C([0,V],\mathbb{R}^n)$ is completely continuous.

Step 4. A priori bounds. Now it remains to show that the set

$$\Lambda = \{\chi \in C([0,V],\mathbb{R}^n) : \chi = \lambda P(\chi), for\ some\ \ 0 < \lambda < 1\}$$

is bounded
Let $\chi = \lambda(P\chi)$ for some $0 < \lambda < 1$. Thus, for each $v \in [0,V]$ we have

$$\chi(v) = \lambda\Big[\sum_{u=0}^{V-1} K(v,u) f(u,\chi(u)) + \Gamma^{-1}\sum_{u=0}^{V-1} g(u,\chi(u))\Big]$$

This implies by (H4) and (H6) (as in Step 2) that for each $v \in [0,V]$,

$$|P(\chi)(v)| \leq \|\Gamma^{-1}\|(SN_1 + N_2)V$$

Thus, for every $v \in [0,V]$ we have

$$\|\chi\| \leq \|\Gamma^{-1}\|(SN_1 + N_2)V$$

This shows that the set Λ is bounded. As a consequence of Schaefer's fixed point theorem, we deduce that P has a fixed point which is a solution of the problem (1.1)-(1.2). □

In the following theorem we give an existence result for the problem (1.1)-(1.2) by means of an application of the Leray-Schauder type nonlinear alternative, where the conditions (H4) and (H6) are weakened.

(H7) There exist $\theta_f \in L_1([0,V],\mathbb{R}^+)$ and continuous and nondecreasing $\phi_f : [0,\infty) \to [0,\infty)$ such that
$$|f(v,\chi)| \leq \theta_f(v)\phi_f(|\chi|),$$
for each $v \in [0,V]$ and all $\chi \in \mathbb{R}$;

(H8) There exist $\theta_g \in L^1([0,V],\mathbb{R}^+)$ and continuous and nondecreasing $\phi_g : [0,\infty) \to [0,\infty)$ such that
$$|g(v,\chi)| \leq \theta_g(v)\phi_g(|\chi|),$$
for each $v \in [0,V]$ and all $\chi \in \mathbb{R}$;

(H9) There exists a number $K > 0$ such that

$$\frac{K}{\|\Gamma^{-1}\|S\phi_f(K)\|\theta_f\|_{L_1} + \phi_g(K)\|\Gamma^{-1}\|\|\theta\|_{L_1}} > 1$$

Theorem 3.3. *Assume that (H3), (H5), (H7)-(H9) hold. Then the boundary-value problem (1.1)-(1.2) has at least one solution on* $[0, V]$.

Proof. Consider the operator P defined above. It can be easily shown that P is continuous and completely continuous. For $\lambda \in [0,1]$ let χ be such that for each $v \in [0, V]$ we have $\chi(v) = \lambda(P\chi)(v)$. Then from (H7) and (H8), for each $v \in [0, V]$ we have

$$|\chi(v)| \leq \sum_{u=0}^{v-1} |K(v,u)|\theta_f(u)\phi(|\chi(u)|) + \|\Gamma^{-1}\| \sum_{u=0}^{V-1} \theta_g(u)\phi_g(|\chi(u)|)$$

$$\leq \|\Gamma^{-1}\| S\phi_f(\|\chi\|) \sum_{u=0}^{V-1} \theta_f(u) + \phi_g(|\chi(u)|)\|\Gamma^{-1}\| \sum_{u=0}^{V-1} \theta_g(u)$$

Thus

$$\frac{\|\chi\|}{\|\Gamma^{-1}\| S\phi_f(\|\chi\|)\|\theta_f\|_{L_1} + \phi_g(\|\chi\|)\|\Gamma^{-1}\|\|\theta\|_{L_1}} \leq 1$$

Then, in view of (H9), there exists K such that $\|\chi\| \neq K$. Let us set

$$U = \{\chi \in C[0, V], R) : \|\chi\| < K\}.$$

Note that the operator $P : \bar{U} \to C([0, V], \mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $\chi \in \partial U$ such that $\chi = \lambda P(\chi)$ for some $\lambda \in (0, 1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [11], we deduce that P has a fixed point χ in $\bar{U}$ which is a solution of the problem (1.1)-(1.2). This completes the proof. □

4 Applications

In this section, we give examples to illustrate the usefulness of our main results.

Example 4.1. Let us consider the following nonlocal boundary-value problem for the system of difference equations

$$\Delta\chi_1(v) = 0.01 \sin \chi_2, \quad v \in [0, 2],$$

$$\Delta\chi_2(v) = \frac{|\chi_1|}{(9 + e^v)(1 + |\chi_1|)}, \tag{4.1}$$

with

$$\chi_1(0) = 1, \quad \chi_2(1) = \sum_{v=0}^{1} 2v\chi_1(v). \tag{4.2}$$

We can rewrite the boundary conditions (4.2) in the equivalent form:

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\chi_1(0)\\ \chi_2(0)\end{pmatrix} - \sum_{v=0}^{1}\begin{pmatrix}0 & 0\\ 2v & 0\end{pmatrix}\begin{pmatrix}\chi_1(v)\\ \chi_2(v)\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\chi_1(1)\\ \chi_2(1)\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}.$$

Obviously,

$$\Gamma = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} - \sum_{v=0}^{1}\begin{pmatrix}0 & 0\\ 2v & 0\end{pmatrix} + \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix},$$

and the matrix Γ is invertible.

Evidently, $\Gamma^{-1} = \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$, $\|\Gamma^{-1}\| = 3$

Hence, the conditions (H1)-(H2) hold with $N = 0.01$, $M = 0, S = 3$. We can easily see that the condition (3.2) is satisfied. Indeed,

$$L = \|\Gamma^{-1}\|SNV = 30 \times 0.01 = 0.3 < 1 \tag{4.3}$$

Then, by Theorem 3.1 the boundary-value problem (4.1)-(4.2) has a unique solution on $[0, 2]$.

Note: The matrix norm is defined as maximum absolute row sum.

Example 4.2. On [0, 2] we consider the boundary-value problem

$$\Delta\chi_1 = \sin\chi_2,$$

$$\Delta\chi_2 = \cos\chi_1,$$

$$\Delta\chi_3 = \frac{1}{3}\Big(\frac{1}{1+\chi_1^2} + \frac{1}{1+\chi_2^2} + \frac{1}{1+\chi_3^2}\Big) \tag{4.4}$$

with

$$\chi_1(0) = 0, \quad \chi_2(0) + \sum_{v=0}^{1} v\chi_3(v) = 2, \quad \chi_3(1) = 1. \tag{4.5}$$

Obviously, the matrix

$$\Gamma = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} + \sum_{v=0}^{1}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & t\\ 0 & 0 & 0\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$$

is invertible, and the function

$$\begin{pmatrix}f_1\\ f_2\\ f_3\end{pmatrix} = \begin{pmatrix}\sin\chi_2\\ \cos\chi_1\\ \frac{1}{3}\Big(\frac{1}{1+\chi_1^2} + \frac{1}{1+\chi_2^2} + \frac{1}{1+\chi_3^2}\Big)\end{pmatrix}$$

is continuous and bounded. Hence, by Theorem 3.2 the boundary-value problem (4.4)-(4.5) has at least one solution in [0, 2].

References

[1] R. P. Agarwal; Difference Equations and Inequalities: Theory, Methods and Applications, Marcel Dekker, New York(1991).

[2] R.P. Agarwal, Focal Boundary Value Problems for Differential and Difference Equations, Kluwer Academic Publishers, Dordrecht, 1998.

[3] Bondar. K. L., Existence and uniqueness of results for first order difference equations, Journal of Modern Methods in Numerical Mathematics 2 (1), 2

[4] Bondar. K. L., Borkar V. C. , Patil S. T. , Some existence and uniqueness results for difference boundary value problems Bulletin of Pure and Applied Sciences Mathematics and Statistics 29 (2), 291-296.

[5] Bondar. K. L., Some scalar difference inequalities, Applied Mathematical Sciences 5 (60), 2951-2956.

[6] Bondar. K. L., Some comparison results for first order difference equation, International Journal of Contemporary Mathematical Sciences 6, 1855-1860.

[7] Done. G. C., Bondar. K. L., Chopade. P. U., Existence and Uniqueness Results on Mixed Type Summation-Difference Equations in Cone Metric Space, J. Math. Comput. Sci. 10 (2020), No. 4, 956-963

[8] Done. G. C., Bondar. K. L., Chopade. P. U., Existence and uniqueness of solution of first order nonlinear difference equation, J. Math. Comput. Sci. 10 (2020), No. 5, 1375-1383

[9] Done. G. C., Bondar. K. L., Chopade. P. U., Existence and Uniqueness of Solution of Summation-Difference Equation of Finite Delay in Cone Metric Space, Communications in Mathematics and Applications, Vol. 11, No. 3, pp. 325-334, (2020).

[10] Done. G. C., Bondar. K. L., Chopade. P. U.,Existence and Uniqueness of Solution of First Order Difference Equation of Finite Delay in Cone Metric Space, Advances in Mathematics: Scientific Journal **9** (2020), no.7, 4415-4424.

[11] Granas A., Dugundji J., Fixed Point Theory, Springer Monographs in Mathematics, Springer, New York, NY, USA, 2003..

[12] Henderson j., Thompson H.B., Existence of multiple solutions for second order discrete boundary value problems, Comput. Math. Appl. 43 (2002), 1239-1248..

[13] Ilin V. A. and Moiseev E. I., Nonlocal boundary-value problem of the rst kind for a Sturm-Liouville operator in its differential and finite difference aspects, J. Differential Equations 23(1987), 803-810..

[14] Reunsumrit J. and Sitthiwirattham T. ,Positive solutions of difference-summation boundary value problem for a second-order difference equation, J. COMPUTATIONAL ANALYSIS AND APPLICATIONS, VOL. 19, NO.2, (2015), 219-314

[15] Kelley W G and Peterson A C, Difference Equation, Academic Press(2001).

[16] Zhang G., Medina R., Three-point boundary value problems for difference equations, Comp. Math. Appl. 48(2004), 1791-1799.