Revisiting Pál-type Hermite-Fejér Interpolation on the unit circle

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Abstract

In this research manuscript, authors brought into consideration the set of non-uniformly distributed nodes on the unit circle to investigate a Pál-type (1;0) interpolation problem in account with Hermite-Fejér boundary condition. These nodes are obtained by projecting vertically the zeros of Jacobi polynomial as well as zeros of its derivative onto the unit circle. Explicitly representing the interpolatory polynomial as well as establishment of convergence theorem are the key highlights of this manuscript. The field of approximation theory entertains the results proved.

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1 Introduction

Ki$\hat{s}$ [8] extended his art of making discoveries by initiating interpolation processes on unit circle. He considered the $(0,2)$ and $(0,1,........,r-2,r)$ interpolation for an integer $r \geq 2$ on the n^{th} roots of unity. Since then, study of the interpolation problems such as Lagrange, classical Hermite and Pál-type interpolation on the complex plane evolved.
Pál [10] considered a modified Hermite-Fejér interpolation problem, where function values are prescribed on set of nodes with n points and those of their derivatives on another set of $(n-1)$ points. To obtain a unique solution, he imposed an extra condition and provided explicit representation of the interpolatory polynomial. Since then, researchers look forward for more general Pál-type interpolation problems which basically consists in determining a polynomial that takes prescribed values at one set of nodes and also their r^{th} derivative $(r>0)$ at another set of nodes.
Dikshit [7] also considered the Pál-type interpolation on non-uniformly distributed nodes on the unit circle. Bruin [5] considered Pál-type interpolation problem and studied the effect of interchanging the value nodes and the derivative nodes on the problem's regularity. Lénárd [9] considered a $(0,2)$ type Pál interpolation problem and obtained regularity and explicit representation for the same.
Bahadur & Shukla [3] read a weighted Pál-type interpolation problem on the vertically projected zeros of $(1-x^2)P_n^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha,\beta)'}(x)$ onto the unit circle. Explicit representation and convergence was studied for analytic functions on the unit disk.

Recently, Bahadur [1,2] lead a series of back-to-back research papers on Pál-type interpolation on non-uniformly nodes on the unit circle. It always seems impossible until its done. So, Bahadur & Varun [4] extended Pál-type Hermite-Fejer interpolation onto the unit circle using the zeros of more general Jacobi polynomial. The present paper is about revisiting [4] on the similar nodes.

2 Preliminaries

This section includes the following results, which we shall use.

The differential equation satisfied by $P_n^{(\alpha,\beta)}(x)$ is

$$(1-x^2)P_n^{(\alpha,\beta)''}(x)+[\beta-\alpha-(\alpha+\beta+2)x]P_n^{(\alpha,\beta)'}(x)+n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)=0,$$

Using the Szegő transformation, $x=\frac{1+z^2}{2z}$

$$\begin{aligned}(z^2-1)^4P_n^{(\alpha,\beta)''}(x)+4z(z^2-1)\big[\{(\alpha+\beta+2)z^2+1\}(z^2-1)-2z^3(\beta-\alpha)\big]P_n^{(\alpha,\beta)'}(x)&\\ -16z^6n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x)&=0,\end{aligned} \tag{2.1}$$

Let $\mathscr{Z}_n$ and $\mathscr{T}_n$ be two distinct set of nodes such that,

$$\mathscr{Z}_n=\{z_k=cos\theta_k+i\,sin\theta_k\,;\,z_{n+k}=\overline{z_k}\,;\,k=1(1)n\}$$

$$\mathscr{T}_n=\{t_k=cos\phi_k+i\,sin\phi_k\,;\,t_{n+k}=\overline{t_k}\,;\,k=1(1)n-1\}$$

which are obtained by projecting vertically the zeros of $P_n^{(\alpha,\beta)}(x)$ and $P_n^{(\alpha,\beta)'}(x)$ respectively on the unit circle, where $P_n^{(\alpha,\beta)}(x)$ stands for Jacobi polynomial of degree n.

$$\mathscr{W}(z)=\prod_{k=1}^{2n}(z-z_k)=K_nP_n^{(\alpha,\beta)}\left(\frac{1+z^2}{2z}\right)z^n, \tag{2.2}$$

$$\mathscr{W}_1(z)=\prod_{k=1}^{2n-2}(z-t_k)=K_n^*P_n^{(\alpha,\beta)'}\left(\frac{1+z^2}{2z}\right)z^{n-1}, \tag{2.3}$$

The fundamental polynomials of Lagrange interpolation on the zeros of $\mathscr{W}(z)$and $\mathscr{W}_1(z)$ are given as

$$\mathscr{L}_k(z)=\frac{\mathscr{W}(z)}{(z-z_k)\mathscr{W}'(z_k)}\qquad ,k=1(1)2n \tag{2.4}$$

$$\mathscr{L}_k^*(z)=\frac{\mathscr{W}_1(z)}{(z-t_k)\mathscr{W}_1'(t_k)}\qquad ,k=1(1)2n-2 \tag{2.5}$$

(2.6) $$\mathscr{M}_k(z) = \int_0^z z^{n+2\alpha}(z^2-1)^{r-2\alpha}\,\mathscr{L}_k^*(z)dz$$

(2.7) $$\mathscr{M}_{1j}(z) = \int_0^z z^{n+2\alpha+j}(z^2-1)^{r-2\alpha-2}\,\mathscr{W}(z)dz \qquad ; j=0,1$$

where, $\mathscr{M}_{1j}(-1) = (-1)^{j+1}\mathscr{M}_{1j}(1)$

For $-1 \le x \le 1$,

(2.8) $$(1-x^2)^{1/2} \mid P_n^{(\alpha,\beta)}(x) \mid = O(n^{\alpha-1}),$$

(2.9) $$\mid P_n^{(\alpha,\beta)}(x) \mid = O(n^{\alpha}),$$

(2.10) $$(1-x^2) \mid P_n^{(\alpha,\beta)'}(x) \mid = O(n^{\alpha+1}),$$

(2.11) $$\mid P_n^{(\alpha,\beta)''}(x) \mid = O(n^{\alpha+4}),$$

Let $x_k = cos\theta_k\,, k=1(1)n$ be the zeros of $P_n^{(\alpha,\beta)}(x)$, such that $1 > x_1 > x_2 > .... > x_n > -1$, then

(2.12) $$(1-x_k^2)^{-1} \sim \left(\frac{k}{n}\right)^{-2},$$

(2.13) $$\mid P_n^{(\alpha,\beta)}(x_k) \mid \sim k^{-\alpha-\frac{1}{2}}n^{\alpha},$$

(2.14) $$\mid P_n^{(\alpha,\beta)'}(x_k) \mid \sim k^{-\alpha-\frac{3}{2}}n^{\alpha+2},$$

(2.15) $$\mid P_n^{(\alpha,\beta)''}(x_k) \mid \sim k^{-\alpha-\frac{5}{2}}n^{\alpha+4}.$$

For more details, refer[12]

3 The Problem

Here, we are interested in determining the convergence of interpolatory polynomial $\mathscr{F}_n(z)$ on the two distinct set of nodes $\mathscr{Z}_n$ and $\mathscr{T}_n$, with Hermite boundary conditions at ± 1 satisfying the conditions.

$$\begin{cases} \mathscr{F}_n(t_k) = \eta_k & ; k = 1(1)2n-2 \\ \mathscr{F}_n^{(1)}(z_k) = \varphi_k & ; k = 1(1)2n \\ \mathscr{F}_n^{(m)}(\pm 1) = 0 & ; m = 0(1)r \end{cases} \tag{3.1}$$

where, η_k and φ_k are complex constants. Also, $\mathscr{F}_n^{(m)}(z)$ denotes m^{th} derivative of $\mathscr{F}_n(z)$.

4 Explicit Representation of Interpolatory Polynomial

We shall write $\mathscr{F}_n(z)$ satisfying (3.1)

$$\mathscr{F}_n(z) = \sum_{k=1}^{2n-2} \eta_k \mathscr{C}_k(z) + \sum_{k=1}^{2n} \varphi_k \mathscr{D}_k(z) \tag{4.1}$$

where $\mathscr{C}_k(z)$ and $\mathscr{D}_k(z)$ are unique polynomials each of degree $\leq 4n + 2r - 1$.

For $k = 1(1)2n-2$

$$\begin{cases} \mathscr{C}_k(t_j) = \delta_{kj} & ; j = 1(1)2n-2 \\ \mathscr{C}_k^{'}(z_j) = 0 & ; j = 1(1)2n \\ \mathscr{C}_k^{(m)}(\pm 1) = 0 & ; m = 0(1)r \end{cases} \tag{4.2}$$

For $k = 1(1)2n$

$$\begin{cases} \mathscr{D}_k(t_j) = 0 & ; j = 1(1)2n-2 \\ \mathscr{D}_k^{'}(z_j) = \delta_{kj} & ; j = 1(1)2n \\ \mathscr{D}_k^{(m)}(\pm 1) = 0 & ; m = 0(1)r \end{cases} \tag{4.3}$$

Theorem 4.1. *For $k = 1(1)2n$*

$$\mathscr{D}_k(z) = (z^2-1)^{2+2\alpha} z^{-n-2\alpha-1} \mathscr{W}_1(z) \Big[g_k \mathscr{M}_k(z) + g_{0k} \mathscr{M}_{10}(z) + g_{1k} \mathscr{M}_{11}(z) \Big] \tag{4.4}$$

(4.5) $$g_k = \frac{z_k}{(z_k^2-1)^{2+r}\mathscr{W}_1(z_k)},$$

(4.6) $$g_{1k} = \frac{-g_k[\mathscr{M}_k(1)+\mathscr{M}_k(-1)]}{2\mathscr{M}_{11}(1)}, and$$

(4.7) $$g_{0k} = \frac{g_k[\mathscr{M}_k(-1)-\mathscr{M}_k(1)]}{2\mathscr{M}_{10}(1)}.$$

Proof. Consider (4.4), where $\mathscr{D}_k(z)$ is atmost of degree $(4n+2r-1)$ satisfying the conditions given in (4.3).
Clearly, $\mathscr{D}_k(t_j) = 0$ for $j = 1(1)2n-2$
and from $\mathscr{D}_k^{'}(z_j) = \delta_{kj}$ for $j = 1(1)2n$. Using (2.6), at $j = k$,we get (4.5).
One can verify the results for $j \neq k$.
Also, from $\mathscr{D}_k^{(m)}(\pm 1) = 0$ for $m = 0(1)r$, we get(4.6) and (4.7) using (2.7).
Hence, the theorem follows. □

Theorem 4.2. *For* $k = 1(1)2n-2$

(4.8) $$\mathscr{C}_k(z) = \frac{(z^2-1)^{r+1}\,\mathscr{L}_k^*(z)\mathscr{W}(z)}{(t_k^2-1)^{r+1}\mathscr{W}(t_k)} + (z^2-1)^{2+2\alpha}z^{-n-2\alpha-1}\mathscr{W}_1(z)\Big[\mathscr{N}_k(z)+h_{0k}\mathscr{M}_{10}(z)+h_{1k}\mathscr{M}_{11}(z)\Big]$$

(4.9) $$\mathscr{N}_k(z) = -\int_0^z \frac{z^{n+2\alpha+1}(z^2-1)^{r-2-2\alpha}}{\mathscr{W}_1^{'}(t_k)(t_k^2-1)^{r+1}\mathscr{W}(t_k)}\left[\frac{(z^2-1)\mathscr{W}^{'}(z)+d_k\mathscr{W}(z)}{(z-t_k)}\right]dz$$

(4.10) $$h_{0k} = \frac{\mathscr{N}_k(1)-\mathscr{N}_k(-1)}{2\mathscr{M}_{10}(1)}$$

(4.11) $$h_{1k} = \frac{-[\mathscr{N}_k(1)+\mathscr{N}_k(-1)]}{2\mathscr{M}_{11}(1)}$$

(4.12) $$d_k = \frac{(1-{t_k}^2)\mathscr{W}'(t_k)}{\mathscr{W}(t_k)}$$

Proof. Consider (4.8), where $\mathscr{C}_k(z)$ is atmost of degree $(4n+2r-1)$ satisfying the conditions given in (4.2).
Clearly, $\mathscr{C}_k(t_j) = \delta_{kj}$ for $j = 1(1)2n-2$ and $\mathscr{C}_k^{'}(z_j) = 0$ for $j = 1(1)2n(j \neq k)$, provides us with a polynomial $\mathscr{N}_k(z)$ of degree $(3n+2r-1)$ given by (4.9).
One can verify $[\mathscr{C}_k^{'}(z)]_{z=z_k} = 0$.
$[(z^2-1)\mathscr{W}^{'}(z) + d_k\mathscr{W}(z)]_{z=z_k} = 0$ provides us (4.12).
Also, from $\mathscr{C}_k^{(m)}(\pm 1) = 0$ for $m = 0(1)r$, we get (4.10) and (4.11).
Hence, the theorem follows. □

5 Estimates of Fundamental Polynomials

We need to calculate estimates in order to obtain the rate of convergence of interpolatory polynomials.

Lemma 1. *Let $\mathscr{C}_k(z)$ be given by (4.8), then*

$$\sum_{k=1}^{2n-2} \mid \mathscr{C}_k(z) \mid = \boldsymbol{O}\Big((1-x^2)^{\frac{r-2}{2}} n^r logn\Big); \tag{5.1}$$

where, $-1 < \alpha \leq \dfrac{r-1}{2}$

Proof. Consider (4.8)

$$\begin{aligned} \sum_{k=1}^{2n-2} \mid \mathscr{C}_k(z) \mid \leq & \sum_{k=1}^{2n-2} \Big| \frac{(z^2-1)^{r+1}\mathscr{L}_k^*(z)\mathscr{W}(z)}{(t_k^2-1)^{r+1}\mathscr{W}(t_k)} \Big| \\ & + \sum_{k=1}^{2n-2} \Big| (z^2-1)^{2+2\alpha} z^{-n-2\alpha-1} \mathscr{W}_1(z)\mathscr{N}_k(z) \Big| \\ & + \sum_{k=1}^{2n-2} \Big| (z^2-1)^{2+2\alpha} z^{-n-2\alpha-1} \mathscr{W}_1(z)(h_{0k}\mathscr{M}_{10}(z) + h_{1k}\mathscr{M}_{11}(z)) \Big|, \end{aligned} \tag{5.2}$$

$$\sum_{k=1}^{2n-2} \mid \mathscr{C}_k(z) \mid \leq I_1 + I_2 + I_3 \tag{5.3}$$

Using (2.2),(2.3),(2.4),(2.7),(2.8),(2.10),(2.12) and (2.14), we have

$$I_1 = \mathbf{O}\Big((1-x^2)^{\frac{r-2}{2}} n^{r-1} logn\Big) \tag{5.4}$$

$$I_2 = \mathbf{O}\Big((1-x^2)^{\frac{r-2}{2}} n^r logn\Big) \tag{5.5}$$

$$I_3 = \mathbf{O}\Big((1-x^2)^{\frac{r-2}{2}} n^r logn\Big) \tag{5.6}$$

Combining (5.4), (5.5) and (5.6) give our desired lemma. □

Lemma 2. *Let $\mathscr{D}_k(z)$ be given by (4.4), then*

$$\sum_{k=1}^{2n} \mid \mathscr{D}_k(z) \mid = \boldsymbol{O}\Big((1-x^2)^{\frac{r-2}{2}} n^{r-1} logn\Big); \tag{5.7}$$

where, $\quad -1 < \alpha \leq \dfrac{r-1}{2}$

Proof. Consider (4.4)

$$\mid \mathscr{D}_k(z) \mid = \Big|(z^2-1)^{2+2\alpha} z^{-n-2\alpha-1} \mathscr{W}_1(z)\left[g_k \mathscr{M}_k(z) + g_{0k}\mathscr{M}_{10}(z) + g_{1k}\mathscr{M}_{11}(z)\right]\Big| \tag{5.8}$$

$$\begin{aligned}\sum_{k=1}^{2n} \mid \mathscr{D}_k(z) \mid \leq & \sum_{k=1}^{2n} \Big|(z^2-1)^{2+2\alpha} z^{-n-2\alpha-1} \mathscr{W}_1(z) g_k \mathscr{M}_k(z)\Big| \\ & + \sum_{k=1}^{2n} \Big|(z^2-1)^{2+2\alpha} z^{-n-2\alpha-1} \mathscr{W}_1(z)[g_{0k}\mathscr{M}_{10}(z) + g_{1k}\mathscr{M}_{11}(z)]\Big|\end{aligned} \tag{5.9}$$

$$\sum_{k=1}^{2n} \mid \mathscr{D}_k(z) \mid \leq I_1 + I_2 \tag{5.10}$$

Using (2.2), (2.4), (2.6),(2.7), (2.8), (2.10), (2.11), (2.12), (2.13) and (2.15), we have

$$I_1 = \mathbf{O}\Big((1-x^2)^{\frac{r-2}{2}} n^{r-1} logn\Big) \tag{5.11}$$

$$I_2 = \mathbf{O}\Big((1-x^2)^{\frac{r-2}{2}} n^{r-1} logn\Big) \tag{5.12}$$

Combining (5.11) and (5.12) give our desired lemma.

□

6 Convergence

In this section, we shall need following.

Remark 1. *Let $f(z)$ be continous for $\mid z \mid\leq 1$ and analytic for $\mid z \mid< 1$ and $f^{(r)}\epsilon\, Lip\,\nu$, $\nu > 0$, then the sequence $\{\mathscr{F}_n(z)\}$ converges uniformly to $f(z)$ in $\mid z \mid\leq 1$, which follows as*

$$\omega_r(f, n^{-1}) = O(n^{-r+1-\nu}) \qquad ,0 < \nu < 1, \tag{6.1}$$

where $\omega_r(f, n^{-1})$ be the r^{th} modulus of continuity of $f(z)$.

Remark 2. *Let $f(z)$ be continous for $\mid z \mid\leq 1$ and analytic for $\mid z \mid< 1$. Then, there exists a polynomial $F_n(z)$ of degree $\leq 4n + 2r - 1$ satisfying Jackson's inequality.*

$$\mid f(z) - F_n(z) \mid \leq C\, \omega_{r+1}(f, n^{-1}) \tag{6.2}$$

and also an inequality by Kiŝ

$$\mid F_n^{(m)}(z) \mid \leq C\, n^m\, \omega_{r+1}(f, n^{-1}) \qquad ,m\,\varepsilon\, \mathbb{Z}^+ \tag{6.3}$$

where C is a constant independent of z.

Theorem 6.1. *Let $f(z)$ be continous for $\mid z \mid\leq 1$ and analytic for $\mid z \mid< 1$. Let the arbitary numbers φ_k's be such that*

$$\mid \varphi_k \mid= O(n\, \omega_{r+1}(f, n^{-1})) \qquad ,k = 1(1)2n - 2 \tag{6.4}$$

Then sequence $\{\mathscr{F}_n(z)\}$ defined by

$$\mathscr{F}_n(z) = \sum_{k=1}^{2n-2} f(z_k)\mathscr{C}_k(z) + \sum_{k=1}^{2n} \varphi_k \mathscr{D}_k(z) \tag{6.5}$$

satisfies the relation,

$$\mid \mathscr{F}_n(z) - f(z) \mid= O((1 - x^2)^{\frac{r-2}{2}} n^r \omega_{r+1}(f, n^{-1}) logn) \tag{6.6}$$

where $\omega_{r+1}(f, n^{-1})$ be the $(r + 1)^{th}$ modulus of continuity of $f(z)$.

Proof. Since $\mathscr{F}_n(z)$ be the uniquely determined polynomial of degree $\leq 4n + 2r - 1$ and the polynomial $F_n(z)$ satisfying equation (6.2) can be expressed as

$$F_n(z) = \sum_{k=1}^{2n-2} F_n(z_k)\mathscr{C}_k(z) + \sum_{k=1}^{2n} F_n'(z_k)\mathscr{D}_k(z) \tag{6.7}$$

Then

$$\mid \mathscr{F}_n(z) - f(z) \mid \leq \mid \mathscr{F}_n(z) - F_n(z) \mid + \mid F_n(z) - f(z) \mid, \tag{6.8}$$

$$\begin{aligned} \mid \mathscr{F}_n(z) - f(z) \mid \leq & \sum_{k=1}^{2n-2} \mid f(z_k) - F_n(z_k) \mid\mid \mathscr{C}_k(z) \mid \\ & + \sum_{k=1}^{2n} \{\mid \varphi_k \mid + \mid F_n'(z_k) \mid\} \mid \mathscr{D}_k(z) \mid + \mid F_n(z) - f(z) \mid \end{aligned} \tag{6.9}$$

Using equations (6.2), (6.3), (6.4), Lemma 1 and Lemma 2, we have Theorem 6.1 □

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