An Application of q-Hypergeometric Series

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Abstract

With the use of q-Hypergeometric Series, a q-analogue of Dziok-Srivastava operator $H_s^r[\alpha_1; q]$ for a normalized class $\mathcal{A}$ of analytic functions is considered and by involving this operator a class $S_q^{r,s}([\alpha_1], A, B)$ of functions $f \in \mathcal{A}$ is defined and studied. Some equivalent conditions for the class $S_q^{r,s}([\alpha_1], A, B)$ are obtained first. A sufficient coefficient condition for functions f to be in the class $S_q^{r,s}([\alpha_1], A, B)$ is obtained and it is proved that this coefficient condition is necessary for the functions in its subclass $TS_q^{r,s}([\alpha_1], A, B)$ of functions with negative coefficients. Further, convexity and radius results are obtained for the subclass $TS_q^{r,s}([\alpha_1], A, B)$.

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1 Introduction and Preliminaries

Let $\mathcal{A}$ be the class of functions which are analytic in the unit disk

$$\mathbb{D} = \{z : z \in \mathbb{C}, |z| < 1\}$$

normalised by the conditions $f(0) = 0, f'(0) = 1$. The function f in the class $\mathcal{A}$ has the power series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let S be a subclass of **univalent functions** $f \in \mathcal{A}$. A *starlike function* [4] is a conformal mapping of unit disk $\mathbb{D}$ onto a domain starlike with respect to origin. Let S^* denotes the subclass of S consisting of *starlike functions.* Then $f \in S^*$ if and only if

$$\Re e \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \ (z \in \mathbb{D}).$$

For $0 \leq \alpha < 1$, the class $S^*(\alpha)$ of starlike functions of order α, consists of $f \in \mathcal{A}$ satisfying the condition:

$$\Re e\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \ (z \in \mathbb{D}).$$

For $q \in (0,1)$ and for a non-negative integer n, $[n]_q$ is defined by

$$[n]_q = \frac{1-q^n}{1-q} = \begin{cases} 1+q+\cdots+q^{n-1}, & \text{if } n \in \mathbb{N} \\ 0, & \text{if } n = 0 \end{cases}$$

and the q-derivative operator ∂_q [2] (see also [5]) for $f \in \mathcal{A}$ is defined by

$$\partial_q f(z) = \begin{cases} \frac{f(qz)-f(z)}{z(q-1)} & (z \neq 0), \\ 1 & (z = 0). \end{cases}$$

Clearly,

$$\lim_{q \to 1^-} [n]_q = n$$

and

$$\partial_q z^n = [n]_q z^{n-1}.$$

The *convolution* or *Hadamard product* of two power series

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \quad \text{and } g(z) = \sum_{n=1}^{\infty} b_n z^n$$

convergent in the unit disk $\mathbb{D}$ is defined as the function $h = f * g$ with convergent power series

$$h(z) = \sum_{n=1}^{\infty} a_n b_n z^n.$$

A q-hypergeometric series [5], denoted by ${}_r\phi_s(\alpha_i; \beta_j; q, z)$ is given by

$$(1.2) \qquad {}_r\phi_s(\alpha_i; \beta_j; q, z) = \sum_{n=0}^{\infty} \frac{(\alpha_1;q)_n(\alpha_2;q)_n \cdots (\alpha_r;q)_n}{(q;q)_n(\beta_1;q)_n(\beta_2;q)_n \cdots (\beta_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n,$$

where $\binom{n}{2} = \frac{n(n-1)}{2}$, (in case $r > s+1$) $q \neq 0$, $\alpha_i's$ $(i = 1, 2, ..., r)$ and $\beta_i's$ $(i = 1, 2, ...s)$ are complex numbers, $\beta_i \neq q^{-m}(m = 0, 1, ...)$ are such that the denominator factors in the series are never zero. The symbol

$$(\alpha_i; q)_n = \begin{cases} 1 & \text{for } n = 0 \\ (1-\alpha_i)(1-\alpha_i q) \cdots (1-\alpha_i q^{n-1}) & \text{for } n = 1, 2, 3, ... \end{cases}$$

is the q-shifted factorial. We also have

$$(\alpha_i; q)_\infty = \prod_{n=0}^{\infty} (1 - \alpha_i q^n)$$

and the q-Gamma function $\Gamma_q(n)$ is given by [5]

$$\begin{aligned} \Gamma_q(n) &= \frac{(q;q)_\infty}{(q^n;q)_\infty} (1-q)^{1-n} \\ &= [1]_q [2]_q \cdots [n-1]_q. \end{aligned}$$

In terms of q-Gamma function, the q-shifted factorial is given by

$$(q^a;q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \quad a \neq -n \quad (n = 0,1,2,...)$$

and

$$\lim_{q\to 1^-} \frac{(q^a;q)}{(1-q)^n} = (a)_n = a(a+1)(a+2)\cdots(a+n-1) \text{ for } n > 0.$$

is well known Pochammer symbol.

Note that the series ${}_r\phi_s$ converges absolutely for all z if $r \leq s$ and for $|z| < 1$ if $r = s+1$. Further, note that

$$\lim_{q\to 1^-} {}_r\phi_s(q^{\alpha_i};q^{\beta_j};q,(q-1)^{1+s-r}z) = {}_rF_s(\alpha_i;\beta_j;z)$$

which is a well known generalized hypergeometric function [5].

Corresponding to the q-hypergeometric series ${}_r\phi_s(\alpha_i;\beta_j;q,z)$ given by (1.2), we define a linear operator $H_s^r[\alpha_1;q] \equiv H_s^r(\alpha_i;\beta_j;q,z) : \mathcal{A} \to \mathcal{A}$ by

$$H_s^r[\alpha_1;q]f(z) = z \; {}_r\phi_s(\alpha_i;\beta_j;q,z) * f(z). \tag{1.3}$$

For a function $f(z)$ of the form (1.1), the series expansion of $H_s^r[\alpha_1;q]f(z)$ is given by

$$\begin{aligned} H_s^r[\alpha_1;q]f(z) &= z + \sum_{n=2}^{\infty} \frac{(\alpha_1;q)_{n-1}(\alpha_2;q)_{n-1}\cdots(\alpha_r;q)_{n-1}}{(q;q)_{n-1}(\beta_1;q)_{n-1}(\beta_2;q)_{n-1}\cdots(\beta_s;q)_{n-1}} \times \\ & \left[(-1)^{n-1}q^{\binom{n-1}{2}}\right]^{1+s-r} a_n z^n \end{aligned} \tag{1.4}$$

which converges absolutely in $\mathbb{D}$ if $r \leq s+1$. The operator $H_s^r[\alpha_1;q]$ is called a q-analogue of Dziok-Srivastava operator [3] .

Let us write

$$\Gamma_{n-1} = \frac{(\alpha_2;q)_{n-1}(\alpha_3;q)_{n-1}\cdots(\alpha_r;q)_{n-1}}{(q;q)_{n-1}(\beta_1;q)_{n-1}(\beta_2;q)_{n-1}\cdots(\beta_s;q)_{n-1}} \left[(-1)^{n-1}q^{\binom{n-1}{2}}\right]^{1+s-r}. \tag{1.5}$$

Then, (1.4) reduces to

$$H_s^r[\alpha_1;q]f(z) = z + \sum_{n=2}^{\infty}(\alpha_1;q)_{n-1}\Gamma_{n-1}a_n z^n.$$

Let f and g be analytic in $\mathbb{D}$. We say that f is subordinate to g in $\mathbb{D}$ written as $f \prec g$ for $z \in \mathbb{D}$ if there is an analytic function w defined on $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. Also, $f \prec g \Rightarrow f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$ [7].

Definition 1. *By involving the operator $H_s^r[\alpha_1;q]$ and the subordination, we now define a subclass $S_q^{r,s}([\alpha_1],A,B)$ consists of $f \in \mathcal{A}$ satisfying the subordination*

$$\frac{z\partial_q(H_s^r[\alpha_1;q]f(z))}{H_s^r[\alpha_1;q]f(z)} \prec \frac{1+Az}{1+Bz} \qquad (-B \leq A < B \leq 1; z \in \mathbb{D}). \tag{1.6}$$

On giving special values $A = (1+q)\alpha - 1$ $(0 \leq \alpha < 1)$ and $B = q$, we denote the class $S_q^{r,s}([\alpha_1],A,B)$ by $S_q^*(r,s,\alpha)$ which we again denote by $S_q^*(\alpha)$ when we take the values $\alpha_1 = q$, $r = s+1$, $\alpha_2 = \beta_1$,...,$\alpha_r = \beta_s$. Note that the class $S_q^*(0) \equiv S_q^*$ is the class of q-starlike functions in $\mathbb{D}$ and was studied by *Ismail et al. in* [6] (see also [8, 9]).

2 Main Results

We first find certain equivalent conditions for the class $S_q^{r,s}([\alpha_1], A, B)$ which are as follows:

Theorem 2.1. *If $f \in S_q^{r,s}([\alpha_1], A, B)$, then f satisfies*

$$\left|\frac{(1-B^2)-(1-q)(1-AB)}{(1-B^2)} - \frac{(H_s^r[\alpha_1;q]f(qz))}{(H_s^r[\alpha_1;q]f(z))}\right| < (1-q)\frac{B-A}{1-B^2} \quad (B \neq 1)$$

and in case $B = 1$,

$$\Re e\left(\frac{H_s^r[\alpha_1;q]f(qz)}{H_s^r[\alpha_1;q]f(z)}\right) < 1 - \frac{(1+A)(1-q)}{2}.$$

Proof. Let $f \in S_q^{r,s}([\alpha_1], A, B)$. Then from the class condition (1.6) we have

$$\left|\frac{z\partial_q(H_s^r[\alpha_1;q]f(z))}{H_s^r[\alpha_1;q]f(z)} - \frac{1-AB}{1-B^2}\right| < \frac{B-A}{1-B^2} \ (B \neq 1) \tag{2.1}$$

and in case $B = 1$,

$$\Re e\left(\frac{z\partial_q(H_s^r[\alpha_1;q]f(z))}{H_s^r[\alpha_1;q]f(z)}\right) > \frac{1+A}{2}. \tag{2.2}$$

Further, we have

$$\frac{z\partial_q(H_s^r[\alpha_1;q]f(z))}{H_s^r[\alpha_1;q]f(z)} = \frac{1}{q-1}\left(\frac{H_s^r[\alpha_1;q]f(qz)}{H_s^r[\alpha_1;q]f(z)} - 1\right)$$

which together with conditions (2.1) and (2.2) yields the desired result. □

Putting $A = (1+q)\alpha - 1$ and $B = q$ in Theorem 2.1, we get the following corollary:

Corollary 1. *Let $f \in S_q^*(r, s, \alpha)$. Then*

$$\left|\frac{H_s^r[\alpha_1;q]f(qz)}{H_s^r[\alpha_1;q]f(z)} - q\alpha\right| < 1 - \alpha \tag{2.3}$$

holds.

Remark 1. *If we put $\alpha_1 = q$,$r = s+1$,$\alpha_2 = \beta_1$,...,$\alpha_r = \beta_s$ in the above Corollary 1, the class $S_q^*(r, s, \alpha)$ reduces to $S_q^*(\alpha)$ and the result (2.3) reduces to the result for the class $S_q^*(\alpha)$ which was obtained earlier by Agrawal and Sahoo [1, Theorem 2.3 p. 5].*

We now give our next equivalent condition as a convolution condition:

Theorem 2.2. *A function $f \in S_q^{r,s}([\alpha_1], A, B)$ if and only if*

$$f(z) * h(z;\zeta) \neq 0 \qquad (\zeta \in \mathbb{C}, |\zeta| = 1, z \in \mathbb{D}\backslash\{0\}), \tag{2.4}$$

where

$$h(z;\zeta) = z\ {}_r\phi_s(\alpha_i;\beta_j;q,z) * \phi(z;\zeta),$$

${}_r\phi_s(\alpha_i;\beta_j;q,z)$ is given by (1.2) and

$$\phi(z;\zeta) = \frac{(B-A)\zeta z + q(1+A\zeta)z^2}{(1-qz)(1-z)}.$$

Proof. Let $f \in S_q^{r,s}([\alpha_1], A, B)$. Then from the condition (2.1) we have

$$\frac{z\partial_q(H_s^r[\alpha_1;q]f(z))}{H_s^r[\alpha_1;q]f(z)} \neq \frac{1+A\zeta}{1+B\zeta} \quad (\zeta \in \mathbb{C}, |\zeta| = 1, z \in \mathbb{D}\backslash\{0\})\,. \tag{2.5}$$

On writing

$$\begin{aligned} z\partial_q(H_s^r[\alpha_1;q]f(z)) &= H_s^r[\alpha_1;q]f(z) * \frac{z}{(1-qz)\,(1-z)} \\ &= f(z) * H_s^r[\alpha_1;q]\frac{z}{(1-qz)\,(1-z)}, \end{aligned}$$

and

$$\begin{aligned} H_s^r[\alpha_1;q]f(z) &= H_s^r[\alpha_1;q]f(z) * \frac{z}{1-z} \\ &= f(z) * H_s^r[\alpha_1;q]\frac{z}{1-z}, \end{aligned}$$

condition (2.5) can equivalently be written by

$$f(z) * \left[(1+B\zeta)\,H_s^r[\alpha_1;q]\frac{z}{(1-qz)\,(1-z)} - (1+A\zeta)\,H_s^r[\alpha_1;q]\frac{z}{1-z}\right] \neq 0,$$

where the operator $H_s^r[\alpha_1;q]$ is defined by (1.3) in terms of a q-hypergeometric series ${}_r\phi_s(\alpha_i;\beta_j;q,z)$. This proves the result (2.4). □

Our next results are based on coefficient inequality.

Theorem 2.3. *Let f be of the form (1.1) and let $-B \le A < B \le 1$. If*

$$\sum_{n=2}^{\infty} C_n|a_n| \le B - A, \tag{2.6}$$

where

$$C_n = [(1+B)[n]_q - (1+A)](\alpha_1;q)_{n-1}\,|\Gamma_{n-1}|\,, \tag{2.7}$$

Γ_{n-1} is given by (1.5) with $r \le s+1$ and real parameters $\alpha_i < 1$ $(i = 1, 2, ..., r)$, $\beta_i < 1$ $(i = 1, 2, ...s)$, then $f \in S_q^{r,s}([\alpha_1], A, B)$. The inequality (2.6) is sharp for the function

$$f_n(z) = z - \frac{B-A}{C_n}z^n \ (n \ge 2),$$

where $-B \le A < B \le 1$ and C_n is given by (2.7).

Proof. To prove $f \in S_q^{r,s}([\alpha_1], A, B)$, we need to show for an analytic function w such that $w(0) = 0$, $|w(z)| < 1$,

$$\frac{z\partial_q(H_s^r[\alpha_1;q]f(z))}{H_s^r[\alpha_1;q]f(z)} = \frac{1+Aw(z)}{1+Bw(z)} \ (z \in \mathbb{D}) \tag{2.8}$$

which is true at $z = 0$. Suppose $z \neq 0$. Then to show condition (2.8) or equivalently

$$\left|\frac{H_s^r[\alpha_1;q]f(z)}{z} - \partial_q(H_s^r[\alpha_1;q]f(z))\right| - \left|B\partial_q(H_s^r[\alpha_1;q]f(z)) - A\frac{H_s^r[\alpha_1;q]f(z)}{z}\right| < 0,$$

consider

$$\begin{aligned}
&\left|\frac{H_s^r[\alpha_1;q]f(z)}{z} - \partial_q(H_s^r[\alpha_1;q]f(z))\right| - \left|B\partial_q(H_s^r[\alpha_1;q]f(z)) - A\frac{H_s^r[\alpha_1;q]f(z)}{z}\right| \\
&= \left|\sum_{n=2}^{\infty}(\alpha_1;q)_{n-1}\Gamma_{n-1}\left([n]_q - 1\right)a_n z^{n-1}\right| \\
&- \left|(B-A) + \sum_{n=2}^{\infty}(\alpha_1;q)_{n-1}\Gamma_{n-1}\left(B[n]_q - A\right)a_n z^{n-1}\right| \\
&\leq \sum_{n=2}^{\infty}(\alpha_1;q)_{n-1}\left|\Gamma_{n-1}\right|\left([n]_q - 1\right)|a_n|\,|z^{n-1}| - (B-A) \\
&+ \sum_{n=2}^{\infty}(\alpha_1;q)_{n-1}\left|\Gamma_{n-1}\right|(B[n]_q - A)|a_n||z^{n-1}| \\
&< \sum_{n=2}^{\infty}(\alpha_1;q)_{n-1}\left|\Gamma_{n-1}\right|\{(B+1)[n]_q - (A+1)\}|a_n| - (B-A) \\
&= \sum_{n=2}^{\infty}C_n|a_n| - (B-A) \leq 0
\end{aligned}$$

using (2.6), where C_n is given by (2.7). □

Now to show that the condition (2.6) is both necessary and sufficient, we consider functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty}|a_n|z^n \qquad (z \in \mathbb{D}) \tag{2.9}$$

and consider for $r \leq s+1$ with real parameters $\alpha_i < 1$ $(i = 1,2,...,r)$, $\beta_i < 1$ $(i = 1,2,...s)$, a q-hypergeometric series

$${}_r\Phi_s(\alpha_i;\beta_j;q,z) = \sum_{n=0}^{\infty}\frac{(\alpha_1;q)_n(\alpha_2;q)_n\cdots(\alpha_r;q)_n}{(q;q)_n(\beta_1;q)_n(\beta_2;q)_n\cdots(\beta_s;q)_n}q^{\binom{n}{2}(1+s-r)}z^n. \tag{2.10}$$

Then the operator $H_s^r[\alpha_1;q]$ is defined by

$$H_s^r[\alpha_1;q]f(z) = z\;{}_r\Phi_s(\alpha_i;\beta_j;q,z) * f(z) \tag{2.11}$$

and the series expansion of $H_s^r[\alpha_1;q]f(z)$ is given by

$$H_s^r[\alpha_1;q]f(z) = z - \sum_{n=2}^{\infty}(\alpha_1;q)_{n-1}\left|\Gamma_{n-1}\right|a_n z^n, \tag{2.12}$$

where Γ_{n-1} has the value (1.5).

Denote by $TS_q^{r,s}([\alpha_1], A, B)$ a subclass of analytic functions $f \in S_q^{r,s}([\alpha_1], A, B)$, of the form (2.9) and the operator $H_s^r[\alpha_1; q]$ is defined by (2.11). If $f \in TS_q^{r,s}([\alpha_1], A, B)$, then $H_s^r[\alpha_1; q]f(z)$ has a series (2.12) with negative coefficients. Further, we denote by $TS_q^*(r, s, \alpha)$ and $TS_q^*(\alpha)$ the subclass of functions $f \in S_q^*(r, s, \alpha)$ and $f \in S_q^*(\alpha)$, respectively, when f is of the form (2.9) and the operator $H_s^r[\alpha_1; q]$ is defined by (2.11).

Theorem 2.4. *Let f be of the form (2.9) and let $r \leq s+1$ with real parameters $\alpha_i < 1$ $(i = 1, 2, ..., r)$, $\beta_i < 1$ $(i = 1, 2, ...s)$, then $f \in TS_q^{r,s}([\alpha_1], A, B)$ if and only if the condition (2.6) holds.*

Proof. By Theorem 2.3, the condition (2.6) is sufficient for functions $f \in TS_q^{r,s}([\alpha_1], A, B)$. Now to prove the necessary part, we need to show that if $f \in TS_q^{r,s}([\alpha_1], A, B)$ with $H_s^r[\alpha_1; q]f(z)$ of the form (2.12), then the condition (2.6) holds. Assume that $f \in TS_q^{r,s}([\alpha_1], A, B)$ with $H_s^r[\alpha_1; q]f(z)$ of the form (2.12), then by class condition (1.6), we have

$$\frac{z\partial_q(H_s^r[\alpha_1; q]f(z))}{H_s^r[\alpha_1; q]f(z)} = \frac{z - \sum\limits_{n=2}^{\infty}[n]_q(\alpha_1; q)_{n-1}\,|\Gamma_{n-1}|\,|a_n|z^n}{z - \sum\limits_{n=2}^{\infty}(\alpha_1; q)_{n-1}\,|\Gamma_{n-1}|\,|a_n|z^n} = \frac{1 + Aw(z)}{1 + Bw(z)} \qquad (z \in \mathbb{D}),$$

where w is analytic in $\mathbb{D}$ such that $w(0) = 0$, $|w(z)| < 1$. Hence, we have for all $z \in \mathbb{D}$,

$$\begin{aligned} & \left|\sum_{n=2}^{\infty}(\alpha_1; q)_{n-1}\,|\Gamma_{n-1}|\,([n]_q - 1)\,|a_n|z^{n-1}\right| \\ & - \left|(B - A) - \sum_{n=2}^{\infty}(\alpha_1; q)_{n-1}\,|\Gamma_{n-1}|\,(B[n]_q - A)\,|a_n|z^{n-1}\right| \\ < \;& 0 \end{aligned}$$

which implies that

$$\sum_{n=2}^{\infty}(\alpha_1; q)_{n-1}\,|\Gamma_{n-1}|\,([n]_q - 1)\,|a_n| - (B - A) + \sum_{n=2}^{\infty}(\alpha_1; q)_{n-1}\,|\Gamma_{n-1}|\,(B[n]_q - A)|a_n| \leq 0$$

or

$$\sum_{n=2}^{\infty} C_n|a_n| \leq B - A,$$

where C_n is given by (2.7). This completes the proof. □

Putting $A = (1 + q)\alpha - 1$ and $B = q$ in Theorem 2.4, we get the following result.

Corollary 2. *Let f be of the form (2.9). A necessary and sufficient condition for the function f to be in class $TS_q^*(r, s, \alpha)$ is*

$$\sum_{n=2}^{\infty}([n]_q - \alpha)(\alpha_1; q)_{n-1}\,|\Gamma_{n-1}|\,|a_n| \leq 1 - \alpha.$$

Further, if we put $\alpha_1 = q, r = s+1, \alpha_2 = \beta_1, \ldots, \alpha_r = \beta_s$ in the Corollary 2, we get our next result for the class $TS_q^*[\alpha]$ of q-starlike functions of order α in $\mathbb{D}$.

Corollary 3. *Let f be of the form (2.9). A necessary and sufficient condition for the function f to be in class $TS_q^*[\alpha]$ is*

$$\sum_{n=2}^{\infty}([n]_q - \alpha)|a_n| \leq 1 - \alpha.$$

Theorem 2.5. *The class $TS_q^{r,s}([\alpha_1], A, B)$ is a convex class.*

Proof. Let $f, g \in TS^{r,s}([\alpha_1], A, B)$ be of the form $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ and $g(z) = z - \sum_{n=2}^{\infty} |b_n| z^n$, respectively. Then for $0 \leq \rho \leq 1$, consider a convex combination F of f and g which can be written as

$$\begin{aligned} F(z) &= \rho f(z) + (1-\rho) g(z) \\ &= \rho \left(z - \sum_{n=2}^{\infty} |a_n| z^n \right) + (1-\rho) \left(z - \sum_{n=2}^{\infty} |b_n| z^n \right) \\ &= z - \sum_{n=2}^{\infty} \left(\rho |a_n| + (1-\rho)|b_n| \right) z^n. \end{aligned}$$

On applying Theorem 2.4, for C_n defined by (2.7), we get

$$\begin{aligned} &\sum_{n=2}^{\infty} C_n(\rho|a_n| + (1-\rho)|b_n|) \\ &= \rho \sum_{n=2}^{\infty} C_n |a_n| + (1-\rho) \sum_{n=2}^{\infty} C_n |b_n| \\ &\leq \rho(B-A) + (1-\rho)(B-A) = B - A. \end{aligned}$$

which again by Theorem 2.4 proves $F \in TS_q^{r,s}([\alpha_1], A, B)$. Thus, the class $TS_q^{r,s}([\alpha_1], A, B)$ is convex. □

Theorem 2.6. *Let $f \in TS_q^{r,s}([\alpha_1], A, B)$. Then the radius of q-starlikeness of order α is given by*

$$r_{S_q^*(\alpha)}\left(TS_q^{r,s}([\alpha_1], A, B)\right) \leq \inf_{n \geq 2} \left[\frac{1-\alpha}{B-A} \min \left(\frac{C_n}{[n]_q - \alpha} \right) \right]^{\frac{1}{n-1}}, \tag{2.13}$$

where C_n is given by (2.7).

Proof. Let $f \in TS_q^{r,s}([\alpha_1], A, B)$. Then from Theorem 2.4, we have

$$\sum_{n=2}^{\infty} \frac{C_n}{B-A} |a_n| \leq 1.$$

Let r_0 be the radius of q-starlikness of order α for functions in the class $TS_q^{r,s}([\alpha_1], A, B)$. Then from Corollary 3, $\frac{f(r_0 z)}{r_0} \in TS_q^*(\alpha)$ if and only if

$$\sum_{n=2}^{\infty}([n]_q - \alpha)|a_n| r_0^{n-1} \leq 1 - \alpha$$

which is true if

$$\frac{([n]_q - \alpha)}{1-\alpha} r_0^{n-1} \leq \frac{C_n}{B-A} \qquad (n = 2, 3, ...)$$

or if,

$$r_0 \leq \left\{ \frac{1-\alpha}{B-A} \min\left(\frac{C_n}{[n]_q - \alpha} \right) \right\}^{\frac{1}{n-1}} \qquad (n = 2, 3, ...)$$

it follows that the radius $r_0 = r_{S_q^*(\alpha)}\left(TS_q^{r,s}([\alpha_1], A, B)\right)$ satisfies (2.13). □

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