Invariant Submanifolds in Complex Contact Metric Structure Manifolds

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Abstract

In the present article, we have studied of invariant submanifolds in complex contact metric structure manifolds. In first section, we have studied the some literature related to the complex contact metric structure manifolds and their properties. Again the second section, we have continued the first discussion as well as preliminaries the complex contact metric manifolds and a brief treatment of invariants submanifolds. Section third, we define invariant and anti-invariant (CC-totally real) submanifolds in such manifolds in complex contact metric structure manifolds and start the study of their basic properties. Also, section fourth, we establish the Chen first inequalities and Chen inequalities for the invariant $\delta(2,2)$ and CC-totally real submanifolds in a complex contact metric space and obtaining relationships between extrinsic and intrinsic invariants submanifolds.

Subject Classification:[2010] 53C40, 53C25
Keywords: Complex Manifold, Contact manifold, Submanifolds, invariants, Metric space.

1 Introduction

Complex contact metric structure manifolds are rarely examined, with only a few articles in the mathematical relevant literature. However, [1], D.E. Blair accomplishes a comprehensive review of known results on (normal) complex contact manifolds. In [2], D.E. Blair and the first author in this work proved that a locally symmetric normal complex contact metric manifold is locally isometric to the complex projective space $CP2n+1(4)$ of constant holomorphic curvature. They also worked at reflections in the vertical subbundle of a normal complex contact manifold's integral submanifolds, finding that when such reflections are isometries, the manifold fibres throughout a locally symmetric space. If a normal complex contact manifold is Khler, the manifold fibres over a quaternionic symmetric space. The manifold fibres over a locally symmetric complex symplectic manifold are also represented by a global holomorphic contact form if the complex contact structure is defined by a global holomorphic contact form. In [3], D.E. Blair is the same authors studied the homogeneity and local symmetry of complex (k,m)-spaces. It was proved that for $k<1$, a complex (k,m)-space is locally homogeneous and GH-locally symmetric.

In other word, we can say the theory of submanifolds is play extremely important role in differential geometry. Present until, there seem to be no publications on submanifolds in normal complex contact manifolds. We propose invariant and anti-invariant (CC-totally real) submanifolds of normal complex contact metric manifolds and attempt to analyze their basic properties in this study. In addition, the theory of Chen invariants is one of the most suitable techniques for finding relationships between a submanifold's extrinsic and intrinsic invariants. B.Y. Chen trends survey on Chen invariants and Chen inequalities, as well as their applications, in [9] B.Y. Chen. We establish the Chen first inequality and Chen inequality for the invariant $\delta(2,2)$ for CC-totally real submanifolds in a normal complex contact space form and give the characterizations of the equality cases.

2 Preliminaries

A Riemannian metric is an associated metric for a complex contact manifold M of odd complex dimension $(2n+1)$ equipped with a real 1-forms $u, v = uoJ$, (1,1)-tensor fields $N, P = NJ$, unit vector a fields $U, V = -JU$ such that

$$N^2 = P^2 = -I + u \otimes U + v \otimes V, \tag{2.1}$$

$$g(NX,Y) = -g(X,NY), \qquad u(X) = g(U,X), \quad JN = -NJ, \quad NU = 0. \tag{2.2}$$

And on the overlaps, the given metric tensor change as

$$u' = mu - nv, \qquad v' = nu + mv, \qquad N' = mN - nP, \qquad P' = nN + mP, \tag{2.3}$$

where m, n be the functions defined with $m^2 + n^2 = 1$.
A complex contact manifold allows a complex contact metric structure for which the contact form ϕ is $u - iv$ to within a non-vanishing complex valued function. The metric tensor fields N and P implies du and dv such that

$$du(X,Y) = g(X,NY) + (\sigma\Lambda v)(X,Y), dv(X,Y) = g(X,PY) - (\sigma\Lambda u)(X,Y), \tag{2.4}$$

$$\hat{N}(X,Y) = g(X,NY), \qquad \hat{P}(X,Y) = g(X,PY) \tag{2.5}$$

Also, on a contact metric structure manifold on M , one has $U'\Lambda V' = U\Lambda V_i$ and $TM = v \bigoplus h$ it follows that there is a global vertical bundle v orthogonal to h on M , locally spanned by $U, V = -JU$ and is usually assumed to be integrable. In this case, $\sigma(X) = g(D_X U, V)$, where D_X being the Levi-Civita connection of g. In the case of a strict complex contact structure, u, v may be taken globally such that $\phi = u - iv$ and $\sigma = 0$.

Ishihara and Konishi [16] introduced a notion of normality for complex contact structures. Their notation is the neglecting of the two tensor field S and T given by

$$\begin{aligned} S(X,Y) = {} & [N,N](X,Y) + 2g(X,NY)U - 2g(X,PY)V + 2(v(Y)PX \\ & -v(X)PY) + \sigma(NY)PX - \sigma(NX)PY + \sigma(X)NPY - \sigma(Y)NPX, \end{aligned} \tag{2.6}$$

$$T(X,Y) = [P,P](X,Y) - 2g(X,NY)U + 2g(X,PY)V + 2(u(Y)NX - u(X)NY) + \sigma(PX)NY - \sigma(PY)NY + \sigma(X)NPY - \sigma(Y)NPX, \tag{2.7}$$

where $[N,N]$ and $[P,P]$ denote the Nijenhuis tensor of N and P respectively.
A complex contact metric structure is normal [17] if N and P neglect, then

$$S(X,Y) = T(X,Y) = 0, \forall X,Y \in h \quad \text{and} \quad S(U,Y) = T(V,Y) = 0, \forall X \in h. \tag{2.8}$$

The expressions for the covariant derivatives of the structures tensors on a normal complex contact metric manifold are

$$D_X U = -NX + \sigma(X)V \quad \text{and} \quad D_X V = -PX + \sigma(X)U. \tag{2.9}$$

Uniformly, a complex contact metric structure manifold is normal, if and only if the covariant derivatives of N and P have defined the following forms:

$$g((D_X N)Y,Z) = \sigma(X)g(PY,Z) + v(X)d\sigma(NZ,NY) - 2v(X)g(PNY,Z) - u(Y)g(X,Z) - v(Y)g(JX,Z) + u(Z)g(X,Y) + v(Z)g(JX,Y), \tag{2.10}$$

$$g((D_X P)Y,Z) = \sigma(X)g(NY,Z) - u(X)d\sigma(PZ,PY) - 2u(X)g(NPY,Z) + u(Y)g(JX,Z) - v(Y)g(X,Z) + u(Z)g(X,JY) + v(Z)g(X,Y), \tag{2.11}$$

for a Hermitian structure J, we have

$$g((D_X J)Y,Z) = u(X)(d\sigma(Z,NY) - 2g(PY,Z)) + v(X)(d\sigma(Z,PY) + 2g(NY,Z)), \tag{2.12}$$

for $p \in \hat{M}$ and a unit vector $X \in h_p$ the plane in $T_p\hat{M}$ spanned by X and $Y = \lambda NX + \mu PX$, $\lambda, \mu \in \Re, \lambda^2 + \mu^2 = 1$, is called a NP-plane section. The sectional curvature of the NP-plane section is denoted by $\hat{K}(X,Y)$. For a tangent vector X, the sectional curvature $\hat{K}(X,Y)$ is free of the vector Y in the plane of NX and PX. This is equivalent to $\hat{K}(X,NX) = \hat{K}(X,PX)$ and $g(\hat{R}_{x(NX)}PX,X) = 0$.

According to [17], B. Korkmaz and the convention used by Blair, the curvature tensor defined as

$$
\begin{aligned}
\hat{R}(X, YZ, W) &= g(\hat{R}_{XY}Z, W) = g(R(X, \hat{Y})Z, W) \\
&= \frac{c+3}{4}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W) + g(Z,JY)g(JX,W) \\
&- g(Z,JX)g(JY,W) + 2g(X,JY)g(JZ,W)] + \frac{c-1}{4}[-(u(Y)u(Z) \\
&+ v(Y)v(Z))g(X,W) + (u(x)u(Z) + v(x)v(Z))g(Y,W) \\
&+ 2u\Lambda v(Z,Y)g(JX,W) - 2u\Lambda v(Z,X)g(JY,W) + 4u\Lambda v(X,Y)g(JZ,W) \\
&+ g(Z,NY)g(NX,W) - g(Z,NX)g(NY,W) + 2g(X,NY)g(NZ,W) \\
&+ g(Z,PY)g(PX,W) - g(Z,PX)g(PY,W) + 2g(X,PY)g(PZ,W) \\
&+ (-u(X)g(Y,Z) + u(Y)g(X,Z)) + (v(X)g(JY,Z) - v(Y)g(JX,Z)) \\
&+ 2v(Z)g(X,JY)g(U,W) + (-v(X)g(Y,Z) + v(Y)g(X,Z)) \\
&+ u(X)g(JY,Z) - u(Y)g(JX,Z) + 2u(Z)g(X,JY)g(V,W) \\
&+ \frac{4}{3}(g(U,JV) + (c+1))[(v(X)u\Lambda v(Z,Y) - v(Y)u\Lambda v(Z,X) \\
&+ 2u(Z)u\Lambda v(X,Y))g(V,W)], \qquad \text{where} \qquad g(U,JV) = \Omega(U,V)
\end{aligned} \tag{2.13}
$$

Theorem 2.1. *Let $\hat{M}$ be a complex contact metric structure manifolds. Then $\hat{M}$ has constant sectional curvature c if and only if for X horizontal, the holomorphic sectional curvature of the plane spanned by X and JX is $c+3$.*

Theorem 2.2. *Let $\hat{M}$ be a complex contact metric structure manifolds of constant NP-sectional curvature $+1$ and satisfying $(V,U) = 2$ Then $\hat{M}$ has constant holomorphic sectional curvature c.*

3 Invariant Submanifolds

Let M be an invariant submanifolds of a complex contact metric structure manifolds $\hat{M}$. Also let us assume that U and V are vector fields to M. For $p \in M$ and $X, Y \in T_pM$, we have

$$g(NX,Y) = -g(\hat{D}_X U, Y) = g(U, \hat{X}Y) = g(U, h(X,Y)) \tag{3.1}$$

Since the first term of the equation (3.1)is skew-symmetric and last term is symmetric in (X,Y) , then

$$g(NX,Y) = 0 \qquad \text{and} \qquad g(PX,Y) = 0 \tag{3.2}$$

Remark 1. *If the vector fields U and V are normal to M then for any $p \in M$, $N(T_pM) \subset T_p^{\perp}M$ and $P(T_pM) \subset T_p^{\perp}M$.*

Remark 2. *The above conditions do not imply $J(T_pM) \subset T_p^{\perp}M$ for all $p \in M$.*

Definition 1. *An invariant submanifolds M of a complex contact metric structure manifolds $\hat{M}$ is known as invariant submanifolds if $N(T_pM) \subset T_p^{\perp}M$ and $P(T_pM) \subset T_p^{\perp}M$ for all $p \in M$.*

Remark 3. *It is just that, in the case when M is invariant, T_pM is invariant by J, too.*

Suppose that U is not tangent to the invariant submanifolds and decompose U into its tangential and normal parts, say $U = U^T + U^T$, then $0 = NU = NU^T + NU^{\perp}$; since the N-invariant of the tangent space indicates the N-invariant of the normal space, both NU^T and $NU^{\perp}$ neglect.
Let X be a horizontal tangent vector field; then

$$g(U, [X, NX]) = u([X, NX]) = -2du(X, NX) = -2g(X, N^2X) = 2g(X, X) \tag{3.3}$$

Now $[X, NX]$ is tangent to the invariant submanifolds and therefore

$$\begin{aligned} g(U, [X, NX]) &= g(U^T, [X, NX]) = g(U^T, \hat{D}NX - \hat{D}_{NX}X) \\ &= g(U^T, (\hat{D}_X N)X + (\hat{D}_{NX}N)NX). \end{aligned} \tag{3.4}$$

Using the equation (2.10) for the covariant derivatives of N for a normal complex contact metric manifold, this becomes

$$\begin{aligned} g(U, [X, NX]) &= u(U^T)g(X, X) + u(U^T)g(NX, NX) \\ &= 2g(U^T, U^T)g(X, X) \end{aligned} \tag{3.5}$$

Comparing with (3.3), $g(U^T, U^T) = 1$ i.e., U^T is a unit as is U; therefore $U^{\perp} = 0$, contradicting the supposition that U was not tangent.
For this reason, an orthonormal basis of M, $dimM = 4n + 2$ can be written as

$$\{E_1 ....E_n, NE_1 ....NE_n, JE_1 ....JE_n, U, V\}. \tag{3.6}$$

The proof of the minimality uses the formula of the covariant derivative given by [15], B. Foreman.
Let $pr : TM \longrightarrow h$ denote the projection to the horizontal subbundle and $J' = pr \circ J$. We have

$$\begin{aligned} 2g((\hat{D}_X N)Y, Z) &= g([N, N](Y, Z), NX) - 3v\Lambda d\sigma(X, Y, Z) \\ &- 2\sigma(X)g(Y, PZ) + 4v(X)g(Y, JZ) - \sigma(Y)g(Z, PX) \\ &+ \sigma(NY)g(Z, JX) - 2u(Y)g(X, pZ) - 2u(Y)g(Z, JX) \\ &+ \sigma(Z)g(Y, PX) - \sigma(NZ)g(Y, JX) + 2u(Z)g(X, pY) \\ &+ 2v(Z)g(Y, JX). \end{aligned} \tag{3.7}$$

Now taking X as horizontal and $Y = X$ we make the following computation

$$\begin{aligned} 2g((\hat{D}_X N)Y, Z) + 2g((\hat{D}_{NX}N)NY, Z) &= g([N, N](X, Z), NX) \\ &- 2\sigma(X)g(X, PZ) - 2\sigma(X)g(Z, PX) + \sigma(NX)g(Z, JX) \\ &+ 2u(Z)g(X, X) - g(([N, N])(NX, Z), X) \\ &- 2\sigma(NX)g(NX, PZ) + \sigma(NX)g(PZ, NX) + \sigma(X)g(Z, PX) \\ &+ 2u(Z)g(X, X). \end{aligned} \tag{3.8}$$

Expanding the Nijenhuis torsion terms and cancelling as appropriate, we get

$$g((\hat{D}_X N)Y, Z) + g((\hat{D}_{NX} N)NY, Z) = \sigma(X)g(PX, Z) + \sigma(NX)g(JX, Z) + 2u(Z)g(X, X). \tag{3.9}$$

Now suppose that Z is normal to the invariant submanifold, then

$$g(h(X, NX) - Nh(X, X), Z) + g(h(NX, -X) - Nh(NX, NX), Z) \tag{3.10}$$

Giving $h(X, X) + h(NX, NX) = 0$.

Theorem 3.1. *Any invariant submanifolds of a complex contact metric structure manifolds is minimal.*

Definition 2. *An invariant submanifolds M of a complex contact metric structure manifolds $\hat{M}$ is called a CC- totally real (anti-invariant) submanifolds if*
(a) U, V are normal to M ; (b) M is totally real submanifold of $\hat{M}$ (with respect to J).

Theorem 3.2. *Let M be a invariant submanifold of a compact regular Sasakian manifold $\hat{M}$. Then M is regular and M/Z_n is a invariant submanifold of $\hat{M}/Z_n$.*

4 Chen Inequalities

We have studied the Chen first inequalities and Chen inequalities for the invariant $\delta(2,2)$ for CC-totally real submanifolds in a normal complex almost contact space form $\hat{M}(c)$. Let (M, g) be a Riemannian manifold of dimension $n \geq 2$ and denote by *kappa* and τ the sectional curvature and scalar curvature of M, respectively.

Chen first invariant is $\delta_M = \tau - inf\kappa.$

Chen invariant is $\delta(2,2) = \tau - inf[\kappa(\pi_1) + \kappa(\pi_2)],$
where π_1 and π_2 are mutually orthogonal plane sections. In order to prove Chen first inequality, we state the following lemma.

Lemma 4.1: Let $n \geq 3$ be an integer and let $a_1, ...a_n$ be n real numbers [11], B.Y. Chen. Then one has

$$\sum_{1 \leq i < j \leq n} a_i a_j - a_1 a_2 \leq \frac{n-2}{2(n-1)} \left(\sum_{i=1}^{n} a_i \right)^2 . \tag{4.1}$$

Moreover, the equality holds if and only if $a_1 + a_2 = a_3 = ... = a_n$.
Let M be an n-dimensional CC-totally real submanifold of a normal complex contact space form of arbitrary codimension m. The Gauss equation for M in $\hat{M(c)}$ is given by

$$\hat{R}(X, Y, Z, W) = R(X, Y, Z, W) - g(h(X, W), h(Y, Z)) + g(h(X, Z), h(Y, W)), \tag{4.2}$$

For any $X, Y, Z, W \in \Gamma(TM)$, where h is the second fundamental form.
Let $p \in M$, π a plane section in T_pM, $\{E_1, E_2\}$ an orthonormal basis of π, $\{E_1, E_2, ...E_n\}$

an orthonormal basis of T_pM and $\{E_{n+1}, ...E_m\}$ an orthonormal basis of $T_p^{\perp}M$. The Gauss equation implies

$$\begin{aligned}\kappa(\pi) = R(E_1, E_2, E_2, E_1) &= \frac{c+3}{4} + g\left(h(E_1, E_1), h(E_2, E_2)\right) \\ &- g\left((E_1), E_2), h(E_1, E_2)\right) \\ &= \frac{c+3}{4} + \sum_{\alpha=n+1}^{m} \left[h_{11}^{\alpha} - (h_{12}^{\alpha})^2\right],\end{aligned} \tag{4.3}$$

where $h_{ij}^{\alpha} = g\left(h(E_i, E_j), E_\alpha\right), i, j \in 1, ..., n, \alpha \in \{n+1, ..., m\}$. On the other hand,

$$\begin{aligned}\sum_{1\leq i<j\leq n} R(E_i, E_j, E_j, E_i) &= n(n-1)\frac{c+3}{8} + \sum_{1\leq i<j\leq n} [g(h(E_i, E_i), h(E_j, E_j)] \\ \tau &= n(n-1)\frac{c+3}{8} + \sum_{\alpha=n+1}^{m} \sum_{1\leq i<j\leq n} \left[h_{ii}^{\alpha} - (h_{jj}^{\alpha}) - (h_{ij}^{\alpha})^2\right].\end{aligned} \tag{4.4}$$

Subtracting from (4.4) in (4.3), we get

$$\begin{aligned}\tau - \kappa(\pi) &= (n-2)(n+1)\frac{c+3}{8} + \sum_{\alpha=n+1}^{m} \left(\sum_{1\leq <j\leq n} h_{ii}^{\alpha} h_{jj}^{\alpha} - h_{11}^{\varepsilon} h_{22}^{\alpha}\right) \\ &- \sum_{\alpha=n+1}^{m} \left[\sum_{1\leq i<j\leq n} (h_{ij}^{\alpha})^2 - (h_{12}^{\varepsilon})^2\right] \\ &\leq (n-2)(n+1)\frac{c+3}{8} + \sum_{\alpha=n+1}^{m} \left(\sum_{1\leq i<j\leq n} h_{ii}^{\alpha} - h_{jj}^{\alpha} - h_{11}^{\varepsilon} h_{22}^{\alpha}\right).\end{aligned} \tag{4.5}$$

By applying Lemma (4.1), we obtain for all $\alpha \in \{n+1, ...m\}$:

$$\sum_{1\leq i<j\leq n} h_{ii}^{\alpha} - h_{jj}^{\alpha} - h_{11}^{\alpha} h_{22}^{\alpha} \leq \frac{n-2}{2(n-1)} \left(\sum_{i=1}^{n} h_{ii}^{\alpha}\right)^2 = \frac{n^2(n-2)}{2(n-1)} (H^{\alpha})^2. \tag{4.6}$$

By summing the above relations, we get

$$\tau - \kappa(\pi) \leq (n-2)(n+1)\frac{c+3}{8} + \frac{n^2(n-2)}{2(n-1)} \|\hat{H}\|^2, \tag{4.7}$$

where $\hat{H}$ is the mean curvature vector.
The equality holds at a point $p \in m$ if and only if for any $\alpha \in \{n+1, ..., m\}$,

$$\begin{aligned}h_{11}^{\alpha} + h_{22}^{\alpha} = h_{33}^{\alpha} .... = h_{nn}^{\alpha} \\ h_{ij}^{\alpha} = 0, \forall \quad 1 \leq i < j <\leq n, (i, j) \neq (1, 2)\end{aligned} \tag{4.8}$$

If we take E_{n+1} parallel to $\hat{P}(p)$ and E_1, E_2 such that $h_{12}^{n+1} = 0$, the shape operators take the forms given in the following:

Theorem 4.1. *Let$\hat{M}(c)$ be a complex contact metric space from and M an n-dimensional $(n \geq 3)$ CC-totally real invariant submanifolds. Then we have*

$$\delta_M = \tau\kappa(\pi) \leq \frac{(n-2)}{2}\left[\frac{n^2}{n-1} \parallel \hat{H}^2 \parallel + (n+1)\frac{c+3}{4}\right]. \tag{4.9}$$

Moreover, the equality case of the inequality holds at a point $p \in M$ if and only if $\exists$ an orthonormal basis $\{E_1, E_2, ...E_n\}$ of T_pM and an orthonormal basis $\{E_{n+1}, ...E_m\}$ of $T_p^{\perp}$ such that the shape operators take the following forms:

$$\begin{aligned} A_{n+1} &= \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & \mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu \end{pmatrix}, a+b=\mu \\ A_n &= \begin{pmatrix} h_{11}^r & h_{12}^r & 0 & \cdots & 0 \\ h_{12}^r & h_{11}^r & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, r \in \{n+2, ..., m\}, \end{aligned} \tag{4.10}$$

where $dimM = n$ & $dim\hat{M} = 4n+2$.
If M is of maximum dimension, i.e., $dim \ \ M = n$ and $dim \ \ \hat{M} = 4n+2$. (the analogue of a Legendrian invariant submanifold in the real case), we obtain:

Theorem 4.2. *Let $\hat{M}(c)$ be a complex contact metric space from of $\hat{M} = 4n+2$. Then any n-dimensional CC-totally real submanifolds M are satisfying the equality case of Chen first inequality, identically, is minimal.*

Proof. Now the equation (4.8) can be written as ☐

$$\begin{aligned} h(E_1, E_1) + h(E_3, E_3) &= ...h(E_n, E_n) \\ h(E_i, E_j) = 0, \quad \forall 1 \leq i < j &\leq n, \quad (i,j) \neq (1,2) \end{aligned} \tag{4.11}$$

Then by using (2.9) and (2.10), we get

$$\begin{aligned} g(h(E_3, E_3), U) = g(\tilde{D}_{E_3}E_3, U) &= -g((E_3, NE_3 - \sigma(E_3)V) = 0 \\ g(h(E_3, h_3), V) = 0, \quad \text{and} \quad &h(E_3, E_3) = 0. \end{aligned} \tag{4.12}$$

It follows by (4.11) that

$$\vec{H} = \frac{n-1}{n}h(E_3, E_3) = 0, \qquad \text{i.e. } M \text{ is a minimal submanifolds.} \tag{4.13}$$

Lemma 4.2: Let $n \geq 4$ be an integer and let $a_1, ...a_n$ be n real numbers [18], A. Mihai and I. Mihai. Then one has:

$$\sum_{1\leq i<j\leq n} a_i a_j - a_1 a_2 - a_3 a_4 \leq \frac{n-3}{2(n-2)} \left(\sum_{i=1}^{n} a_i\right)^2 . \tag{4.14}$$

Moreover, the equality holds if and only if $a_1 + a_2 = a_3 + a_4 = a_5 = ... = a_n$.
Let $p \in M$, π_1 and π_2 mutually orthogonal plane sections in T_pM, $\{E_1, E_2\}$ and $\{E_3, E_4\}$ orthonormal basis of π_1 and π_2 and $\{E_1, ...E_n\}$ an orthonormal basis of T_pM.
Then the Gauss equation implies

$$\kappa(\pi_1) = \frac{c+3}{4} + \sum_{\alpha=n+1}^{m} \left[h_{11}^{\alpha} + h_{22}^{\alpha} - (h_{12}^{\alpha})^2\right], \tag{4.15}$$

$$\kappa(\pi_2) = \frac{c+3}{4} + \sum_{\alpha=n+1}^{m} \left[h_{33}^{\alpha} + h_{44}^{\alpha} - (h_{34}^{\alpha})^2\right]. \tag{4.16}$$

By subtracting equations (4.15) and (4.16) from (4.4), we get

$$\tau - \kappa(\pi_1) - \kappa(\pi_2) \leq (n^2 - n - 4)\frac{c+3}{4} + \sum_{\alpha=n+1}^{m} \left[h_{ii}^{\alpha} h_{jj}^{\alpha} - h_{11}^{\alpha} h_{22}^{\alpha} - h_{33}^{\alpha} h_{44}^{\alpha}\right] \tag{4.17}$$

By applying Lemma (4.2), we get for all $\alpha \in \{n+1, ..., m\}$:

$$\sum_{1\leq i<j\leq n} h_{ii}^{\alpha} h_{jj}^{\alpha} - h_{11}^{\alpha} h_{22}^{\alpha} - h_{33}^{\alpha} h_{44}^{\alpha} \leq \frac{n-3}{2(n-2)} \left(\sum_{i=1}^{n} h_{ii}^{\alpha}\right)^2 = \frac{n^2(n-2)}{2(n-1)} (H^{\alpha})^2 . \tag{4.18}$$

By summing the above equations, we obtain

$$\tau - \kappa(\pi_1) - \kappa(\pi_2) \leq (n^2 - n - 4)\frac{c+3}{8} + \frac{n^2(n-3)}{2(n-2)} \|\vec{H}\|^2. \tag{4.19}$$

Theorem 4.3. *Let $\tilde{M}(c)$ be an m-dimensional complex contact metric space from and M an n-dimensional $(n \geq 4)$ CC-totally real submanifolds. Then we get*

$$\delta(2,2) \leq (n^2 - n - 4)\frac{c+3}{8} + \frac{n^2(n-3)}{2(n-2)} \|\vec{H}\|^2. \tag{4.20}$$

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