Generalized form of Dunkl analogue of Szász operators

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Abstract

The content of present article is modified form of Szász operators via generalized form of exponential function in form of Dunkl modification. We study the error of estimation via different approximation tools like first order modulus of continuity, second order modulus of continuity using Steklov mean. Furthermore, we obtain the error of approximation for functions belonging to class of functions of derivatives of bounded variation.

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1 Introduction

For $\mu \geq 0,\ n \in \mathbb{N}$, Sucu [13] characterized the accompanying operator created with the help of generalized form of exponential function as

$$S_n^*(\mathfrak{f}; y) = \frac{1}{e_\mu(ny)} \sum_{p=0}^{\infty} \frac{(ny)^p}{\gamma_\mu(p)} \mathfrak{f}\left(\frac{p + 2\mu\theta_p'}{n}\right), \tag{1.1}$$

where $y \geq 0$ and $\mathfrak{f} \in C[0, \infty)$. The operator characterized by (1.1) is named as Dunkl form of Szász operators.
$e_\mu(z) = \sum_{p=0}^{\infty} \frac{z^p}{\gamma_\mu(p)}$ is modified form of exponential function defined by Rosenblum [12] using generalization of factorial γ_μ defined as:

$$\gamma_\mu(2p) = \frac{2^{2p} p! \Gamma(p + \mu + 1/2)}{\Gamma(\mu + 1/2)}$$

and

$$\gamma_\mu(2p + 1) = \frac{2^{2p+1} p! \Gamma(p + \mu + 3/2)}{\Gamma(\mu + 1/2)}$$

for $p \in \mathbb{N}_0$ and $\mu > -1/2$.
The underneath recurrence relation is fulfilled by γ_μ,

$$\gamma_\mu(2p+1) = (p+1+2\mu\theta'_{p+1})\gamma_\mu(p), \quad p \in \mathbb{N}_0, \tag{1.2}$$

İcöz and Çekim ([5],[6]) enlarged the approach of Dunkl generalization of linear positive operators in q-calculus and measured the approximation properties of Dunkl modification of some linear positive operators. Including a sequence $r_n(x) = x - \frac{1}{2n},\ n \in \mathbb{N}$, Mursaleen et al. [8] presented Dunkl generalization of Sz*á*sz operators and established some direct results for estimate properties of these operators. Afterwards, many researchers considered Dunkl modification for estimation theory for linear positive operators, refer to [[1],[9], [10], [15]] and reference therein. Deshwal et al. [7] used the operators defined by Paltanea [11] to construct a Dunkl generalization and set up some immediate outcomes for approximation properties of these operators. In 2019, Taşdelen et al. [14] motivated themself in this approach to define Dunkl-Gamma type operators including Appell polynomials and studied approximation properties of these operators.
The present article contains the generalized case of the operators (1.1). In consequence, we assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are increasing unbounded sequences of positive real numbers such that $\dfrac{\beta_n}{\alpha_n} \to 0$, as $n \to \infty$. The article is categorized as:

- In module 2, the modified form of operator (1.1) is constructed and calculated the values of polynomial functions and moments for the same.
- In module 3, our intention is to scrutinize estimated results for operators defined in module 1 via different tools of approximation process like modulus of continuity, the second order modulus of continuity via Steklov mean. We discuss the rate of convergence of the operators for the functions having a derivative of bounded variation on $[0, \infty)$.

2 Construction of operators and moments estimates

For $\tau \geq 0,\ n \in \mathbb{N},\ y \geq 0$ and $\mathfrak{f} \in C_\tau[0,\infty) := \{f \in C[0,\infty) : |\mathfrak{f}(w)| \leq \mathcal{M}_\mathfrak{f}(1+w^\tau)$, favoured with the norm $||\mathfrak{f}|| = \sup_{0\leq w<\infty} \dfrac{|\mathfrak{f}(w)|}{1+w^\tau}$. Here, $\mathcal{M}_\mathfrak{f}$ is a constant depends only on choice of function $\mathfrak{f}$. We instigated the generalized form of the operators (1.1) as

$$\mathcal{S}_n^{\mathcal{G}}(\mathfrak{f}; y) = \frac{1}{e_\mu(\frac{\alpha_n}{\beta_n}y)} \sum_{p=0}^{\infty} \frac{\left(\frac{\alpha_n}{\beta_n}y\right)^p}{\gamma_\mu(p)} \mathfrak{f}\left(\frac{(p+2\mu\theta'_p)\beta_n}{\alpha_n}\right). \tag{2.1}$$

The operator (2.1) may also be written in the following form

$$\mathcal{S}_n^{\mathcal{G}}(\mathfrak{f}; y) = \int_0^\infty \mathfrak{f}(s) \frac{\partial}{\partial s} \mathcal{Q}_{n,\mu}(y,s) ds \tag{2.2}$$

where,

$$\mathcal{Q}_{n,\mu}(y,s)=\begin{cases}\displaystyle\sum_{\frac{(p+2\mu\theta_p')\beta_n}{\alpha_n}\le s} e_\mu^{-1}\left(\frac{\alpha_n}{\beta_n}y\right)\frac{\left(\frac{\alpha_n}{\beta_n}y\right)^p}{\gamma_\mu(p)}, & \text{if } 0<s<\infty,\\ 0, & \text{if } s=0.\end{cases}$$

Lemma 1. *For each and every $y\in[0,\infty)$ and sufficiently large n, we have the following inequalities:*

(i) $\theta_{n,\mu}(y,w)=\int_0^w \frac{\partial}{\partial s}\mathcal{Q}_{n,\mu}(y,s)ds\le \dfrac{1}{(y-w)^2}(1+2\mu)\dfrac{\beta_n y}{\alpha_n},\quad 0\le w\le y,$

(ii) $1-\theta_{n,\mu}(y,z)=\int_z^\infty \frac{\partial}{\partial s}\mathcal{Q}_{n,\mu}(y,s)ds\le \dfrac{1}{(z-y)^2}(1+2\mu)\dfrac{\beta_n y}{\alpha_n},\quad y\le z<\infty.$

Lemma 2. *$\mathcal{S}_n^G(.;y)$ satisfies the following equalities:*

(i) $\mathcal{S}_n^{\mathcal{G}}(1;y)=1,$

(ii) $\mathcal{S}_n^{\mathcal{G}}(w;y)=y,$

(iii) $\mathcal{S}_n^{\mathcal{G}}(w^2;y)=y^2+\left(2\mu\dfrac{e_\mu(\frac{-\alpha_n y}{\beta_n})}{e_\mu(\frac{\alpha_n y}{\beta_n})}+1\right)\dfrac{\beta_n y}{\alpha_n},$

(iv) $\mathcal{S}_n^{\mathcal{G}}(w^3;y)=y^3+\left(3-2\mu\dfrac{e_\mu(\frac{-\alpha_n y}{\beta_n})}{e_\mu(\frac{\alpha_n y}{\beta_n})}\right)\dfrac{\beta_n y^2}{\alpha_n}+\left(1+4\mu^2+4\mu\dfrac{e_\mu(\frac{-\alpha_n y}{\beta_n})}{e_\mu(\frac{\alpha_n y}{\beta_n})}\right)\dfrac{\beta_n^2 y^2}{\alpha_n^2},$

(v) $\mathcal{S}_n^{\mathcal{G}}(w^4;y)=y^4+\left(6+4\mu\dfrac{e_\mu(\frac{-\alpha_n y}{\beta_n})}{e_\mu(\frac{\alpha_n y}{\beta_n})}\right)+\left(7+4\mu^2-8\mu\dfrac{e_\mu(\frac{-\alpha_n y}{\beta_n})}{e_\mu(\frac{\alpha_n y}{\beta_n})}\right)\dfrac{\beta_n^2 y}{\alpha_n^2}+\Bigg(1+12\mu^2+$
$2\mu(3+4\mu^2)\dfrac{e_\mu(\frac{-\alpha_n y}{\beta_n})}{e_\mu(\frac{\alpha_n y}{\beta_n})}\Bigg)\dfrac{\beta_n^3 y}{\alpha_n^3}.$

Proof. By the importance of operators $\mathcal{S}_n^{\mathcal{G}}(.;y)$ and $e_\mu(z)$, one can undoubtedly get Lemma 2. Hence details are omitted. □

$\theta_{n,m}(y)=\mathcal{S}_n^{\mathcal{G}}((w-y)^m;y)$ is used as m-th order moment for the operator $\mathcal{S}_n^{\mathcal{G}}(.;y)$ throughout the article.

As a result of Lemma 2, we obtain:

Lemma 3. *For the operators $\mathcal{S}_n^G(.;y)$ the following equalities hold:*

(i) $\theta_{n,1}(y)=\mathcal{S}_n^{\mathcal{G}}(w-y;y)=0;$

(ii) $\theta_{n,2}(y)=\mathcal{S}_n^{\mathcal{G}}((w-y)^2;x)=\left(1+2\mu\dfrac{e_\mu(\frac{-\alpha_n y}{\beta_n})}{e_\mu(\frac{\alpha_n y}{\beta_n})}\right)\dfrac{\beta_n y}{\alpha_n}\le\dfrac{\beta_n}{\alpha_n}(1+2\mu)y;$

(iii) $\theta_{n,4}(y) = \mathcal{S}_n^{\mathcal{G}}((w-y)^4;x) = 24\mu \frac{e_\mu(\frac{-\alpha_n y}{\beta_n})}{e_\mu(\frac{\alpha_n y}{\beta_n})} \frac{\beta_n y^3}{\alpha_n} + \left(3 - 12\mu^2 - 24\mu \frac{e_\mu(\frac{-\alpha_n y}{\beta_n})}{e_\mu(\frac{\alpha_n y}{\beta_n})}\right) \frac{\beta_n^2 y^2}{\alpha_n^2} + \left(1 + 12\mu^2 + 2\mu(3+4\mu^2) \frac{e_\mu(\frac{-\alpha_n y}{\beta_n})}{e_\mu(\frac{\alpha_n y}{\beta_n})}\right) \frac{\beta_n^3 y}{\alpha_n^3} \leq 24\mu \frac{\beta_n}{\alpha_n} y^3 + 3\frac{\beta_n^2}{\alpha_n^2} y^2 + (1 + 2\mu^2 + 6\mu) \frac{\beta_n^3}{\alpha_n^3} y.$

Proof. Taking linearity property of operators $\mathcal{S}_n^{\mathcal{G}}(.;y)$, Lemma 2 and definition of $\mathcal{S}_n^{\mathcal{G}}(.;y)$ into account, one can easily arrive at conclusions. □

Keeping in mind Lemma 3, we obtain:

Lemma 4. *Let* $\frac{e_\mu(-\frac{\alpha_n y}{\beta_n})}{e_\mu(\frac{\alpha_n y}{\beta_n})} \to l_{1,\mu}(y)$ *as* $\frac{\alpha_n}{\beta_n} \to \infty$ *and* $\frac{\alpha_n}{\beta_n} \frac{e_\mu(-\frac{\alpha_n y}{\beta_n})}{e_\mu(\frac{\alpha_n y}{\beta_n})} \to l_{2,\mu}(y)$ *as* $\frac{\alpha_n}{\beta_n} \to \infty$. *Then for every* $y \in [0,\infty)$, *we have*

(i) $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} \mathcal{S}_n^{\mathcal{G}}((w-y)^2;x) = (1 + 2\mu l_{1,\mu}(y))y;$

(ii) $\lim_{n\to\infty} \frac{\alpha_n^2}{\beta_n^2} \mathcal{S}_n^{\mathcal{G}}((w-y)^4;x) = 24\mu x^3 l_{2,\mu}(y) + (3 - 12\mu^2 - 24\mu l_{1,\mu}(y))y^2.$

3 main results

Theorem 3.1. *For* $\mathfrak{f} \in C_\tau(\mathbb{R}^+)$ *and* $\mathcal{A}$ *compact subset of* $[0,\infty)$,

$$\lim_{n\to\infty} \mathcal{S}_n^{\mathcal{G}}(\mathfrak{f};y) = \mathfrak{f}(y),$$

uniformly on each $\mathcal{A}$.

Proof. In the light of Lemma 2,

$$\mathcal{S}_n^{\mathcal{G}}(w^i;y) \to y^i, \text{ as } n \to \infty, \text{ uniformly on } \mathcal{A}, \text{ for } i = 0,1,2.$$

As needs be, the fundamental result goes to by looking for the Bohman-Korovkin theorem. □

We consider $C_B[0,\infty)$ as a normed linear space of all bounded and uniformly continuous functions defined over $[0,\infty)$ favoured the norm $||\mathfrak{f}|| = \sup_{y\in[0,\infty)} |\mathfrak{f}(y)|$.
Succeeding we define

$$\omega(\mathfrak{f},\delta) = \sup_{t,v,w\in[0,\infty),|v-w|\leq\delta} |\mathfrak{f}(t+v) - f(t+w)|$$

and

$$\omega_2(\mathfrak{f},\delta) = \sup_{t,v,w\in[0,\infty),|v-w|\leq\delta} |f(t+2v) - 2f(t+v+w) + f(t+2w)|, \quad \delta > 0.$$

as the modulus of continuity of first and second order respectively. Contiguous, underneath theorem provides an error of estimation of operators $\mathcal{S}_n^{\mathcal{G}}(.;y)$ in expression of modulus of continuity of first order.

Theorem 3.2. *For* $\mathfrak{f} \in C_B[0,\infty)$, *the error of estimation of operator* $\mathcal{S}_n^{\mathcal{G}}(.;y)$ *in terms of* $\omega(\mathfrak{f};\delta)$, $\delta > 0$, *is given by*

$$|\mathcal{S}_n^{\mathcal{G}}(\mathfrak{f};y) - \mathfrak{f}(y)| \leq \left(1 + \sqrt{(1+2\mu)y}\right)\omega\left(\mathfrak{f}, \sqrt{\frac{\beta_n}{\alpha_n}}\right).$$

Proof. Alongside $\omega(\mathfrak{f};\delta)$, Lemma 3 and Cauchy-Schwarz inequality, we come by following inequality

$$\begin{aligned}
|\mathcal{S}_n^{\mathcal{G}}(\mathfrak{f};y) - \mathfrak{f}(y)| &\leq \mathcal{S}_n^{\mathcal{G}}(|\mathfrak{f}(v) - \mathfrak{f}(y)|;y) \\
&\leq \left(1 + \frac{1}{\delta}\mathcal{S}_n^{\mathcal{G}}(|v-y|;y)\right)\omega(\mathfrak{f};\delta) \\
&\leq \left(1 + \frac{1}{\delta}\sqrt{\theta_{n,2}(x);x}\right)\omega(\mathfrak{f};\delta) \\
&\leq \left(1 + \frac{1}{\delta}\sqrt{\frac{\beta_n}{\alpha_n}(1+2\mu)y}\right)\omega(\mathfrak{f};\delta).
\end{aligned}$$

Now, opting $\delta = \sqrt{\dfrac{\beta_n}{\alpha_n}}$, we immediate arrive to result. □

For $\mathfrak{f} \in C_B[0,\infty)$, the Steklov mean is characterized as

$$\mathfrak{f}_h(y) = \frac{4}{h^2}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}[2\mathfrak{f}(y+u+v) - \mathfrak{f}(y+2(u+v))]dvdw. \tag{3.1}$$

Remark 1. *[4] The following properties are fulfilled by the Steklov mean* $\mathfrak{f}_h(x)$ *:*

(i) $||\mathfrak{f}_h - \mathfrak{f}|| \leq \omega_2(\mathfrak{f},h)$,

(ii) $\mathfrak{f}'_h, \mathfrak{f}''_h \in C_B[0,\infty)$ *and*

$$||\mathfrak{f}'_h|| \leq \frac{5}{h}\omega(\mathfrak{f},h), \quad ||\mathfrak{f}''_h|| \leq \frac{9}{h^2}\omega_2(\mathfrak{f},h).$$

Next, we proceed the study of approximating the rate of convergence using operators by concerning the second order modulus of continuity with regards to steklov mean.

Theorem 3.3. *Let* $\mathfrak{f} \in C_B[0,\infty)$. *Then for each* $y \in [0,\infty)$, *we have*

$$|\mathcal{S}_n^{\mathcal{G}}(\mathfrak{f};y) - \mathfrak{f}(y)| \leq \left(2 + \frac{9}{2}\left\{(1+2\mu)y\right\}\right)\omega_2\left(\mathfrak{f};\sqrt{\frac{\beta_n}{\alpha_n}}\right).$$

Proof. By implementing Lemma 2 and Remark 1, one can come by following

$$|\mathcal{S}_n^{\mathcal{G}}(\mathfrak{f} - \mathfrak{f}_h;y)| \leq ||\mathfrak{f} - \mathfrak{f}_h|| \leq \omega_2(\mathfrak{f};h).$$

Using Taylor's expansion for $\mathfrak{f}_h^{''} \in C_B[0,\infty)$,

$$\mathfrak{f}_h(w) = \mathfrak{f}_h(x) + (w-x)\mathfrak{f}_h'(x) + \int_x^w (w-s)\mathfrak{f}_h''(s)ds.$$

Operating $\mathcal{S}_n^{\mathcal{G}}(.;y)$ on the upwards equality, we come into underneath inequality

$$|\mathcal{S}_n^{\mathcal{G}}(\mathfrak{f}_h(w) - \mathfrak{f}_h(y);y)| \le ||\mathfrak{f}_h'||\ |\mathcal{S}_n^{\mathcal{G}}(w-y;y)| + \frac{||\mathfrak{f}_h''||}{2}\mathcal{S}_n^{\mathcal{G}}((w-y)^2;y).$$

Hence using Lemma 3 and Lemma 1, we have

$$\begin{aligned}
|\mathcal{S}_n^{\mathcal{G}}(\mathfrak{f};y) - \mathfrak{f}(y)| &\le |\mathcal{S}_n^{\mathcal{G}}(\mathfrak{f} - \mathfrak{f}_h;y)| + |\mathcal{S}_n^{\mathcal{G}}(\mathfrak{f}_h - \mathfrak{f}_h(y);y)| + |\mathfrak{f}_h(y) - \mathfrak{f}(y)| \\
&\le \omega_2(\mathfrak{f};h) + \frac{||\mathfrak{f}_h^{''}||}{2}\left(\frac{\beta_n}{\alpha_n}(1+2\mu)y\right) + ||\mathfrak{f}_h - \mathfrak{f}|| \\
&\le \left(2 + \frac{9}{2h^2}\left\{\frac{\beta_n}{\alpha_n}(1+2\mu)y\right\}\right)\omega_2(\mathfrak{f};h).
\end{aligned}$$

Finally selecting $h = \sqrt{\dfrac{\beta_n}{\alpha_n}}$, the essential outcome is established. □

In the following theorem, we obtain the estimate of error in approximation for continuously differentiable functions.

Theorem 3.4. *For* $\mathfrak{f}^{'} \in C_B[0,\infty)$, *we have*

$$|\mathcal{S}_n^{\mathcal{G}}(\mathfrak{f};y) - \mathfrak{f}(y)| \le \omega\left(\mathfrak{f}^{'}; \frac{\beta_n}{\alpha_n}(1+2\mu)y\right)\left(1 + \sqrt{\frac{\beta_n}{\alpha_n}(1+2\mu)y}\right),$$

where $\omega(\mathfrak{f}^{'};\delta)$ *signifies the modulus of continuity of* $\mathfrak{f}^{'}$.

Proof. Since $\mathfrak{f}^{'} \in C_B[0,\infty)$, using mean value theorem , one may write

$$\begin{aligned}
\mathfrak{f}(w) &= \mathfrak{f}(y) + (w-y)\mathfrak{f}'(\zeta) \\
&= \mathfrak{f}(y) + (w-y)\mathfrak{f}^{'}(y) + (w-y)(\mathfrak{f}'(\zeta) - \mathfrak{f}'(y)),
\end{aligned}$$

where ζ lies between w and y.
Operating $\mathcal{S}_n^{\mathcal{G}}(.;y)$ to both sides of the upwards equality and afterward utilizing Lemma 3 and Cauchy-Schwarz inequality, we assure that

$$\begin{aligned}
|\mathcal{S}_n^{\mathcal{G}}(\mathfrak{f};y) - \mathfrak{f}(y)| &\le |\mathfrak{f}^{'}(y)||\mathcal{S}_n^{\mathcal{G}}(w-y;y)| + \mathcal{S}_n^{\mathcal{G}}(|w-y||\mathfrak{f}'(\zeta) - \mathfrak{f}'(y)|;y) \\
&\le \mathcal{S}_n^{\mathcal{G}}(|w-y||\mathfrak{f}'(\zeta) - \mathfrak{f}'(y)|;y) \\
&\le \mathcal{S}_n^{\mathcal{G}}\left(|w-y|\omega(f',\delta)\left(1 + \frac{|w-y|}{\delta}\right);y\right) \\
&\le \omega(\mathfrak{f}',\delta)\mathcal{S}_n^{\mathcal{G}}\left(|w-y| + \frac{(w-y)^2}{\delta};y\right) \\
&\le \omega(\mathfrak{f}',\delta)\sqrt{\theta_{n,2}(y)} + \frac{\omega(\mathfrak{f}',\delta)}{\delta}\theta_{n,2}(y).
\end{aligned}$$

Choosing $\delta = \dfrac{\beta_n}{\alpha_n}(1+2\mu)y$, we arrive to conclusion. $\square$

Next, we define $DBV[0,\infty)$ as the class of functions in $C_2[0,\infty)$, having a derivative of bounded variation on every finite subinterval of $[0,\infty)$.It transpires that for $\mathfrak{f} \in DBV[0,\infty)$ and a function $g(\tau)$ of bounded variation on each finite subinterval of $[0,\infty)$, we have

$$\mathfrak{f}(y) = \int_0^y g(\tau)d\tau + \mathfrak{f}(0),$$

Theorem 3.5. *Let $\mathfrak{f} \in DBV[0,\infty)$. Then for every $y \in (0,\infty)$ and sufficiently large n, we have*

$$\begin{aligned}
|\mathcal{S}_n^{\mathcal{G}}(\mathfrak{f};y) - \mathfrak{f}(y)| \leq\ & \left|\frac{\mathfrak{f}'(y+) - \mathfrak{f}'(y-)}{2}\right| \left\{\frac{(1+2\mu)\beta_n y}{\alpha_n}\right\}^{1/2} + \frac{y}{\sqrt{n}} \bigvee_{y-\frac{y}{\sqrt{n}}}^{y+\frac{y}{\sqrt{n}}} (\mathfrak{f}'_y) \\
& + \frac{(1+2\mu)\beta_n}{\alpha_n} \sum_{k=1}^{[\sqrt{n}\,]} \bigvee_{y-\frac{y}{k}}^{y+\frac{y}{k}} (\mathfrak{f}'_y) + |\mathfrak{f}'(y+)| \left\{\frac{(1+2\mu)\beta_n y}{\alpha_n}\right\}^{1/2} \\
& + |\mathfrak{f}(2y) - \mathfrak{f}(y) - y\mathfrak{f}'(y+)| \frac{(1+2\mu)\beta_n}{\alpha_n y} + (M(1+4y^2) + |\mathfrak{f}(y)|) \frac{(1+2\mu)\beta_n}{\alpha_n y},
\end{aligned}$$

where

$$\mathfrak{f}'_y(w) = \begin{cases} \mathfrak{f}'(w) - \mathfrak{f}'(y-), & 0 \leq w < y, \\ 0, & w = y, \\ \mathfrak{f}'(w) - \mathfrak{f}'(y+), & y < w < \infty. \end{cases}$$

$\bigvee_a^b(\mathfrak{f}'_y)$ is the total variation of $\mathfrak{f}'_y$ on $[a,b]$.

Proof. If $\mathfrak{f} \in DBV[0,\infty)$, then one can correspond to

$$\begin{aligned}
\mathfrak{f}'(w) =\ & \mathfrak{f}'(y-) + \frac{1}{2}\Big(\mathfrak{f}'(y+) + \mathfrak{f}'_y(w)\Big) + \frac{1}{2}\Big(\mathfrak{f}'(y+) - \mathfrak{f}'(y-)\Big) sgn(w-y) \\
& + \delta_y(w)\Big(\mathfrak{f}'(w) - \frac{1}{2}\Big(\mathfrak{f}'(y+) + \mathfrak{f}'(y-)\Big)\Big), \qquad (3.2)
\end{aligned}$$

where $\delta_y(w) = 1,$ for $w = y$ and $\delta_y(w) = 0,$ for $w \neq y$ Out of equations (2.2) and (3.2), we have

$$\begin{aligned}
\mathcal{S}_n^{\mathcal{G}}(\mathfrak{f};y) - \mathfrak{f}(y) &= \int_0^\infty \frac{\partial}{\partial w}\{\mathcal{Q}_{n,\mu}(y,w)\}\mathfrak{f}(w)dw - \mathfrak{f}(y) = \int_0^\infty (\mathfrak{f}(w) - \mathfrak{f}(y))\frac{\partial}{\partial w}\{\mathcal{Q}_{n,\mu}(y,w)\}dw \\
&= \int_0^y (\mathfrak{f}(w) - \mathfrak{f}(y))\frac{\partial}{\partial w}\{\mathcal{Q}_{n,\mu}(y,w)\}dw + \int_y^\infty (\mathfrak{f}(w) - \mathfrak{f}(y))\frac{\partial}{\partial w}\{\mathcal{Q}_{n,\mu}(y,w)\}dw \\
&= -\int_0^y \left(\int_w^y \mathfrak{f}'(u)du\right)\frac{\partial}{\partial w}\{\mathcal{Q}_{n,\mu}(y,w)\}dw + \int_y^\infty \left(\int_y^w \mathfrak{f}'(u)du\right)\frac{\partial}{\partial w}\{\mathcal{Q}_{n,\mu}(y,w)\}dw \\
&=: -\mathcal{A}_1(y,n,\mu) + \mathcal{A}_2(y,n,\mu),\ \ \textit{say}.
\end{aligned}$$

Make use of equation (3.2), we achieve

$$\mathcal{A}_1(y,n,\mu) = \int_0^y \Big\{ \int_w^y \Big(\frac{1}{2}\Big(\mathfrak{f}'(y+)+\mathfrak{f}'(y-)\Big) + \mathfrak{f}'_y(u) + \frac{1}{2}\Big(\mathfrak{f}'(y+)-\mathfrak{f}'(y-)\Big) sgn(u-y) + \delta_y(u)\Big(\mathfrak{f}'(u) - \frac{1}{2}\Big(\mathfrak{f}'(y+)+\mathfrak{f}'(y-)\Big)\Big)\Big) du \Big\} \frac{\partial}{\partial w}\{\mathcal{Q}_{n,\mu}(y,w)\}\, dw.$$

Considering $\int_y^w \delta_y(u)du = 0$,

$$\mathcal{A}_1(y,n,\mu) = \frac{1}{2}\Big(\mathfrak{f}'(y+)+\mathfrak{f}'(y-)\Big)\int_0^y (y-w)\frac{\partial}{\partial w}\{\mathcal{Q}_{n,\mu}(y,w)\}\, dw + \int_0^y \Big(\int_w^y \mathfrak{f}'_y(u)du\Big)\frac{\partial}{\partial w}\{\mathcal{Q}_{n,\mu}(y,w)\} - \frac{1}{2}\Big(\mathfrak{f}'(y+)-\mathfrak{f}'(y-)\Big)\int_0^y (y-w)\frac{\partial}{\partial w}\{\mathcal{Q}_{n,\mu}(y,w)\}\, dw. \tag{3.3}$$

In the same way, we can obtain

$$\mathcal{A}_2(y,n,\mu) = \frac{1}{2}\Big(\mathfrak{f}'(y+)+\mathfrak{f}'(y-)\Big)\int_y^\infty (w-y)\frac{\partial}{\partial w}\{\mathcal{Q}_{n,\mu}(y,w)\}\, dw + \int_y^\infty \Big(\int_y^w \mathfrak{f}'_y(u)du\Big)\frac{\partial}{\partial w}\{\mathcal{Q}_{n,\mu}(y,w + \frac{1}{2}\Big(\mathfrak{f}'(y+)-\mathfrak{f}'(y-)\Big)\int_y^\infty (w-y)\frac{\partial}{\partial w}\{\mathcal{Q}_{n,\mu}(y,w)\}\, dw. \tag{3.4}$$

Merging (3.3) and (3.4), we get

$$\begin{aligned}\mathcal{S}_n^{\mathcal{G}}(\mathfrak{f},y) - \mathfrak{f}(y) &= \frac{1}{2}\Big(\mathfrak{f}'(y+)+\mathfrak{f}'(y-)\Big)\int_0^\infty (w-y)\frac{\partial}{\partial w}\{\mathcal{Q}_{n,\mu}(y,w)\}\, dw \\ &+ \frac{1}{2}\Big(\mathfrak{f}'(y+)-\mathfrak{f}'(y-)\Big)\int_0^\infty |w-y|\frac{\partial}{\partial w}\{\mathcal{Q}_{n,\mu}(y,w)\}\, dw \\ &- \int_0^y \Big(\int_w^y \mathfrak{f}'_y(u)du\Big)\frac{\partial}{\partial w}\{\mathcal{Q}_{n,\mu}(y,w)\}\, dw + \int_y^\infty \Big(\int_y^w \mathfrak{f}'_y(u)du\Big)\frac{\partial}{\partial w}\{\mathcal{Q}_{n,\mu}(y,w)\}\, dw.\end{aligned}$$

Hence
$|\mathcal{S}_n^{\mathcal{G}}(\mathfrak{f},y) - \mathfrak{f}(y)|$

$$\begin{aligned}&\leq \left|\frac{\mathfrak{f}'(y+)+\mathfrak{f}'(y-)}{2}\right| |\mathcal{S}_n^{\mathcal{G}}(w-y;y)| + \left|\frac{\mathfrak{f}'(y+)-\mathfrak{f}'(y-)}{2}\right| \mathcal{S}_n^{\mathcal{G}}(|w-y|;y) \\ &+ \left|\int_0^y \Big(\int_w^y \mathfrak{f}'_y(u)du\Big)\frac{\partial}{\partial w}\{\mathcal{Q}_{n,\mu}(y,w)\}\, dw\right| + \left|\int_y^\infty \Big(\int_y^w \mathfrak{f}'_y(u)du\Big)\frac{\partial}{\partial w}\{\mathcal{Q}_{n,\mu}(y,w)\}\, dw\right|.\end{aligned} \tag{3.5}$$

Utilizing Lemma 1 and integration by parts, we achieve

$$\int_0^y \left(\int_w^y \mathfrak{f}'_y(u)du\right)\frac{\partial}{\partial w}\left\{\mathcal{Q}_{n,\mu}(y,w)\right\}dw = \int_0^y \left(\int_w^y \mathfrak{f}'_y(u)du\right)\frac{\partial}{\partial w}\theta_{n,\mu}(y,w)dw = \int_0^y \mathfrak{f}'_y(w)\theta_{n,\mu}(y,w)dw.$$

Thus,

$$\begin{aligned}\left|\int_0^y \left(\int_w^y \mathfrak{f}'_y(u)du\right)\frac{\partial}{\partial w}\left\{\mathcal{Q}_{n,\mu}(y,w)\right\}dw\right| &\leq \int_0^y |\mathfrak{f}'_y(w)|\theta_{n,\mu}(y,w)dw \\ &\leq \int_0^{y-\frac{y}{\sqrt{n}}} |\mathfrak{f}'_y(w)|\theta_{n,\mu}(y,w)dw + \int_{y-\frac{y}{\sqrt{n}}}^y |\mathfrak{f}'_y(w)|\theta_{n,\mu}(y,w)dw.\end{aligned}$$

Again, considering $\mathfrak{f}'_y(y)=0$ and $\theta_{n,\mu}(y,w)\leq 1$, we get

$$\begin{aligned}\int_{y-\frac{y}{\sqrt{n}}}^y |\mathfrak{f}'_y(w)|\theta_{n,\mu}(y,w)dw &= \int_{y-\frac{y}{\sqrt{n}}}^y |\mathfrak{f}'_y(w)-\mathfrak{f}'_y(y)|\theta_{n,\mu}(y,w)dw \leq \int_{y-\frac{y}{\sqrt{n}}}^y \bigvee_w^y(\mathfrak{f}'_y)dw \\ &\leq \bigvee_{y-\frac{y}{\sqrt{n}}}^y(\mathfrak{f}'_y)\int_{y-\frac{y}{\sqrt{n}}}^y dt = \frac{y}{\sqrt{n}}\bigvee_{y-\frac{y}{\sqrt{n}}}^y(\mathfrak{f}'_y).\end{aligned}$$

Similarly, applying Lemma 1 and putting $w=y-\frac{y}{u}$, we get

$$\begin{aligned}\int_0^{y-\frac{y}{\sqrt{n}}} |\mathfrak{f}'_y(w)|\theta_{n,\mu}(y,w)dt &\leq \frac{(1+2\mu)\beta_n y}{\alpha_n}\int_0^{y-\frac{y}{\sqrt{n}}}|\mathfrak{f}'_y(w)|\frac{dw}{(y-w)^2} \\ &\leq \frac{(1+2\mu)\beta_n y}{\alpha_n}\int_0^{y-\frac{y}{\sqrt{n}}}\bigvee_w^y(\mathfrak{f}'_y)\frac{dw}{(y-w)^2} \\ &= \frac{(1+2\mu)\beta_n}{\alpha_n}\int_1^{\sqrt{n}}\bigvee_{y-\frac{y}{u}}^y(\mathfrak{f}'_y)du \leq \frac{(1+2\mu)\beta_n}{\alpha_n}\sum_{k=1}^{[\sqrt{n}\,]}\bigvee_{y-\frac{y}{k}}^y(f'_y).\end{aligned}$$

On this account we have,

$$\left|\int_0^y \left(\int_w^y \mathfrak{f}'_y(u)du\right)\frac{\partial}{\partial w}\left\{\mathcal{Q}_{n,\mu}(y,w)\right\}dw\right| \leq \frac{y}{\sqrt{n}}\bigvee_{y-\frac{y}{\sqrt{n}}}^y(\mathfrak{f}'_y) + \frac{(1+2\mu)\beta_n}{\alpha_n}\sum_{k=1}^{[\sqrt{n}\,]}\bigvee_{y-\frac{y}{k}}^y(\mathfrak{f}'_y).$$

(3.6)

Also, we have

$$\left|\int_y^\infty\left(\int_y^w \mathfrak{f}_y'(u)du\right)\frac{\partial}{\partial w}\left\{\mathcal{Q}_{n,\mu}(y,w)\right\}dw\right|$$

$$\begin{aligned}
&\leq \left|\int_y^{2y}\left(\int_y^w \mathfrak{f}_y'(u)du\right)\frac{\partial}{\partial w}(1-\theta_{n,\mu}(y,w))dw\right| \\
&+ \left|\int_{2y}^\infty\left(\int_y^w \mathfrak{f}_y'(u)du\right)\frac{\partial}{\partial w}\left\{\mathcal{Q}_{n,\mu}(y,w)\right\}dw\right| \\
&\leq \left|\int_{2y}^\infty(\mathfrak{f}(w)-\mathfrak{f}(y))\frac{\partial}{\partial w}\left\{\mathcal{Q}_{n,\mu}(y,w)\right\}dt\right|+|\mathfrak{f}'(y+)|\left|\int_{2y}^\infty(w-y)\frac{\partial}{\partial w}\left\{\mathcal{Q}_{n,\mu}(y,w)\right\}dt\right| \\
&+ \left|\int_y^{2y}\mathfrak{f}_y'(u)du\right||1-\theta_{n,\mu}(y,2y)|+\int_y^{2y}|\mathfrak{f}_y'(w)|(1-\theta_{n,\mu}(y,w))dw.
\end{aligned} \tag{3.7}$$

We may write

$$\begin{aligned}
\int_y^{2y}|\mathfrak{f}_y'(w)|(1-\theta_{n,\mu}(y,w))dw &= \int_y^{y+\frac{y}{\sqrt{n}}}|\mathfrak{f}_y'(w)|(1-\theta_{n,\mu}(y,w))dw+\int_{y+\frac{y}{\sqrt{n}}}^{2y}|\mathfrak{f}_y'(w)|(1-\theta_{n,\mu}(y,w))dw \\
&=: I_1+I_2,\quad (\text{say}).
\end{aligned} \tag{3.8}$$

Since $\mathfrak{f}_y'(y)=0$ and $(1-\theta_{n,\mu}(y,w))\leq 1$, we have

$$\begin{aligned}
I_1 &= \int_y^{y+\frac{y}{\sqrt{n}}}|\mathfrak{f}_y'(w)-\mathfrak{f}_y'(y)|(1-\theta_{n,\mu}(y,w))dw \\
&\leq \int_y^{y+\frac{y}{\sqrt{n}}}\bigvee_y^{y+\frac{y}{\sqrt{n}}}(\mathfrak{f}_y')dw \\
&= \frac{y}{\sqrt{n}}\bigvee_y^{y+\frac{y}{\sqrt{n}}}(\mathfrak{f}_y').
\end{aligned} \tag{3.9}$$

Next, we estimate I_2. Applying Lemma 1 and putting $w = y + \frac{y}{u}$ we have

$$\begin{aligned} I_2 &= \int_{y+\frac{y}{\sqrt{n}}}^{2y} |\mathfrak{f}'_y(w)|(1-\theta_{n,\mu}(y,w))dw \\ &\leq \frac{(1+2\mu)\beta_n y}{\alpha_n} \int_{y+\frac{y}{\sqrt{n}}}^{2y} |\mathfrak{f}'_y(w)-\mathfrak{f}'_y(y)| \frac{dt}{(y-w)^2} \\ &\leq \frac{(1+2\mu)\beta_n y}{\alpha_n} \int_{y+\frac{y}{\sqrt{n}}}^{2y} \bigvee_y^w (\mathfrak{f}'_y) \frac{dw}{(y-w)^2} \\ &= \frac{(1+2\mu)\beta_n y}{\alpha_n} \int_1^{\sqrt{n}} \bigvee_y^{y+\frac{y}{u}} (\mathfrak{f}'_y) du \\ &\leq \frac{(1+2\mu)\beta_n y}{\alpha_n} \sum_{k=1}^{[\sqrt{n}\,]} \int_k^{k+1} \bigvee_y^{y+\frac{y}{u}} (\mathfrak{f}'_y) du \\ &\leq \frac{(1+2\mu)\beta_n}{\alpha_n} \sum_{k=1}^{[\sqrt{n}\,]} \bigvee_y^{y+\frac{y}{k}} (\mathfrak{f}'_y). \end{aligned} \tag{3.10}$$

Putting the values of I_1 and I_2 in (3.8), we get

$$\int_y^{2y} |\mathfrak{f}'_y(t)|(1-\theta_{n,\mu}(y,w))dt \leq \frac{y}{\sqrt{n}} \bigvee_y^{y+\frac{y}{\sqrt{n}}} (\mathfrak{f}'_y) + \frac{(1+2\mu)\beta_n}{\alpha_n} \sum_{k=1}^{[\sqrt{n}\,]} \bigvee_y^{y+\frac{y}{k}} (\mathfrak{f}'_y). \tag{3.11}$$

In view of (2.2), (3.7), (3.11) and Lemma 1, we get
$\left| \int_{2y}^{\infty} (\mathfrak{f}(w)-\mathfrak{f}(y)) \frac{\partial}{\partial w} \{\mathcal{Q}_{n,\mu}(y,w)\} dw \right| + |\mathfrak{f}'(y+)| \left| \int_{2y}^{\infty} (w-y) \frac{\partial}{\partial w} \{\mathcal{Q}_{n,\mu}(y,w)\} dw \right| + \left| \int_y^{2y} \mathfrak{f}'_y(u) du \right| |1-\theta_{n,\mu}(y,2y)|$

$$\begin{aligned} &\leq M \int_{2y}^{\infty} (1+w^2) \frac{\partial}{\partial w} \{\mathcal{Q}_{n,\mu}(y,w)\} dw + |\mathfrak{f}(y)| \int_{2y}^{\infty} \frac{\partial}{\partial w} \{\mathcal{Q}_{n,\mu}(y,w)\} dw \\ &\quad + |\mathfrak{f}'(y+)| \left\{ \frac{(1+2\mu)\beta_n y}{\alpha_n} \right\}^{1/2} + \frac{(1+2\mu)\beta_n}{\alpha_n y} |\mathfrak{f}(2y) - \mathfrak{f}(y) - y\mathfrak{f}'(y+)|. \end{aligned} \tag{3.12}$$

Since $t \leq 2(w-y)$ and $y \leq w-y$ when $w \geq 2y$, using equations (2.2) and (3.7), we obtain

$$M\int_{2y}^{\infty}(1+w^2)\frac{\partial}{\partial w}\left\{\mathcal{Q}_{n,\mu}(y,w)\right\}dw+|\mathfrak{f}(y)|\int_{2y}^{\infty}\frac{\partial}{\partial w}\left\{\mathcal{Q}_{n,\mu}(y,w)\right\}dw$$

$$\leq M\left(\frac{1}{y^2}+2^2\right)\left(\int_{0}^{\infty}(w-y)^2\frac{\partial}{\partial w}\left\{\mathcal{Q}_{n,\mu}(y,w)\right\}dw\right)+\frac{|\mathfrak{f}(y)|}{y^2}\int_{0}^{\infty}\frac{\partial}{\partial w}\left\{\mathcal{Q}_{n,\mu}(y,w)\right\}(w-y)^2dt$$

$$\leq (M(1+4y^2)+|f(y)|)\frac{(1+2\mu)\beta_n}{\alpha_n y}. \quad (3.13)$$

By associating (3.5), (3.6), (3.12) and (3.13), we get the required result. □

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References

[1] Aktaş, R. Çekim B. and Taşdelen, F. (2017), A Dunkl Analogue of Operators Including Two-variable Hermite polynomials, Arxiv: 1704.08183v1.

[2] İbikli E. and E. Gadjieva, A., (1995) The order of approximation of some unbounded functions by the sequences of positive linear operators, Turkish J. Math. 19(3) 331-337.

[3] Gadzhiev, A.D. (1974), The convergence problem for a sequence of positive linear operators on unbounded sets and theorem analogue to that of P.P. Korovkin, Soviet Math./ Dokl. 15(5) 1433-1436.

[4] Gupta, V. (2015) (p,q)- Szász- Mirakyan- Baskakov Operators, Complex Anal. Oper. Theory, DOI: 10.1007/s11785-015-0521-4.

[5] İcöz, G. and Çekim, B. (2015) Dunkl generalization of Szász operators via q-calculus, J. Inqual Appl., DOI 10.1186/s13660-015-0809.

[6] İcöz, G. and Çekim B., (2015), Stancu type generalization of Dunkl analogue of Szász-Kantorovich operators, Math. Meth. Appl. Sci., DOI: 10.1002/mma.3602.

[7] Deshwal, S., Agrawal, P..N. and Araci, S. (2018), Dunkl analogue of Szsz-mirakjan operators of blending type, Open Math., 16(1), 1344-1356.

[8] Mursaleen, M., Khan, T. and Nasiruzzaman, Md.,(2016), Approximating properties of generalized dunkl analogue of Szász operators, Appl. Math. Inf. Sci. 10 (6), 2303-2310.

[9] Mursaleen, M. and Nasiruzzaman, Md. (2016), Dunkl generalization of Kantorovich type Szász-Mirakjan operators via q-calculus, Asian-European J. Math., DOI: http://dx.doi.org/10.1142/S1793557117500772.

[10] Mursaleen, M., Rahman S. and Alotaibi, A. (2016), Dunkl generalization of q-Szàsz-Mirakjan Kantorovich operators which preserve some test functions, J. Inequal. Appl., 317, 1-18.

[11] Păltanea, R.(2008), Modified Szász-Mirakjan operators of integral form, Carpathian J. Math. 24 (3), 378-385.

[12] Rosenblum, M. (1994), Generalized Hermite polynomials and the Bose-like oscillator calculus, Oper. Theory. Adv. Appl. 73, 369-396.

[13] Sucu, S.(2014), Dunkl analogue of Szasz operators, Appl. Math. Comput., 244 , 42-48.

[14] Taşdelen, F., Söylemez, D. and Aktas, R. (2019), Dunkl-Gamma Type Operators Including Appell Polynomials, Complex Anal. Oper. Theory , DOI: 10.1007/s11785-019-00942-x.

[15] Wafi, A. and Rao, N. (2017), Szász-Durrmeyer Operators Based on Dunkl Analogue, Complex Anal. Oper. Theory, DOI 10.1007/s11785-017-0647-7.