An Application of $q$-Hypergeometric Series

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Abstract

With the use of $q$-Hypergeometric Series, a $q$-analogue of Dziok-Srivastava operator $H_{r,s}^{[\alpha_1]}[\alpha; q]$ for a normalized class $\mathcal{A}$ of analytic functions is considered and by involving this operator a class $S_{q}^{r,s}([\alpha_1], A, B)$ of functions $f \in \mathcal{A}$ is defined and studied. Some equivalent conditions for the class $S_{q}^{r,s}([\alpha_1], A, B)$ are obtained first. A sufficient coefficient condition for functions $f$ to be in the class $S_{q}^{r,s}([\alpha_1], A, B)$ is obtained and it is proved that this coefficient condition is necessary for the functions in its subclass $TS_{q}^{r,s}([\alpha_1], A, B)$ of functions with negative coefficients. Further, convexity and radius results are obtained for the subclass $TS_{q}^{r,s}([\alpha_1], A, B)$.

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1 Introduction and Preliminaries

Let $\mathcal{A}$ be the class of functions which are analytic in the unit disk

$$D = \{ z : z \in \mathbb{C}, |z| < 1 \}$$

normalised by the conditions $f(0) = 0, f'(0) = 1$. The function $f$ in the class $\mathcal{A}$ has the power series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

Let $\mathcal{S}$ be a subclass of univalent functions $f \in \mathcal{A}$. A starlike function [4] is a conformal mapping of unit disk $D$ onto a domain starlike with respect to origin. Let $\mathcal{S}^*$ denotes the subclass of $\mathcal{S}$ consisting of starlike functions. Then $f \in \mathcal{S}^*$ if and only if

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in D).$$
For $0 \leq \alpha < 1$, the class $S^{*}(\alpha)$ of starlike functions of order $\alpha$, consists of $f \in A$ satisfying the condition:

\[ \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in \mathbb{D}). \]

For $q \in (0, 1)$ and for a non-negative integer $n$, $[n]_q$ is defined by

\[ [n]_q = \begin{cases} \frac{1 - q^n}{1 - q}, & \text{if } n \in \mathbb{N} \\ 0, & \text{if } n = 0 \end{cases} \]

and the $q$-derivative operator $\partial_q [2]$ (see also [5]) for $f \in A$ is defined by

\[ \partial_q f(z) = \begin{cases} \frac{f(qz) - f(z)}{z(q-1)}, & (z \neq 0) \\ 1, & (z = 0) \end{cases} \]

Clearly,

\[ \lim_{q \to 1} [n]_q = n \]

and

\[ \partial_q z^n = [n]_q z^{n-1}. \]

The convolution or Hadamard product of two power series

\[ f(z) = \sum_{n=1}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \]

convergent in the unit disk $\mathbb{D}$ is defined as the function $h = f \ast g$ with convergent power series

\[ h(z) = \sum_{n=1}^{\infty} a_n b_n z^n. \]

A $q$-hypergeometric series [5], denoted by $r \phi_s(\alpha_i; \beta_j; q, z)$ is given by

\[ r \phi_s(\alpha_i; \beta_j; q, z) = \sum_{n=0}^{\infty} \frac{(\alpha_1; q)_n (\alpha_2; q)_n \cdots (\alpha_r; q)_n}{(q; q)_n (\beta_1; q)_n (\beta_2; q)_n \cdots (\beta_s; q)_n} \left[ (-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n, \]

where $\binom{n}{2} = \frac{n(n-1)}{2}$, (in case $r > s + 1$) $q \neq 0$, $\alpha_i's \ (i = 1, 2, ..., r)$ and $\beta_j's \ (i = 1, 2, ..., s)$ are complex numbers, $\beta_i \neq q^{-m} (m = 0, 1, ...)$ are such that the denominator factors in the series are never zero. The symbol

\[ (\alpha_1; q)_n = \begin{cases} 1 & \text{for } n = 0 \\ (1 - \alpha_1)(1 - \alpha_2)(1 - \alpha_3)(1 - \alpha_4)(1 - \alpha_5)(1 - \alpha_6) & \text{for } n = 1, 2, 3, ... \end{cases} \]

is the $q$-shifted factorial. We also have

\[ (\alpha_1; q)_\infty = \prod_{n=0}^{\infty} (1 - \alpha_1 q^n) \]

and the $q$-Gamma function $\Gamma_q(n)$ is given by [5]

\[ \Gamma_q(n) = \frac{(q; q)_\infty}{(q^n; q)_\infty} (1 - q)^{1-n} = [1]_q [2]_q \cdots [n-1]_q. \]
In terms of $q$-Gamma function, the $q$-shifted factorial is given by

$$(q^n)_q = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, \quad a \neq -n \quad (n = 0, 1, 2, \ldots)$$

and

$$\lim_{q \to 1} \frac{(q^n)_q}{(1-q)^n} = (a)_n = a(a+1)(a+2)\cdots(a+n-1) \text{ for } n > 0.$$ 

is well known Pochammer symbol.

Note that the series $r\phi_s$ converges absolutely for all $z$ if $r \leq s$ and for $|z| < 1$ if $r = s + 1$.

Further, note that

$$\lim_{q \to 1} r\phi_s(q^n; q, (q-1)^{1+s-r}z) = rF_s(\alpha; \beta, z)$$

which is a well known generalized hypergeometric function [5].

Corresponding to the $q$-hypergeometric series $r\phi_s(\alpha; \beta, q, z)$ given by (1.2), we define a linear operator $H^*_s[\alpha; q] \equiv H^*_s(\alpha; \beta; q, z) : A \to A$ by

$$(1.3) \quad H^*_s[\alpha; q]f(z) = z \cdot r\phi_s(\alpha; \beta, q, z)*f(z).$$

For a function $f(z)$ of the form (1.1), the series expansion of $H^*_s[\alpha; q]f(z)$ is given by

$$(1.4) \quad H^*_s[\alpha; q]f(z) = z + \sum_{n=2}^{\infty} \frac{(\alpha_1; q)_{n-1}(\alpha_2; q)_{n-1}\cdots(\alpha_r; q)_{n-1}}{(q; q)_{n-1}(\beta_1; q)_{n-1}(\beta_2; q)_{n-1}\cdots(\beta_s; q)_{n-1}} \times (-1)^{n-1}q^{(n-1)}(\frac{n-1}{2})^{1+s-r}a_nz^n$$

which converges absolutely in $\mathbb{D}$ if $r \leq s + 1$. The operator $H^*_s[\alpha; q]$ is called a $q$-analogue of Dziok-Srivastava operator [3].

Let us write

$$(1.5) \quad \Gamma_{n-1} = \frac{(\alpha_2; q)_{n-1}(\alpha_3; q)_{n-1}\cdots(\alpha_r; q)_{n-1}}{(q; q)_{n-1}(\beta_1; q)_{n-1}(\beta_2; q)_{n-1}\cdots(\beta_s; q)_{n-1}} \times (-1)^{n-1}q^{(n-1)}(\frac{n-1}{2})^{1+s-r}.$$ 

Then, (1.4) reduces to

$$H^*_s[\alpha; q]f(z) = z + \sum_{n=2}^{\infty} (\alpha_1; q)_{n-1}\Gamma_{n-1}a_nz^n.$$

Let $f$ and $g$ be analytic in $\mathbb{D}$. We say that $f$ is subordinate to $g$ in $\mathbb{D}$ written as $f \prec g$ for $z \in \mathbb{D}$ if there is an analytic function $w$ defined on $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ for all $z \in \mathbb{D}$. Also, $f \prec g \Rightarrow f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$ [7].

**Definition 1.** By involving the operator $H^*_s[\alpha; q]$ and the subordination, we now define a subclass $S^q_{\alpha,\beta}([\alpha_1], A, B)$ consists of $f \in A$ satisfying the subordination

$$(1.6) \quad \frac{z \partial_q[H^*_s[\alpha; q]f(z)]}{H^*_s[\alpha; q]f(z)} < \frac{1 + A z}{1 + B z} \quad (-B \leq A < B \leq 1; z \in \mathbb{D}).$$

On giving special values $A = (1 + q)\alpha - 1 \ (0 \leq \alpha < 1)$ and $B = q$, we denote the class $S^q_r,\alpha([\alpha_1], A, B)$ by $S^q_{\alpha,\beta}([\alpha_1], A, B)$ by which we again denote by $S^q_{\alpha}([\alpha_1], A, B)$ when we take the values $\alpha_1 = q, r = s + 1, \alpha_2 = \beta_1, \ldots, \alpha_r = \beta_s$. Note that the class $S^q_{\alpha}(0) \equiv S^q_{\alpha}$ is the class of $q$-starlike functions in $\mathbb{D}$ and was studied by Ismail et al. in [6] (see also [8, 9]).
2 Main Results

We first find certain equivalent conditions for the class $S^*_{q}(\{\alpha\}, A, B)$ which are as follows:

**Theorem 2.1.** If $f \in S^*_{q}(\{\alpha\}, A, B)$, then $f$ satisfies

$$\left|\frac{(1 - B^2) - (1 - q)(1 - AB)}{(1 - B^2)} - \frac{H_\infty^r[\alpha; q]f(qz)}{H_\infty^r[\alpha; q]f(z)}\right| < (1 - q)\frac{B - A}{1 - B^2} \quad (B \neq 1)$$

and in case $B = 1$,

$$\Re \left(\frac{H_\infty^r[\alpha; q]f(qz)}{H_\infty^r[\alpha; q]f(z)}\right) < 1 - \frac{(1 + A)(1 - q)}{2}.$$

**Proof.** Let $f \in S^*_{q}(\{\alpha\}, A, B)$. Then from the class condition (1.6) we have

$$\left(1 - q\right)H_\infty^r[\alpha; q]f(qz) = \frac{1}{H_\infty^r[\alpha; q]f(z)}(1 - B^2) - (1 - q)(1 - AB) \quad (\alpha \neq 1)$$

and in case $B = 1$,

$$\Re \left(\frac{z\partial_q(\alpha; q) f(z)}{H_\infty^r[\alpha; q]f(z)}\right) > \frac{1 + A}{2}.$$

Further, we have

$$\frac{z\partial_q(\alpha; q) f(z)}{H_\infty^r[\alpha; q]f(z)} = \frac{1}{q - 1} \left(\frac{H_\infty^r[\alpha; q]f(qz)}{H_\infty^r[\alpha; q]f(z)} - 1\right)$$

which together with conditions (2.1) and (2.2) yields the desired result. \hfill \Box

Putting $A = (1 + q)\alpha - 1$ and $B = q$ in Theorem 2.1, we get the following corollary:

**Corollary 1.** Let $f \in S^*_{q}(r, s, \alpha)$. Then

$$\left|\frac{H_\infty^r[\alpha; q]f(qz)}{H_\infty^r[\alpha; q]f(z)} - q\alpha\right| < 1 - \alpha$$

holds.

**Remark 1.** If we put $\alpha_1 = q,r = s + 1,\alpha_2 = \beta_1,...,\alpha_r = \beta_s$ in the above Corollary 1, the class $S^*_{q}(r, s, \alpha)$ reduces to $S^*_{q}(\alpha)$ and the result (2.3) reduces to the result for the class $S^*_{q}(\alpha)$ which was obtained earlier by Agrawal and Sahoo [1, Theorem 2.3 p. 5].

We now give our next equivalent condition as a convolution condition:

**Theorem 2.2.** A function $f \in S^*_{q}(\{\alpha\}, A, B)$ if and only if

$$f(z) * h(z; \zeta) \neq 0 \quad (\zeta \in \mathbb{C}, |\zeta| = 1, z \in \mathbb{D}\setminus\{0\})$$

where

$$h(z; \zeta) = z^{-r\phi_s(\alpha_i; \beta_j; q, z)} \ast \phi(z; \zeta),$$

$r\phi_s(\alpha_i; \beta_j; q, z)$ is given by (1.2) and

$$\phi(z; \zeta) = \frac{(B - A)\zeta + q(1 + A\zeta)z^2}{(1 - qz)(1 - z)}.$$
Proof. Let \( f \in S_{q}^{r,s}([\alpha_1], A, B) \). Then from the condition (2.1) we have

\[
\frac{z \partial_q (H_r^s[\alpha_1; q]f(z))}{H_r^s[\alpha_1; q]f(z)} \neq \frac{1 + A \zeta}{1 + B \zeta} \quad (\zeta \in \mathbb{C}, |\zeta| = 1, z \in \mathbb{D} \setminus \{0\}).
\]

On writing

\[
z \partial_q (H_r^s[\alpha_1; q]f(z)) = H_r^s[\alpha_1; q]f(z) * \frac{z}{(1 - qz)(1 - z)},
\]

and

\[
H_r^s[\alpha_1; q]f(z) = f(z) \ast H_r^s[\alpha_1; q] * \frac{z}{1 - z},
\]

condition (2.5) can equivalently be written by

\[
f(z) * \left[ (1 + B \zeta) H_r^s[\alpha_1; q] - (1 + A \zeta) H_r^s[\alpha_1; q] * \frac{z}{1 - z} \right] \neq 0,
\]

where the operator \( H_r^s[\alpha_1; q] \) is defined by (1.3) in terms of a \( q \)-hypergeometric series \( r \phi_s(\alpha_i; \beta_j; q, z) \). This proves the result (2.4). \( \square \)

Our next results are based on coefficient inequality.

**Theorem 2.3.** Let \( f \) be of the form (1.1) and let \(-B \leq A < B \leq 1\). If

\[
\sum_{n=2}^{\infty} C_n |a_n| \leq B - A,
\]

where

\[
C_n = [(1 + B)[n]_q - (1 + A)(\alpha_1; q)_{n-1} |\Gamma_{n-1}|],
\]

\( \Gamma_{n-1} \) is given by (1.5) with \( r \leq s+1 \) and real parameters \( \alpha_i < 1 \) \( (i = 1, 2, ..., r) \), \( \beta_i < 1 \) \( (i = 1, 2, ..., s) \), then \( f \in S_{q}^{r,s}([\alpha_1], A, B) \). The inequality (2.6) is sharp for the function

\[
f_n(z) = z - \frac{B - A}{C_n} z^n \quad (n \geq 2),
\]

where \(-B \leq A < B \leq 1\) and \( C_n \) is given by (2.7).

Proof. To prove \( f \in S_{q}^{r,s}([\alpha_1], A, B) \), we need to show for an analytic function \( w \) such that \( w(0) = 0 \), \( |w(z)| < 1 \),

\[
\frac{z \partial_q (H_r^s[\alpha_1; q]f(z))}{H_r^s[\alpha_1; q]f(z)} = \frac{1 + A w(z)}{1 + B w(z)} \quad (z \in \mathbb{D})
\]
which is true at \( z = 0 \). Suppose \( z \neq 0 \). Then to show condition (2.8) or equivalently

\[
\left| \frac{H_s^r[\alpha_1; q]f(z)}{z} - \partial_q(H_s^r[\alpha_1; q]f(z)) \right| < 0,
\]

consider

\[
\left| \frac{H_s^r[\alpha_1; q]f(z)}{z} - \partial_q(H_s^r[\alpha_1; q]f(z)) \right| - \left| B \partial_q(H_s^r[\alpha_1; q]f(z)) - A \frac{H_s^r[\alpha_1; q]f(z)}{z} \right|
\]

\[
= \left| \sum_{n=2}^\infty (\alpha_1; q)_{n-1} \Gamma_{n-1} ([n]_q - 1) a_n z^{n-1} \right| - (B - A) \sum_{n=2}^\infty (\alpha_1; q)_{n-1} \Gamma_{n-1} ([n]_q - 1) a_n z^{n-1}
\]

\[
\leq \sum_{n=2}^\infty (\alpha_1; q)_{n-1} \Gamma_{n-1} \{ ([n]_q - 1) |a_n| |z|^{n-1} \} - (B - A)
\]

\[
+ \sum_{n=2}^\infty (\alpha_1; q)_{n-1} \Gamma_{n-1} ([n]_q - 1) |a_n| |z|^{n-1}
\]

\[
< \sum_{n=2}^\infty (\alpha_1; q)_{n-1} \Gamma_{n-1} \{ ([n]_q - (A + 1)) |a_n| - (B - A)
\]

\[
= \sum_{n=2}^\infty C_n |a_n| - (B - A) \leq 0
\]

using (2.6), where \( C_n \) is given by (2.7). \qed

Now to show that the condition (2.6) is both necessary and sufficient, we consider functions \( f \) of the form

\[
f(z) = z - \sum_{n=2}^\infty |a_n| z^n \quad (z \in \mathbb{D})
\]

and consider \( r \leq s + 1 \) with real parameters \( \alpha_i < 1 \ (i = 1, 2, ..., r) \), \( \beta_i < 1 \ (i = 1, 2, ..., s) \), a \( q \)-hypergeometric series

\[
\Phi_s(\alpha_1; \beta_j; q, z) = \sum_{n=0}^\infty \frac{(\alpha_1; q)_n (\alpha_2; q)_n \cdots (\alpha_r; q)_n (\beta_1; q)_n (\beta_2; q)_n \cdots (\beta_s; q)_n}{(q; q)_n (\beta_1; q)_n (\beta_2; q)_n \cdots (\beta_s; q)_n} q^n |z|^{(1+s-r)} z^n.
\]

Then the operator \( H_s^r[\alpha_1; q] \) is defined by

\[
H_s^r[\alpha_1; q]f(z) = z \Phi_s(\alpha_1; \beta_j; q, z) \ast f(z)
\]

and the series expansion of \( H_s^r[\alpha_1; q]f(z) \) is given by

\[
H_s^r[\alpha_1; q]f(z) = z - \sum_{n=2}^\infty (\alpha_1; q)_{n-1} |\Gamma_{n-1}| a_n z^n,
\]
where \( \Gamma_{n-1} \) has the value (1.5).

Denote by \( TS^r_q([\alpha_1], A, B) \) a subclass of analytic functions \( f \in S^r_q([\alpha_1], A, B) \), of the form (2.9) and the operator \( H^*_r[\alpha_1; q] \) is defined by (2.11). If \( f \in TS^r_q([\alpha_1], A, B) \), then \( H^*_r[\alpha_1; q]f(z) \) has a series (2.12) with negative coefficients. Further, we denote by \( TS^*_q(r,s, \alpha) \) and \( TS^*_q(\alpha) \) the subclass of functions \( f \in S^*_q(r,s, \alpha) \) and \( f \in S^*_q(\alpha) \), respectively, when \( f \) is of the form (2.9) and the operator \( H^*_r[\alpha_1; q] \) is defined by (2.11).

**Theorem 2.4.** Let \( f \) be of the form (2.9) and let \( r \leq s + 1 \) with real parameters \( \alpha_i < 1 \) (\( i = 1, 2, \ldots, r \)), then \( f \in TS^r_q([\alpha_1], A, B) \) if and only if the condition (2.6) holds.

**Proof.** By Theorem 2.3, the condition (2.6) is sufficient for functions \( f \in TS^r_q([\alpha_1], A, B) \). Now to prove the necessary part, we need to show that if \( f \in TS^r_q([\alpha_1], A, B) \) with \( H^*_r[\alpha_1; q]f(z) \) of the form (2.12), then the condition (2.6) holds. Assume that \( f \in TS^r_q([\alpha_1], A, B) \) with \( H^*_r[\alpha_1; q]f(z) \) of the form (2.12), then by class condition (1.6), we have

\[
\frac{z\partial_q(H^*_r[\alpha_1; q]f(z))}{H^*_r[\alpha_1; q]f(z)} = \frac{z - \sum_{n=2}^{\infty} [n]_q (\alpha_1; q)_{n-1} |\Gamma_{n-1}| |a_n| z^n}{z - \sum_{n=2}^{\infty} (\alpha_1; q)_{n-1} |\Gamma_{n-1}| |a_n| z^n} = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (z \in \mathbb{D}),
\]

where \( w \) is analytic in \( \mathbb{D} \) such that \( w(0) = 0, \ |w(z)| < 1 \). Hence, we have for all \( z \in \mathbb{D}, \)

\[
|\sum_{n=2}^{\infty} (\alpha_1; q)_{n-1} |\Gamma_{n-1}| ([n]_q - 1) |a_n| z^{n-1}| - \sum_{n=2}^{\infty} (\alpha_1; q)_{n-1} |\Gamma_{n-1}| |B[n]_q - A| |a_n| z^{n-1}| \leq 0
\]

which implies that

\[
\sum_{n=2}^{\infty} (\alpha_1; q)_{n-1} |\Gamma_{n-1}| ([n]_q - 1) |a_n| - (B - A) + \sum_{n=2}^{\infty} (\alpha_1; q)_{n-1} |\Gamma_{n-1}| |B[n]_q - A| |a_n| \leq 0
\]

or

\[
\sum_{n=2}^{\infty} C_n |a_n| \leq B - A,
\]

where \( C_n \) is given by (2.7). This completes the proof. \( \square \)

**Corollary 2.** Let \( f \) be of the form (2.9). A necessary and sufficient condition for the function \( f \) to be in class \( TS^*_q(r,s,\alpha) \) is

\[
\sum_{n=2}^{\infty} ([n]_q - \alpha)(\alpha_1; q)_{n-1} |\Gamma_{n-1}| |a_n| \leq 1 - \alpha.
\]
Further, if we put \( \alpha_1 = q, r = s + 1, \alpha_2 = \beta_1, \ldots, \alpha_r = \beta_s \) in the Corollary 2, we get our next result for the class \( TS^*_{q}[\alpha] \) of \( q \)-starlike functions of order \( \alpha \) in \( D \).

\textbf{Corollary 3.} Let \( f \) be of the form (2.9). A necessary and sufficient condition for the function \( f \) to be in class \( TS^*_{q}[\alpha] \) is

\[ \sum_{n=2}^{\infty} ([n]_q - \alpha)|a_n| \leq 1 - \alpha. \]

\textbf{Theorem 2.5.} The class \( TS^*_{q}([\alpha_1], A, B) \) is a convex class.

\textbf{Proof.} Let \( f, g \in TS^*_{q}([\alpha_1], A, B) \) be of the form \( f(z) = z - \sum_{n=2}^{\infty} |a_n|z^n \) and \( g(z) = z - \sum_{n=2}^{\infty} |b_n|z^n \), respectively. Then for \( 0 \leq \rho \leq 1 \), consider a convex combination \( F \) of \( f \) and \( g \) which can be written as

\[ F(z) = \rho f(z) + (1 - \rho)g(z) = \rho \left( z - \sum_{n=2}^{\infty} |a_n|z^n \right) + (1 - \rho) \left( z - \sum_{n=2}^{\infty} |b_n|z^n \right) = z - \sum_{n=2}^{\infty} \left( \rho |a_n| + (1 - \rho) |b_n| \right) z^n. \]

On applying Theorem 2.4, for \( C_n \) defined by (2.7), we get

\[ \sum_{n=2}^{\infty} C_n (\rho |a_n| + (1 - \rho) |b_n|) \leq \rho (B - A) + (1 - \rho) (B - A) = B - A. \]

which again by Theorem 2.4 proves \( F \in TS^*_{q}([\alpha_1], A, B) \). Thus, the class \( TS^*_{q}([\alpha_1], A, B) \) is convex.

\textbf{Theorem 2.6.} Let \( f \in TS^*_{q}([\alpha_1], A, B) \). Then the radius of \( q \)-starlikeness of order \( \alpha \) is given by

\[ (2.13) \quad r_{S^*_q(\alpha)}(TS^*_{q}([\alpha_1], A, B)) \leq \inf_{n \geq 2} \left[ \frac{1 - \alpha}{B - A} \min \left( \frac{C_n}{[n]_q - \alpha} \right) \right] \frac{1}{\alpha}, \]

where \( C_n \) is given by (2.7).

\textbf{Proof.} Let \( f \in TS^*_{q}([\alpha_1], A, B) \). Then from Theorem 2.4, we have

\[ \sum_{n=2}^{\infty} \frac{C_n}{B - A} |a_n| \leq 1. \]
Let $r_0$ be the radius of $q$-starlikness of order $\alpha$ for functions in the class $TS^{r,s}_q([\alpha], A, B)$. Then from Corollary 3, $\frac{f(r_0z)}{z} \in TS^{r,s}_q(\alpha)$ if and only if

$$\sum_{n=2}^{\infty} ([n]_q - \alpha) |a_n| r_0^{n-1} \leq 1 - \alpha$$

which is true if

$$\frac{([n]_q - \alpha)}{1 - \alpha} r_0^{n-1} \leq \frac{C_n}{B - A} \quad (n = 2, 3, ...)
$$

or if,

$$r_0 \leq \left\{ \frac{1 - \alpha}{B - A} \min \left( \frac{C_n}{[n]_q - \alpha} \right) \right\}^{\frac{1}{n-1}} \quad (n = 2, 3, ...)
$$

it follows that the radius $r_0 = r_{S^{r,s}_q(\alpha)} (TS^{r,s}_q([\alpha], A, B))$ satisfies (2.13).

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References


