Hub-integrity and hub edge-integrity of some wheel related graphs

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Abstract

The *hub-integrity* of a connected graph $G = (V(G), E(G))$ is denoted as $HI(G)$ and defined by $HI(G) = min\{|S| + m(G - S)\}$, where S is a hub set and $m(G - S)$ is the order of a maximum components of $G - S$. The *hub edge-integrity* of a connected graph G is denoted as $HEI(G)$ and defined by $HEI(G) = min\{|S| + m(G - S)\}$, where S is an edge hub set and $m(G - S)$ is the order of a maximum components of $G - S$. In this paper, we give results for the hub-integrity and hub edge-integrity of some wheel related graphs.

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1 Introduction

The vulnerability of a graph is a determination that includes certain properties of the graph not to be damaged after the removal of a number of vertices or edges. In a graph theory, most studied graph parameters, is the connectivity of a graph. Connectivity (or edge connectivity) measures the vulnerability of a graph (or network). If the network is reconstruct by a graph, then the integrity measures how easy it is to disconnect the graph (or the network) into several small parts by deleting as few vertices(edges) as possible. More about integrity one can refer [2, 3].

Hub number of a graph G was introduced in [15], which is defined as to study a network related problem. Let G be a graph with vertex set $V(G)$ and S be a subset in a graph G such that $S \subseteq V(G)$ and let $x, y \in V(G)$. An S-path between x and y is a path where all intermediate vertices are from S. A set $S \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G) \setminus S$ there is an S-path in G between x and y. The minimum cardinality of a hub set is called *hub number* and is denoted by $h(G)$. Recently, Sultan Mahde et al. [11] have introduced the concept of hub-integrity of a graph as a new measures of vulnerability, is defined as, $HI(G) = min\{|S| + m(G - S)\}$, where S is a hub set and $m(G - S)$ is the order of a largest component of $G - S$. It is denoted as $HI(G)$. Further more Sultan Mahde and Veena Mathad [12] have studied hub-integrity of some operations of graphs.

Sultan Mahde and Veena Mathad [13], have defined the concept of hub edge-integrity of a graph G. Let $e = (u, v)$ and $f = (u', v')$, a path between the two edges e and f is a path between one end vertex from e and another end vertex from f such that $d(e, f) = min\{d(u, u'), d(u, v'), d(v, u'), d(v, v')\}$. Internal edges of a path between two edges e and f are all the edges of the path except e and f. A subset $S \subset E(G)$ is called an edge hub set of G if every pair of edges $e, f \in E - S$ are connected by a path where all intermediate edges are from S. The minimum cardinality of an edge hub set is called edge hub number of G, and is denoted by $h_e(G)$. The hub edge-integrity of a graph G [13], which is defined as $HEI(G) = min\{|S| + m(G - S)\}$, where S is an edge hub set of G and $m(G - S)$ denotes the order of a largest components of $G - S$. It is denoted by $HEI(G)$. Integrity and edge-integrity parameters were introduced in [3]. Grauman et al. [9] studied hub number of a graphs. One can refer hub number related parameters of graph [4, 5, 6].

2 Preliminaries

Here we have used only nontrivial, simple, connected and undirected graphs. Let G be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and edge set $E(G) = \{e_1, e_2, ..., e_m\}$. Thus $|V(G)| = n$ and $|E(G)| = m$ where, n and m are called *order* and *size* of graph G respectively. The *subdivision graph* $S(G)$ of a graph G [10] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and other is an edge of G incident with it. A path, a cycle, a complete graph of order n are denoted by P_n, C_n and K_n respectively. For undefined terminology and notations refer [7, 10].

The fan graph $F_n, (n \geq 3)$ is defined as the graph $K_1 + P_{n-1}$, where K_1 is a singleton graph and P_{n-1} is the path of $n - 1$ vertices. The wheel graph is obtained from a cycle graph C_{n-1} by adding a new vertex. That is $W_n = K_1 + C_{n-1}$ is a graph with n vertices and $2n$ edges, where the vertex v with degree n in the wheel graph is called the central vertex while the vertices on the cycle C_n are called rim vertices. The gear graph G_n is the graph obtained from wheel graph W_n by adding a vertex between each pair of adjacent vertex of the outer circle (rim). The helm H_n is a graph obtained from wheel W_n with central vertex v, by attaching a pendant edge at each vertex of outer circle. A closed helm CH_n is the graph with central vertex v, obtained from a helm by joining each pendant vertex to form a cycle. The flower Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the central vertex v of the helm. The sunflower graph SF_n is a graph obtained from a wheel with central vertex v, n-cycle $v_0, v_1, ..., v_{n-1}$ and additional n vertices $w_0, w_1, ..., w_{n-1}$ such that w_i is joined by edges to v_i, v_{i+1} for $i = 0, 1, ..., n - 1$, where $i + 1$ is taken modulo n. The friendship graph f_n is a collection of n-triangles with common vertex. A web graph is the graph obtained from closed helm by joining a pendant edge to each vertex on the outer cycle. The crown (or sun) CW_n is a corona of form $C_{n-1} \circ K_1$, where $n \geq 3$.

The *duplication* of an edge $e = uv$ by a new vertex v' in a graph G produces a new graph G' by adding a new vertex v' such that $N(v') = \{u, v\}$. Consider a wheel $W_n = C_{n-1} + K_1$ with $v_1, v_2, ..., v_{n-1}$ as its rim vertices and x as its central vertex. Let $e_1, e_2, ..., e_{n-1}$ be the rim edges of W_n which are duplicated by new vertices $w_1, w_2, ..., w_{n-1}$, respectively and let $y_1, y_2, ..., y_{n-1}$ be the spoke edges of W_n which are duplicated by the vertices $u_1, u_2, ..., u_{n-1}$, respectively. The resultant graph is called *duplication of the wheel* denoted by DuW_n [14]. In other words, duplication of the wheel graph is also called semi-total point graph of wheel. The definitions of above said graphs found in [8].

Proposition 1. [3] *The integrity*

i) For path P_p *is* $I(P_p) = \lceil 2\sqrt{p+1} \rceil - 2$,
ii) For cycle C_p *is* $I(C_p) = \lceil 2\sqrt{p} \rceil - 1$.

Proposition 2. [1] *The edge-integrity*
i) For path P_p *is* $I'(P_p) = \lceil 2\sqrt{p} \rceil - 1$,
ii) For cycle C_p *is* $I'(C_p) = \lceil 2\sqrt{p} \rceil$.

Proposition 3. [11] *The hub-integrity*
i) For complete graph $HI(K_p) = p$,
ii) For path P_p *with* $p \geq 3$, $HI(P_p) = p - 1$,
iii) For cycle C_p,

$$HI(C_p) = \begin{cases} p-1 & if\ p = 4,\ 5; \\ p-2 & if\ p \geq 6. \end{cases}$$

iv) For wheel graph $W_{1,p}$,

$$HI(W_{1,p}) = \begin{cases} p, & if\ p \leq 4; \\ \lceil \frac{p}{3} \rceil + 3, & if\ p \geq 5. \end{cases}$$

3 Hub-integrity of some wheel related graphs

Theorem 3.1. *Let* F_n *be a fan graph of order* $n \geq 3$. *Then*
$$HI(F_n) = \lceil 2\sqrt{n} \rceil - 1.$$

Proof. Let $x_1, x_2, x_3, ..., x_{n-1}$ be the vertices of fan graph with join of a vertex v. Since we know that the fan graph is $K_1 + P_{n-1}$, that is join of one complete graph and another path of order $(n-1)$. Let $S \subset V(F_n)$. Choose a center vertex v is sufficient to forms a hub set. If we remove a center vertex v then we get a path of order $n-1$. Therefore integrity of path $P_n = \lceil 2\sqrt{n+1} \rceil - 2$ [3]. Hence, hub-integrity of fan graph is $HI(F_n) = \lceil 2\sqrt{n} \rceil - 1$. □

Theorem 3.2. *Let* W_n *be a wheel graph of order* $n \geq 4$. *Then*
$$HI(W_n) = \lceil 2\sqrt{n-1} \rceil.$$

Proof. The proof is similar to that of Theorem 3.1. □

Theorem 3.3. *Let* G_n *be a gear graph of order* $2n - 1$, *where* $n \geq 5$. *Then*
$$HI(G_n) = \lceil \tfrac{n-1}{2} \rceil + 4.$$

Proof. Let $x_1, x_2, x_3, ..., x_{n-1}$ be the vertices on the rim of wheel graph with a center vertex v. Gear graph is obtained by subdividing the rim vertices. Let $S \subset V(G_n)$. Choose a center vertex v and $\lceil \frac{n-1}{2} \rceil$ rim vertices in a gear graph which forms a hub set. That is $S = \{v, x_1, x_2, ..., x_{\lceil \frac{n-1}{2} \rceil}\}$. If we remove $\lceil \frac{n-1}{2} \rceil + 1$ vertices in gear graph then we get a largest components of path of order three. Therefore, $|S| = \lceil \frac{n-1}{2} \rceil + 1$ and $m(G - S) = 3$. Hence, hub-integrity of gear graph is $HI(G_n) = \lceil \frac{n-1}{2} \rceil + 4$. □

Theorem 3.4. *Let* H_n *be a helm graph of order* $2n - 1$, *where* $n \geq 4$. *Then*
$$HI(H_n) = n.$$

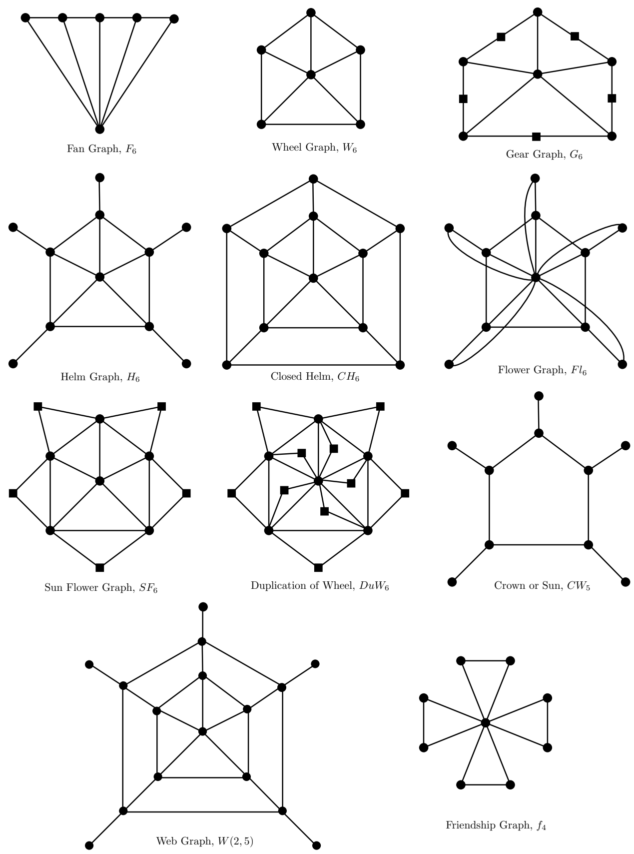

Fig. 1: Examples of some wheel related graphs.

Proof. The proof is similar to that of Theorem 3.3. □

Theorem 3.5. *Let CH_n be a closed helm graph of order $2n-1$, where $n \geq 4$. Then*

$$HI(CH_n) = \begin{cases} n+2 & if\ n\ is\ even, \\ n+1 & if\ n\ is\ odd. \end{cases}$$

Proof. Let $x_1, x_2, x_3, ..., x_{n-1}$ be the vertices on the rim of wheel graph with a center vertex v. A closed helm CH_n is the graph with central vertex v, is obtained from a helm by joining each pendant vertices to form a cycle. Let $S \subset V(CH_n)$. Let n be a even, then choose a center vertex v and alternate rim vertices in a inner and outer cycle of closed helm graph, which forms a hub set. That is $S = \{v, x_1, x_2^{'}, x_3, x_4^{'}, ..., x_{n-1}\}$. If we remove all n vertices in closed helm graph then we get a disconnected graph with largest component order of two. Therefore, $|S| = n$ and $m(G-S) = 2$. Hence, hub-integrity of closed helm graph is $HI(CH_n) = n+2$. Let n be an odd, similarly choose a center vertex v and alternate rim vertices in a inner and outer cycle of closed helm graph, which forms a hub set. That is $S = \{v, x_1, x_2^{'}, x_3, x_4^{'}..., x_{n-1}\}$. Therefore, $|S| = n$ and $m(G-S) = 1$. Hence, hub-integrity of closed helm graph is $HI(CH_n) = n+1$. □

Theorem 3.6. *Let Fl_n be a flower graph of order $2n-1$, where $n \geq 4$. Then*
$$HI(Fl_n) = n + 1.$$

Proof. The proof is similar to that of Theorem 3.3. □

Theorem 3.7. *Let SF_n be a sunflower graph of order $2n-1$, where $n \geq 8$. Then*
$$HI(SF_n) = \left\lceil \frac{n-1}{2} \right\rceil + 4.$$

Proof. The proof is similar to that of Theorem 3.3. □

Theorem 3.8. *Let $W(2,n)$ be the web graph of order $3n-2$, where $n \geq 4$. Then*
$$HI(W(2,n)) = 2n + 1.$$

Proof. The web graph is a graph obtained by joining a pendant edge to each vertex on the outer cycle of the closed helm. $W(t,n)$ is the generalized web with t cycles each of order n. Let $S \subset V(W(2,n))$. Choose a center vertex v and all vertices of an outer cycle in a web graph which forms a hub set. That is $S = \{v, x_1', x_2', ..., x_n'\}$, which gives the required hub set, then we get a largest component of order $n+1$ vertices. Therefore, $|S| = n$ and $m(G-S) = n+1$. Hence, hub-integrity of web graph is $HI(W(2,n)) = 2n+1$. □

Theorem 3.9. *Let DuW_n be the duplication of wheel graph of order $3n-2$, where $n \geq 4$. Then*
$$HI(DuW_n) = n + 1.$$

Proof. The proof is similar to that of Theorem 3.6. □

Theorem 3.10. *Let CW_n be the crown graph of order $n \geq 4$. Then*
$$HI(CW_n) = n + 1.$$

Proof. The proof is similar to that of Theorem 3.6. □

Theorem 3.11. *Let f_n be the friendship graph of order $2n+1$, where $n \geq 3$. Then*
$$HI(f_n) = 3.$$

Proof. Let $x_1, x_2, x_3, ..., x_{n-1}$ be the vertices of the friendship graph with a common vertex v. The friendship graph f_n is a collection of n-triangles with common vertex. Choose a center vertex v, which forms a hub set. That is $S = \{v\}$, which gives the required hub set, then we get a largest component of order two. Therefore, $|S| = 1$ and $m(G-S) = 2$. Hence, hub-integrity of friendship graph is $HI(f_n) = 3$. □

4 Hub edge-integrity of some wheel related graphs

Theorem 4.1. *Let F_n be a fan graph of order $n \geq 4$. Then*
$$HEI(F_n) = \left\lfloor \frac{n-1}{2} \right\rfloor + n.$$

Proof. Let $x_1, x_2, x_3, ..., x_{n-1}$ be the vertices of fan graph with join of a vertex v. Since we know that the fan graph is $K_1 + P_{n-1}$, that is join of one complete graph and another path of order $(n-1)$. Let $S \subset E(F_n)$, choose an edge hub set
$S = \{(x_2, v), (x_4, v), (x_6, v), ..., (x_{\lfloor \frac{n-1}{2} \rfloor}, v)\}$. Therefore, $|S| = \left\lfloor \frac{n-1}{2} \right\rfloor$ and $m(G-S) = n$. Hence, hub edge-integrity of fan graph is $HEI(F_n) = \left\lfloor \frac{n-1}{2} \right\rfloor + n$. □

Theorem 4.2. *Let W_n be a wheel graph of order $n \geq 5$. Then*
$$HEI(W_n) = \lceil \tfrac{n-1}{2} \rceil + n.$$

Proof. The proof is similar to that of Theorem 4.1. □

Theorem 4.3. *Let G_n be a gear graph of order $2n-1$, where $n \geq 4$. Then*
$$HEI(G_n) = \left\lceil 2\sqrt{2(n-1)} \right\rceil + n - 1.$$

Proof. Let $x_1, x_2, x_3, ..., x_{n-1}$ be the vertices on the rim of wheel graph with a center vertex v. Gear graph is obtained by subdividing the rim vertices. Let $S \subset E(G_n)$, choose a set $S = \{(x_1, v), (x_2, v), (x_3, v), ..., (x_{n-1}, v)\}$, which gives the required edge hub set. If we remove edges $\{(x_1, v), (x_2, v), (x_3, v), ..., (x_{n-1}, v)\}$, then we get a cycle of order $2(n-1)$ vertices. Therefore, edge-integrity of cycle C_n is $I'(C_n) = \lceil 2\sqrt{n} \rceil$ [1]. Hence, hub edge-integrity of gear graph is $HEI(G_n) = \left\lceil 2\sqrt{2(n-1)} \right\rceil + n - 1$. □

Theorem 4.4. *Let H_n be a helm graph of order $2n-1$, where $n \geq 4$. Then*
$$HEI(H_n) = 2n.$$

Proof. Let $x_1, x_2, x_3, ..., x_{n-1}$ be the vertices on the rim of wheel graph with a center vertex v. Helm graph is obtained by attaching a pendant edge to each rim vertex of wheel graph. Let $S \subset E(H_n)$, choose a set,
$S = \{(x_1, v), (x_2, v), (x_3, v), ..., (x_{n-1}, v), (x_1, x_2), (x_2, x_3), (x_3, x_4), ..., (x_{n-1}, x_1)\}$, which gives the required edge hub set, then we get a maximum order of two vertices. Therefore, $|S| = 2(n-1)$ and $m(G-S) = 2$. Hence, hub edge-integrity of helm graph is $HEI(H_n) = 2n$. □

Theorem 4.5. *Let CH_n be a closed helm graph of order $2n-1$, where $n \geq 8$. Then*
$$HEI(CH_n) = 2\left(\left\lceil \sqrt{2(n-1)} \right\rceil + n - 1 \right).$$

Proof. The proof is similar to that of Theorem 4.3. □

Theorem 4.6. *Let Fl_n be a flower graph of order $2n-1$, where $n \geq 4$. Then*
$$HEI(Fl_n) = 3n - 2.$$

Proof. Let $x_1, x_2, x_3, ..., x_{n-1}$ be the vertices on the rim of wheel graph with a center vertex v. Choose a set, $S = \{(x_1, v), (x_2, v), (x_3, v), ..., (x_{n-1}, v)\}$, which gives the required edge hub set, then we get a maximum order of $2n-1$ vertices. Therefore, $|S| = n-1$ and $m(G-S) = 2n-1$. Hence, hub edge-integrity of flower graph is $HEI(Fl_n) = 3n-2$. □

Theorem 4.7. *Let SF_n be a sunflower graph of order $2n-1$, where $n \geq 4$. Then*
$$HEI(SF_n) = 3(n-1).$$

Proof. The proof is similar to that of Theorem 4.6. □

Theorem 4.8. *Let $W(2, n)$ be the web graph of order $3n-2$, where $n \geq 3$. Then*
$$HEI(W(2, n)) = 4n.$$

Proof. The web graph is a graph obtained by joining a pendant edge to each vertex on the outer cycle of the closed helm. $W(t,n)$ is the generalized web with t cycles each of order n. Let $S \subset E(W(2,n))$. Choose a set, $S = \{(x_1, v), (x_2, v), (x_3, v), ..., (x_n, v), (x_1, x_1^{'}), (x_2, x_2^{'}), (x_3, x_3^{'}), ..., (x_n, x_n^{'})\}$, which gives the required edge hub set, then we get a largest component of order $2n$ vertices. Therefore, $|S| = 2n$ and $m(G - S) = 2n$. Hence, hub edge-integrity of web graph is $HEI(W(2,n)) = 4n$. □

Theorem 4.9. *Let DuW_n be the duplication of wheel graph of order $3n-2$, where $n \geq 4$. Then*
$$HEI(DuW_n) = 4n - 3.$$

Proof. The proof is similar to that of theorem 4.6. □

Theorem 4.10. *Let CW_n be the crown graph of order $n \geq 3$. Then*
$$HEI(CW_n) = n + 2.$$

Proof. The proof is similar to that of Theorem 4.6. □

Theorem 4.11. *Let f_n be the friendship graph of order $2n+1$, where $n \geq 3$. Then*
$$HEI(f_n) = 2n + 2.$$

Proof. The proof is similar to that of Theorem 4.10. □

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