On Köthe-Toeplitz duals of some new non-absolute type sequence spaces

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Abstract

In the present paper, we generalise the difference sequence spaces obtained by M. Mursaleen and Noman [18] and introduce $c_0^\lambda(\Delta_v)$ and $c^\lambda(\Delta_v)$ of difference sequence spaces, which are BK-spaces of non-absolute type and prove that these spaces are linearly isomorphic to space c_0 and c respectively. Furthermore, we find the Schauder basis for these spaces and also find their Köthe-Toeplitz duals.

Subject Classification:[2010]40C05; 40H05; 46A45; 46B20;
Keywords: Difference sequence spaces, BK-space, Schauder basis, $\alpha-$, $\beta-$ *and* $\gamma-$*duals.*

1 Introduction

We write ω for the set of all sequences $x = (x_k)_{k=0}^{\infty}$ and ϕ, l_∞, c *and* c_0 for the set of all finite, bounded, convergent sequences and sequence converges to naught respectively. Further, by $l_p\,(1 \leq p < \infty)$, we denote the sequence space of all $p-$absolutely convergent series, that is

$$l_p = \left\{ x = (x_k) \in \omega : \sum_{k=0}^{\infty} |x_k|^p < \infty \right\}$$

Moreover we write bs, cs, *and* cs_0 for the sequence space of all bounded, convergent and null series respectively.

By e and $e^n (n = 0, 1, 2, ...)$, we denote the sequence such that

$$\begin{aligned} e_k &= 1 \ for\ k = 0, 1, 2, ... \\ e_n^{(n)} &= 1 \\ and\ e_k^{(n)} &= 0 \ for\ k \neq n. \end{aligned}$$

A sequence space X is called FK-space if it is a complete linear metric space with continuous coordinates $p_n : X \to \mathbb{C}$ $(n \in \mathbb{N})$, where $\mathbb{C}$ denote the complex field and $p_n(x) = x_n$ for all $x = (x_k) \in X$, $\forall\, n \in \mathbb{N}$. A normal FK-space is called a BK-space that is, a BK-space is a Banach sequence space with continuous coordinates.

The sequence spaces l_∞, c, c_0 are BK-spaces with the usual norm given by

$$\|x\|_{l_\infty} = \sup_k |x_k|.$$

Also the space l_p is a BK-space with the usual l_p-norm defined by

$$\|x\|_{l_p} = \left(\sum_{k=0}^{\infty} |x_k|^p\right)^{1/p}, \ (1 \leq p < \infty).$$

A sequence $(b_n)_{n=0}^{\infty}$ in a linear metric space X is called a schauder basis if for each $x \in X$, there exists unique sequence $(\lambda_n)_{n=0}^{\infty}$ of scalars such that $x = \sum_{n=0}^{\infty} \lambda_n.b_n$ i.e.

$$\lim_{m\to\infty} \left\| x - \sum_{n=0}^{m} \lambda_n.b_n \right\| = 0.$$

Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex numbers and $x \in \omega$. Then we write,

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk}x_k, \ (n = 0, 1, 2, ...) \ and \ A(x) = (A_n(x))_{n=0}^{\infty}$$

for any subset X of ω, the set

$$X_A = \{x \in \omega : Ax \in X\} \quad (1.1)$$

is called matrix domain of A in X, which is a sequence space. For instance, if A is the matrix defined by

$$c_{nk} = \begin{cases} 1 & ; 0 \leq k \leq n \qquad \forall \ n = 0, 1, 2, ... \\ 0 & ; k > n \end{cases}$$

then $cs = c_A$ and $bs = (l_\infty)_A$ are the spaces of convergent and bounded series.

In addition, let X, Y be two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then we say that A define a matrix mapping from X into Y if for every sequence $x \in X$ the A-transform of x exists and is in Y. Moreover, we write (X, Y) for the class of all infinite matrices that map X into Y. A sequence x is said to be A-summable to $l \in \mathbb{C}$ if Ax converges to l, which is called as the A-limit of x.

We shall denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$. The approach of constructing a new sequence space by means of the matrix domain of a particular limitation method has been employed by several authors. They introduced the sequence space $(l_\infty)_{N_q}$ and c_{N_q} in [2], $(l_p)_{c_1} = X_p$ and $(l_\infty)_{c_1} = X_\infty$ in [3], $\mu_G = z(u, v; \mu)$ in [4], $(l_\infty)_{R^t} = r^t_\infty$, $c_{R^t} = r^t_c$ and $(c_0)_{R^t} = r^t_0$ in [5], $(l_p)_{R^t} = r^t_p$ in [6], $(c_0)_{E^r} = e^r_0$ and $c_{E^r} = e^r_c$ in [7], $(l_p)_{E^r} = e^r_p$ and $(l_\infty)_{E^r} = e^r_\infty$ in [8], $(c_0)_{A^r} = a^r_0$ and $c_{A^r} = a^r_c$ in [9], $[c_0(u, p)]_{A^r} = a^r_0(u, p)$ and $[c(u, p)]_{A^r} = a^r_c(u, p)$ in [10], $(l_p)_{A^r} = a^r_p$ and $(l_\infty)_{A_r} = a^r_\infty$ in [11] and $(c_0)_{c_1} = \tilde{c}_0 \, and \, c_{c_1} = \tilde{c}$ in [12]; where $N_q, c_1, R^t \ and \ E^r$ denote the Nörlund, Cesàro, Riesz and Euler means respectively, A^r and G are respectively defined in [9, 4].

In the present paper following [17, 18], we introduce the difference sequence spaces $c^\lambda_0(\triangle_v)$ and $c^\lambda(\triangle_v)$ of non-absolute type and derive some related results. We also establish some inclusion relations. Furthermore, we determine the $\alpha-, \beta-, \, and \, \gamma - duals$ of those spaces and construct their basis.

2 Non-absolute type difference sequence spaces $c_0^\lambda(\Delta_v)$ and $c^\lambda(\Delta_v)$

The difference sequence spaces have been studied by several authors in different ways [19, 20]. In this section we introduce the spaces $c_0^\lambda(\Delta_v)$ and $c^\lambda(\Delta_v)$ of difference sequences and show that these spaces are the BK-spaces of non-absolute type which is linearly isomorphic to the spaces c_0 and c respectively.

Throughout this paper, let $\lambda = (\lambda_k)_{k=0}^{\infty}$ be a strictly increasing sequence of positive reals tending to infinity, that is

$$0 < \lambda_0 < \lambda_1 < \lambda_2 < ...... \text{ and } \lambda_k \to \infty \text{ as } k \to \infty \tag{2.1}$$

The sequence spaces c_0^λ and c^λ of non-absolute type have been introduced by Mursaleen and Noman [17] as follows:

$$c_0^\lambda = \left\{ x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k = 0 \right\}$$

and

$$c^\lambda = \left\{ x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) x_k \text{ exists} \right\}$$

Also it has been shown that, the inclusion $c_0 \subset c_0^\lambda,\ c_0^\lambda \subset c,\ c \subset c^\lambda$ hold.

Further, the sequence spaces $c_0^\lambda(\triangle)$ and $c^\lambda(\triangle)$ of non-absolute type have been introduced by Mursaleen and Noman [18] as follows:

$$c_0^\lambda(\Delta) = \left\{ x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \Delta x_k = 0 \right\}$$

and

$$c^\lambda(\Delta) = \left\{ x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \Delta x_k \ \ exist \right\},$$

where $\triangle x_k = x_k - x_{k-1}$.

Also it has been shown that, the inclusion $c_0 \subset c_0^\lambda(\Delta),\ c_0^\lambda(\Delta) \subset c^\lambda(\Delta), c \subset c^\lambda(\Delta)$ hold. Let $v = (v_k)$ be any fixed sequence of non-zero complex numbers. Now we define,

$$c_0^\lambda(\Delta_v) = \left\{ x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \Delta_v x_k = 0 \right\}$$

and

$$c^\lambda(\Delta_v) = \left\{ x = (x_k) \in \omega : \lim_n \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \Delta_v x_k \ \ exists \right\},$$

where $\Delta_v x_k = v_k x_k - v_{k-1} x_{k-1}$.

Here and in sequel, we shall use the convention that any term with a negative subscript is equal to zero e.g. $\lambda_{-1} = 0$ and $v_{-1}x_{-1} = 0$
With the notation of (1.1), we can redefine the spaces $c_0^\lambda(\Delta_v)$ and $c^\lambda(\Delta_v)$ by

$$c_0^\lambda(\Delta_v) = (c_0^\lambda)_{\Delta_v} \text{ and } c^\lambda(\Delta_v) = (c^\lambda)_{\Delta_v} \tag{2.2}$$

where (Δ_v) denotes the band matrix defining the operator i.e. $\Delta_v x = v_k x_k - v_{k-1}x_{k-1}$ for all $x = (x_k) \in \omega$.

If $v = (v_k) = (1, 1, 1, ...)$, these spaces reduced to $c_0^\lambda(\Delta)$ and $c^\lambda(\Delta)$ [18]
We define, the matrix $\tilde{\Lambda} = (\tilde{\lambda}_{nk})$ for all $k, n \in \mathbb{N}$ by

$$\widetilde{\lambda}_{nk} = \begin{cases} \dfrac{(\lambda_k - \lambda_{k-1}) - (\lambda_{k+1} - \lambda_k)}{\lambda_n} & ; (k < n) \\ \dfrac{\lambda_n - \lambda_{n-1}}{\lambda_n} & ; (k = n) \\ 0 & ; (k > n) \end{cases} \tag{2.3}$$

Then it can be easily seen that the equality,

$$\widetilde{\Lambda}_n(x) = \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(v_k x_k - v_{k-1}x_{k-1}) \tag{2.4}$$

holds for all $n \in \mathbb{N}$ and every $x = (x_k) \in \omega$, $v = (v_k)$ be any fixed sequence of non-zero complex numbers, which leads us together with (1.1) to the fact that

$$c_0^\lambda(\Delta_v) = (c_0)_{\tilde{\Lambda}} \text{ and } c^\lambda(\Delta_v) = c_{\tilde{\Lambda}} \tag{2.5}$$

Further for a sequence $x = (x_k)$ and a fixed sequence $v = (v_k)$, we define the sequence $y(\lambda) = \{y_k(\lambda)\}$, which will be frequently used as the $\tilde{\Lambda}-$ transform x i.e. $y(\lambda) = \tilde{\Lambda}(x)$ and so we have that

$$y_k(\lambda) = \sum_{j=0}^{k-1} \frac{(\lambda_j - \lambda_{j-1}) - (\lambda_{j+1} - \lambda_j)}{\lambda_k} v_j x_j + \frac{(\lambda_k - \lambda_{k-1})}{\lambda_k} v_k x_k \ (k \in \mathbb{N}) \tag{2.6}$$

where summation running from 0 to $k - 1$ is equal to zero when $k = 0$.
Further, it is clear by (2.4) that the relation (2.6) can be written as follows

$$y_k(\lambda) = \sum_{j=0}^{k} \left(\frac{\lambda_j - \lambda_{j-1}}{\lambda_k} \right) (v_j x_j - v_{j-1}x_{j-1}), \qquad (k \in \mathbb{N})$$

Now we may begin the following result which is essential in the text.

Theorem 2.1 The difference sequence spaces $c_0^\lambda(\triangle_v)$ and $c^\lambda(\triangle_v)$ are BK-spaces with the norm

$$\|x\|_{c_0^\lambda(\Delta_v)} = \|x\|_{c^\lambda(\Delta_v)} = \|\widetilde{\Lambda}(x)\|_{l_\infty}$$

that is

$$\|x\|_{c_0^\lambda(\Delta_v)} = \|x\|_{c^\lambda(\Delta_v)} = \sup_n |\widetilde{\Lambda}_n(x)|.$$

Proof:- Since (2.5) holds and c_0, c are BK-spaces with respect to their natural norm and the matrix $\widetilde{\Lambda}$ is a triangle; Theorem 4.3.12 of Wilansky [21, p.63] gives the fact that $c_0^\lambda(\Delta_v)$ and $c^\lambda(\Delta_v)$ are BK-spaces with given norm.

Remark 2.2- It can be easily seen that the absolute property does not hold on the spaces $c_0^\lambda(\Delta_v)$ and $c^\lambda(\Delta_v)$, that is

$$\|x\|_{c_0^\lambda(\Delta_v)} \neq \||x|\|_{c_0^\lambda(\Delta_v)} \ and \ \|x\|_{c^\lambda(\Delta_v)} \neq \||x|\|_{c^\lambda(\Delta_v)}$$

for at least one sequence in the spaces $c_0^\lambda(\Delta_v)$ and $c^\lambda(\Delta_v)$, and this shows that $c_0^\lambda(\Delta_v)$ and $c^\lambda(\Delta_v)$ are the sequence spaces of non-absolute type, where $|x| = (|v_k x_k|)$.

Theorem 2.3 The sequence spaces $c_0^\lambda(\Delta_v)$ and $c^\lambda(\Delta_v)$ of non-absolute type are linearly isomorphic to the spaces c_0 and c respectively.

Proof:- We only consider the case $c_0^\lambda(\Delta_v) \cong c_0$ and the case $c^\lambda(\Delta_v) \cong c$ will follow similarly. Thus, to prove the theorem, we must show the existence of linear bijection between $c_0^\lambda(\triangle_v)$ and c_0. For this, consider the transformation T defined with the notation (2.6), from $c_0^\lambda(\triangle_v)$ to c_0 by $x \to y(\lambda) = Tx$. Then $Tx = y(\lambda) = \widetilde{\Lambda}(x) \in c_0$ for every $x \in c_0^\lambda(\Delta_v)$ Also, the linearity of T is clear. Further, it is trivial that $x = 0$ whenever $Tx = 0$ and hence T is injective. Furthermore, let $y = (y_k) \in c_0$ and define the sequence $x = \{x_k(\lambda)\}$ by

$$x_k(\lambda) = \frac{1}{v_k} \sum_{j=0}^{k} \sum_{i=j-1}^{j} (-1)^{j-i} \frac{\lambda_j}{\lambda_j - \lambda_{j-1}} y_i : (k \in \mathbb{N}) \tag{2.7}$$

where (v_k) is fixed sequence of non-zero complex numbers.
Then we obtain that

$$v_k x_k(\lambda) - v_{k-1} x_{k-1}(\lambda) = \sum_{i=k-1}^{k} (-1)^{k-i} . \frac{\lambda_i}{\lambda_k - \lambda_{k-1}} y_i : (k \in \mathbb{N})$$

Thus for every $n \in \mathbb{N}$, we have by (2.4) that-

$$\begin{aligned}
\widetilde{\Lambda}_n(x) &= \frac{1}{\lambda_n} \sum_{k=0}^{n} \sum_{i=k-1}^{k} (-1)^{k-i} \lambda_i y_i \\
&= \frac{1}{\lambda_n} \sum_{k=0}^{n} (\lambda_k y_k - \lambda_{k-1} y_{k-1}) = y_n
\end{aligned}$$

This shows that $\widetilde{\Lambda}(x) = y$ and since $y \in c_0$, we obtain $\widetilde{\Lambda}(x) \in c_0$. Thus, we deduce that $x \in c_0^\lambda(\Delta_v)$ and $Tx = y$. Hence T is surjective.
Moreover, we have for every $x \in c_0^\lambda(\Delta_v)$ that

$$\|Tx\|_0 = \|Tx\|_\infty = \|y(\lambda)\|_\infty = \|\widetilde{\Lambda}(x)\|_\infty = \|x\|_{c_0^\lambda(\Delta_v)}$$

which means that T is norm preserving. Consequently, T is a linear bijection and hence

$$c_0^\lambda(\Delta_v) \cong c_0$$

Similarly, we can prove $c^\lambda(\Delta_v) \cong c$

3 The inclusion relation

In this section, we establish some inclusion relations concerning with the spaces $c_0^\lambda(\Delta_v)$ *and* $c^\lambda(\Delta_v)$.

Theorem 3.1- The inclusion $c_0^\lambda(\Delta_v) \subset c^\lambda(\Delta_v)$ hold strictly.
Proof:- It is obvious that $c_0^\lambda(\Delta_v) \subset c^\lambda(\Delta_v)$ holds. Further, to show that this inclusion is strict, consider the sequence $x = (x_k)$ and $v = (v_k)$ defined by $x_k = k^2$ and $v_k = 1/k$ for all $k \in \mathbb{N}$,
Then we obtained by (2.4) that

$$\widetilde{\Lambda}(x) = \frac{1}{\lambda_n}\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1}) = 1 \quad ; (n \in \mathbb{N})$$

which shows that $\widetilde{\Lambda}(x) = e$ and hence $\widetilde{\Lambda}(x) \in (c - c_0)$ where $e = (1, 1, 1, ...)$. Thus the sequence x is in $c^\lambda(\Delta_v)$ but not in $c_0^\lambda(\Delta_v)$. Hence the inclusion $c_0^\lambda(\Delta_v) \subset c^\lambda(\Delta_v)$ is strict.

Theorem 3.2- The inclusion $c \subset c_0^\lambda(\Delta_v)$ hold strictly.
Proof:- Let $x \in c$ and $v = (v_k)$ be any fixed sequence of non-zero complex number such that $vx \in c$. Then, $\Delta_v x = (v_k x_k - v_{k-1}x_{k-1}) \in c_0$ and hence $\Delta_v x \in c_0^\lambda$. Since the inclusion $c_0 \subset c_0^\lambda$ holds. This shows that $x \in c_0^\lambda(\Delta_v)$. Consequently, the inclusion $c \subset c_0^\lambda(\Delta_v)$ holds. Further, consider the sequence $y = (y_k)$ and $v = (v_k)$ defined by

$$y_k = (k+1)^{3/2},\ v_k = \frac{1}{k+1} \qquad (k \in \mathbb{N})$$

Then it is trivial that $vy \notin c$. On the other hand, it can be easily seen that $\Delta_v y = (v_k y_k - v_{k-1}y_{k-1}) \in c_0$ and hence $\Delta_v y \in c_0^\lambda$ which means that $y \in c_0^\lambda(\Delta_v)$.

Corollary 3.3 The inclusion $c_0 \subset c_0^\lambda(\Delta_v)$ and $c \subset c^\lambda(\Delta_v)$ hold strictly.

Corollary 3.4 Although the spaces l_∞ and $c_0^\lambda(\Delta_v)$ overlap, the space l_∞ does not include the space $c_0^\lambda(\Delta_v)$.

Lemma 3.5.[24] $A \in (l_\infty : c_0)$ if and only if $\lim\limits_n \sum\limits_k |a_{nk}| = 0$

Theorem 3.6- The inclusion $l_\infty \subset c_0^\lambda(\Delta_v)$ strictly holds if and only if $z \in c_0^\lambda$, where the sequence $z = (z_k)$ is defined by,

$$z_k = v_k^{-1}\left|1 - \frac{\lambda_{k+1} - \lambda_k}{\lambda_k - \lambda_{k-1}}\right| : (k \in \mathbb{N})$$

Proof:- Suppose that the inclusion $l_\infty \subset c_0^\lambda(\Delta_v)$. Then we obtain that $\widetilde{\Lambda}(x) \in c_0$ for every $x \in l_\infty$ and hence the matrix $\widetilde{\Lambda} = (\widetilde{\lambda}_{nk})$ is in class $(l_\infty : c_0)$. Thus by lemma 3.5, we have

$$\lim_n \sum_k |\widetilde{\lambda}_{nk}| = 0 \tag{3.1}$$

Now by (2.3), we have for every $n \in \mathbb{N}$ that

$$\sum_k |\widetilde{\lambda}_{nk}| = \frac{1}{\lambda_n} \sum_{k=0}^{n-1} |(\lambda_k - \lambda_{k-1}) - (\lambda_{k+1} - \lambda_k)| + \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} \tag{3.2}$$

Thus, the condition (3.1) implies both

$$\lim_n \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} = 0 \tag{3.3}$$

and

$$\lim_n \frac{1}{\lambda_n} \sum_{k=0}^{n-1} |(\lambda_k - \lambda_{k-1}) - (\lambda_{k+1} - \lambda_k)| = 0 \tag{3.4}$$

Now we have for every $n \geq 1$ that

$$\frac{1}{\lambda_n} \sum_{k=0}^{n-1} |(\lambda_k - \lambda_{k-1}) - (\lambda_{k+1} - \lambda_k)| = \frac{\lambda_{n-1}}{\lambda_n} \left[\frac{1}{\lambda_{n-1}} \sum_{k=0}^{n-1} (\lambda_k - \lambda_{k-1}) v_k z_k \right]$$

and since $\lim_n \frac{\lambda_{n-1}}{\lambda_n} = 1$ by (3.3); we obtain by (3.4) that

$$\lim_n \frac{1}{\lambda_{n-1}} \sum_{k=0}^{n-1} (\lambda_k - \lambda_{k-1}) v_k z_k = 0 \tag{3.5}$$

which shows that $z = (z_k) \in c_0^\lambda$.
Conversely, suppose that $z = (z_k) \in c_0^\lambda$. Then we have that (3.5) holds. Further, for every $n \geq 1$, we prove that

$$\begin{aligned} \frac{1}{\lambda_n} \sum_{k=0}^{n-1} |(\lambda_k - \lambda_{k-1}) - (\lambda_{k+1} - \lambda_k)| &= \frac{1}{\lambda_n} \sum_{k=0}^{n-1} (\lambda_k - \lambda_{k-1}) v_k z_k \\ &\leq \frac{1}{\lambda_{n-1}} \sum_{k=0}^{n-1} (\lambda_k - \lambda_{k-1}) v_k z_k \end{aligned}$$

using above and (3.5) together we have (3.4) holds. On the other hand, we have for every $n \geq 1$ that

$$\begin{aligned}\left|\frac{\lambda_n - \lambda_{n-1} - \lambda_0}{\lambda_n}\right| &= \left|\frac{\lambda_{n-1} - (\lambda_n - \lambda_0)}{\lambda_n}\right| \\ &= \left|\frac{1}{\lambda_n}\sum_{k=0}^{n-1}[(\lambda_k - \lambda_{k-1}) - (\lambda_{k+1} - \lambda_k)]\right| \\ &\leq \frac{1}{\lambda_n}\sum_{k=0}^{n-1}|(\lambda_k - \lambda_{k-1}) - (\lambda_{k+1} - \lambda_k)|\end{aligned}$$

Therefore, it follows by (3.4) that-

$$\lim_n \frac{(\lambda_n - \lambda_{n-1})}{\lambda_n} = \lim_n \frac{\lambda_n - \lambda_{n-1} - \lambda_0}{\lambda_n} = 0$$

which just shows that (3.3) holds. Thus, we deduce by (3.2) that (3.1) holds. This leads us with lemma (3.5) to the consquence that $\widetilde{\Lambda} \in (l_\infty : c_0)$. Hence, the inclusion $l_\infty \subset c_0^\lambda(\triangle_v)$ holds which is strict inclusion by corollary (3.4).

4 The bases for the spaces $c_0^\lambda(\Delta_v)$ and $c_0^\lambda(\Delta_v)$

In this section, we give two sequences of the points of the spaces $c_0^\lambda(\Delta_v)$ and $c_0^\lambda(\Delta_v)$ which form the bases for those spaces.

If normed sequence space X contains a sequence (b_n) with the property that for every $x \in X$ there is a unique sequence (α_n) of scalars such that

$$\lim_n \|x - (\alpha_0 b_0 + \alpha_1 b_1 + \alpha_2 b_2 + \ldots\ldots + \alpha_n b_n)\| = 0$$

Then (b_n) is called a Schauder basis for X. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as

$$x = \sum \alpha_k b_k$$

Now, because of the transformation T defined from $c_0^\lambda(\Delta_v)$ to c_0, in the proof of Theorem 2.3, is onto, the inverse image of the basis $\{e^k\}_{k=0}^\infty$ of the space c_0 is the basis for the new space $c_0^\lambda(\Delta_v)$. Therefore, we have the following result;

Theorem 4.1 Define the sequence $b^{(k)}(\lambda) = \{b_n^{(k)}(\lambda)\}_{n=0}^\infty$ for every fixed $k \in \mathbb{N}$ by

$$b_n^{(k)}(\lambda) = \begin{cases} v_k^{-1}\left(\dfrac{\lambda_k}{\lambda_k - \lambda_{k-1}} - \dfrac{\lambda_k}{\lambda_{k+1} - \lambda_k}\right) & ;(n > k) \\ v_k^{-1}\left(\dfrac{\lambda_k}{\lambda_k - \lambda_{k-1}}\right) & ;(k = n), (n \in \mathbb{N}) \\ 0 & ;(n < k) \end{cases} \tag{4.1}$$

where $v = (v_k)$ be fixed sequence of non-zero complex numbers. Then the sequence $\{b^{(k)}(\lambda)\}_{k=0}^{\infty}$ is basis for the space $c_0^{\lambda}(\Delta_v)$ and every $x \in c_0^{\lambda}(\Delta_v)$ has unique representation of the form

$$x = \sum_k \alpha_k(\lambda) b^k(\lambda); \tag{4.2}$$

where $\alpha_k(\lambda) = \widetilde{\Lambda}_k(x) \quad \forall \quad k \in \mathbb{N}$.

Proof:- It is clear that the inclusion $\{b^{(k)}(\lambda)\} \subset c_0^{\lambda}(\Delta_v)$ holds, Since

$$\widetilde{\Lambda}(b^k(\lambda)) = e^k \in c_0 \quad (k \in \mathbb{N}) \tag{4.3}$$

Further, let $x \in c_0^{\lambda}(\Delta_v)$ be given for every non-negative integer m, we put

$$x^{(m)} = \sum_{k=0}^{m} \alpha_k(\lambda) b^{(k)}(\lambda)$$

Then we obtain by (4.3) that

$$\widetilde{\Lambda}(x^m) = \sum_{k=0}^{m} \alpha_k(\lambda) \widetilde{\Lambda}(b^{(k)}(\lambda)) = \sum_{k=0}^{m} \widetilde{\Lambda}_k(x) e^{(k)}$$

and hence,

$$\widetilde{\Lambda}(x - x^{(m)}) = \begin{cases} 0 & ; 0 \le n \le m \\ \widetilde{\Lambda}_n(x) & ; n > m \end{cases} \qquad (n, m \in \mathbb{N})$$

Now given $\epsilon > 0$, then there is a non-negative integer m_0 such that

$$|\widetilde{\Lambda}_m(x)| < \epsilon/2 \quad \forall \ m \ge m_0$$

Hence, we have for every $m \ge m_0$ that

$$\begin{aligned} \|x - x^{(m)}\|_{c_0^{\lambda}(\Delta_v)} &= \sup_{n>m} |\widetilde{\Lambda}_n(x)| \\ &\le \sup_{n>m_0} |\widetilde{\Lambda}_n(x)| \\ &\le \epsilon/2 < \epsilon. \end{aligned}$$

Thus, we deduce that $\lim_m \|x - x^{(m)}\|_{c_0^{\lambda}(\Delta_v)} = 0$, which shows that $x \in c_0^{\lambda}(\Delta_v)$ is represented as in (4.2).

Finally, let us show the uniqueness of the representation (4.2) of $x \in c_0^{\lambda}(\Delta_v)$. For this, suppose if possible that the another representation

$$x = \sum_k \beta_k(\lambda) b^{(k)}(\lambda)$$

Since the linear transformation T defined from $c_0^\lambda(\Delta_v)$ to c_0, in the proof of Theorem 2.2, is continuous, we have that

$$\begin{aligned}\widetilde{\Lambda}_n(x) &= \sum_k \beta_k(\lambda)\widetilde{\Lambda}_n(b^{(k)}(\lambda)) \\ &= \sum_k \beta_k(\lambda)\delta_{nk} \\ &= \beta_n(\lambda) \qquad ;(n \in \mathbb{N})\end{aligned}$$

which contradict the fact that $\widetilde{\Lambda}_n(x) = \alpha_n(\lambda) \quad \forall \quad n \in \mathbb{N}$. Hence, the representation (4.2) of $x \in c_0^\lambda(\Delta_v)$ is unique.

Theorem 4.2 The sequence $\{b, b^0(\lambda), b^1(\lambda).....\}$ is basis for the space $c^\lambda(\Delta_v)$ and every $x \in c^\lambda(\Delta_v)$ has a unique representation of the form

$$x = lb + \sum_k [\alpha_k(\lambda) - b]b^{(k)}(\lambda) \tag{4.4}$$

where $\alpha_k(\lambda) = \widetilde{\Lambda}_k(x) \quad \forall \quad k \in \mathbb{N}$, the sequence $b = (b_k)$ and $v = (v_k)$ is defined by $b_k = k^2,\ v_k = 1/k \quad \forall \quad k \in \mathbb{N}$, the sequence $b^{(k)}(\lambda) = \{b_n^{(k)}(\lambda)\}_{n=0}^{\infty}$ is defined by (4.1) for every $k \in \mathbb{N}$ and

$$l = \lim_k \widetilde{\Lambda}_k(x) \tag{4.5}$$

Proof:- Since $\{b^{(k)}(\lambda)\} \subset c_0^\lambda(\Delta_v)$ and $\widetilde{\Lambda}(b) = e \in c$; the inclusion $\{b, b^{(k)}(\lambda)\} \subset c^\lambda(\Delta_v)$ trivially hold. Further, let $x \in c^\lambda(\Delta_v)$ be given, then there uniquely exists an l satisfying (4.5). Thus we have that $y \in c_0^\lambda(\Delta_v)$ whenever we set $y = x - lb$. Therefore, we deduce, by theorem 4.1, that the representation

$$y = \sum_k \beta_k(\lambda) b^{(k)}(\lambda)$$

of y is unique, where

$$\beta_k(\lambda) = \widetilde{\Lambda}_k(y) = \widetilde{\Lambda}_k(x - lb) = \widetilde{\Lambda}_k(x) - l = \alpha_k(\lambda) - l; \quad (k \in \mathbb{N})$$

Hence, the representation (4.4) of x is unique.

Finally, it is immediate by Theorem 2.1 that $c_0^\lambda(\Delta_v)$ and $c^\lambda(\Delta_v)$ are banach spaces with their natural norm.

Corollary 4.3. The difference sequence spaces $c_0^\lambda(\Delta_v)$ and $c^\lambda(\Delta_v)$ are separable.

5 The $\alpha-, \beta-,$ *and* $\gamma-$ dual of the spaces $c_0^\lambda(\Delta_v)$ and $c^\lambda(\Delta_v)$

In this section, we state and prove the theorem determining $\alpha-, \beta-$ and $\gamma-$ dual of the spaces $c_0^\lambda(\Delta_v)$ and $c^\lambda(\Delta_v)$ of non-absolute type.

For the space X and Y, define the set

$$S(X:Y) = \{a = (a_k) \in \omega : ax = (a_k x_k) \in Y,\ for\ all\ x = (x_k) \in X \tag{5.1}$$

With the notation of (5.1); the $\alpha-, \beta-$ *and* $\gamma-$duals of sequence space X, which are respectively denoted by $X^\alpha, X^\beta,$ *and* X^γ and are defined by

$$X^\alpha = M(X, l_1), \quad X^\beta = M(X, cs)\ and\ X^\gamma = M(X, bs)$$

Now, we may begin with equating the following lemmas [24] which are used in proving the theorems.

Lemma 5.1 $A \in (c_0 : l_1) = (c : l_1)$ if and only if

$$\sup_{K \in \mathcal{F}} \sum_n \sum_{k \in K} |a_{nk}| < \infty$$

Lemma 5.2 $A \in (c_0 : c)$ if and only if

$$\lim_n a_{nk} \text{ exists for each } \quad k \in \mathbb{N} \tag{5.2}$$

and

$$\sup_n \sum_k |a_{nk}| < \infty \tag{5.3}$$

Lemma 5.3 $A \in (c : c)$ if and only if (5.2) and (5.3) hold and $\lim_n \sum a_{nk}$ exists.

Lemma 5.4 $A \in (c_0 : l_\infty) = (c : l_\infty)$ if and only if (5.3) holds.

Now, we prove the following results.

Theorem 5.5 If $v = (v_k)$ be a fixed sequence of non-zero complex numbers then we define the set

$$b_1^\lambda = \left\{ a = (a_n) \in \omega : \sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} v_k^{-1} b_{nk}^\lambda \right| < \infty \right\}$$

where the matrix $B^\lambda = (b_{nk}^\lambda)$ is defined via the sequence $a = (a_n)$ by

$$b_{nk}^\lambda = \begin{cases} \left(\dfrac{\lambda_k}{\lambda_k - \lambda_{k-1}} - \dfrac{\lambda_k}{\lambda_{k+1} - \lambda_k} \right) a_n & ; (k < n) \\ \left(\dfrac{\lambda_n}{\lambda_n - \lambda_{n-1}} \right) a_n & ; (k = n) \\ 0 & ; (k > n) \end{cases}$$

Then

$$\left[c_0^{\lambda}(\Delta_v)\right]^{\alpha} = \left[c^{\lambda}(\Delta_v)\right]^{\alpha} = b_1^{\lambda} \tag{5.4}$$

Proof:- Let $a = (a_n) \in \omega$. Then, by bearing in mind the equation (2.6) and (2.7), we immediately derive that

$$a_n x_n = \sum_{k=0}^{n} \sum_{j=k-1}^{k} (-1)^{k-j} \frac{\lambda_i}{\lambda_k - \lambda_{k-1}} a_n y_j = B_n^{\lambda}(y) \quad ; n \in N \tag{5.5}$$

Thus, we observe by (5.5) that $ax = (a_n x_n) \in l_1$ whenever $x = (x_k) \in c_0^{\lambda}(\Delta_v)$ or $c^{\lambda}(\Delta_v)$ if and only if $B^{\lambda} y \in l_1$ whenever $y = (y_k) \in c_0$ or c. This means that the sequence $a = (a_n)$ is in $\alpha-$dual of the space $c_0^{\lambda}(\Delta_v)$ or $c^{\lambda}(\Delta_v)$ if and only if $B^{\lambda} \in (c_0 : l_1) = (c : l_1)$. Therefore we obtained by Lemma 5.4 with B^{λ} instead of A that $a \in \left[c_0^{\lambda}(\Delta_v)\right]^{\alpha} = \left[c^{\lambda}(\Delta_v)\right]^{\alpha}$ if and only if

$$\sup_{k \in \mathcal{F}} \sum_n \left| \sum_{k \in K} v_k^{-1} b_{nk}^{\lambda} \right| < \infty$$

which leads us to the consequence that

$$\left[c_0^{\lambda}(\Delta_v)\right]^{\alpha} = \left[c^{\lambda}(\Delta_v)\right]^{\alpha} = b_1^{\lambda}.$$

This completes the proof of theorem.

Theorem 5.6- If $v = (v_k)$ be a fixed sequence of non-zero complex numbers, then we define the following sets as follows.

$$b_2^{\lambda} = \left\{ a = (a_k) \in \omega : \sum_{j=k}^{\infty} v_j^{-1} a_j \text{ exists for each } k \in \mathbb{N} \right\},$$

$$b_3^{\lambda} = \left\{ a = (a_k) \in \omega : \sup_n \sum_{k=0}^{n-1} |\widetilde{a}_k(n)| < \infty \right\},$$

$$b_4^{\lambda} = \left\{ a = (a_k) \in \omega : \sup_k \left| \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} v_k^{-1} a_k \right| < \infty \right\},$$

and

$$b_5^{\lambda} = \left\{ a = (a_k) \in \omega : \sum_k (k+1) v_k^{-1} a_k \quad \text{converges} \right\}$$

where

$$\widetilde{a}_n(n) = \lambda_k \left[\frac{a_k v_k^{-1}}{\lambda_k - \lambda_{k-1}} + \left(\frac{1}{\lambda_k - \lambda_{k-1}} - \frac{1}{\lambda_{k+1} - \lambda_k} \right) \sum_{j=k+1}^{n} a_j v_j^{-1} \right]; (k < n)$$

Then

$$\left[c_0^\lambda(\Delta_v)\right]^\beta = b_2^\lambda \cap b_3^\lambda \cap b_4^\lambda$$

and

$$\left[c^\lambda(\Delta_v)\right]^\beta = b_3^\lambda \cap b_4^\lambda \cap b_5^\lambda.$$

Proof:- Consider the equation

$$\begin{aligned}
\sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n} (v_k x_k)(a_k v_k^{-1}) \\
&= \sum_{k=0}^{n} \left\{ \sum_{j=0}^{k} \left[\sum_{i=j-1}^{j} (-1)^{j-i} \frac{\lambda_i}{\lambda_j - \lambda_{j-1}} y_i \right] \right\} a_k v_k^{-1} \\
&= \sum_{k=0}^{n-1} \lambda_k \left[\frac{a_k v_k^{-1}}{\lambda_k - \lambda_{k-1}} + \left(\frac{1}{\lambda_k - \lambda_{k-1}} - \frac{1}{\lambda_{k+1} - \lambda_k} \right) \sum_{j=k+1}^{n} a_j v_j^{-1} \right] y_k \\
&\quad + \frac{\lambda_n}{\lambda_n - \lambda_{n-1}} (a_n v_n^{-1}) y_n \\
&= \sum_{k=0}^{n-1} \widetilde{a}_k(n) y_k + \frac{\lambda_n}{\lambda_n - \lambda_{n-1}} (a_n v_n^{-1}) y_n \\
&= T_n^\lambda(y) \qquad ; n \in \mathbb{N}
\end{aligned}$$

(5.6) $$\sum_{k=0}^{n} a_k x_k = T_n^\lambda(y) \qquad ; n \in \mathbb{N}$$

where the matrix $T^\lambda = (t_{nk}^\lambda)$ is defined by

$$t_{nk}^\lambda = \begin{cases} \widetilde{a}_k(n) & ; (k < n) \\ \dfrac{\lambda_n}{\lambda_n - \lambda_{n-1}} a_n v_n^{-1} & ; (k = n), \quad (n, k \in N) \\ 0 & ; (k > n) \end{cases}$$

Then, we deduce by (5.6) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in c_0^\lambda(\Delta_v)$ if and only if $T^\lambda y \in c$ whenever $y = (y_k) \in c_0$. This means that $a = (a_k) \in [c_0^\lambda(\Delta_v)]^\beta$ if and only if $T^\lambda y \in (c_0 : c)$. Therefore, by using Lemma 5.2, we derive from (5.2) and (5.3) that

(5.7) $$\sum_{j=k}^{\infty} a_j v_j^{-1} \quad \text{exists for each} \quad k \in \mathbb{N}$$

$$\sup_n \sum_{k=0}^{n-1} |\widetilde{a}_k(n)| < \infty \tag{5.8}$$

and

$$\sup_k \left| \frac{\lambda_k}{\lambda_k - \lambda_{k-1}} a_k v_k^{-1} \right| < \infty \tag{5.9}$$

Hence we conclude that

$$[c_0^\lambda(\Delta_v)]^\beta = b_2^\lambda \cap b_3^\lambda \cap b_4^\lambda$$

Similarly, we deduce from Lemma 5.3 with (5.6) that $a = (a_k) \in [c_0^\lambda(\Delta_v)]^\beta$ if and only if $T^\lambda \in (c : c)$. Therefore we derive from (5.2) and (5.3) that (5.7), (5.8) and (5.9) hold. Further, it can be easily seen that the equality

$$\sum_{k=0}^{n}(k+1)a_k = \sum_{k=0}^{n-1} \widetilde{a}_k(n) + \frac{\lambda_n}{\lambda_n - \lambda_{n-1}} a_k v_k^{-1}, \tag{5.10}$$

which can be written as follows :

$$\sum_{k=0}^{n}(k+1)a_k = \sum_k t_{nk}^\lambda \quad : \quad (n \in \mathbb{N})$$

Consequently, we obtain from (5.4) that $\{(k+1)a_k\} \in cs$.
Thus the condition (5.7) is redundant. Hence, we deduce that

$$[c^\lambda(\Delta_v)]^\beta = b_3^\lambda \cap b_4^\lambda \cap b_5^\lambda$$

Theorem 5.7 The $\gamma-$ dual of the spaces $c_0^\lambda(\Delta_v)$ and $c^\lambda(\Delta_v)$ is the set $b_3^\lambda \cap b_4^\lambda$.
Proof:- This result can be proved similarly as the proof of theorem 5.6 with Lemma 5.2.

Aknowledgemet:- The first author gratefully aknowledges CSIR for financial in the form of JRF.

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