Fuzzy optimization using gH-symmetrically derivative of fuzzy functions

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Abstract

The goal of this study is to obtain optimality conditions of optimization problems that have fuzzy-valued objective functions. In this paper, we proposed new differentiability known as generalized Hukuhara symmetric(gHs) differentiability of fuzzy function. Using the concept of generalized Hukuhara differentiable and pseudo-invex fuzzy-valued functions, we establish KKT optimality requirements for the class of fuzzy optimization problems. As a result, our findings are more general than those provided previously for this type of situation. In application, we applied the gHs derivative to find the non-dominated solutions of the fuzzy optimization problem with KKT criteria. Moreover, we explain the proposed method with examples and show that this method generalizes the existing optimality conditions.

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1 Introduction

In optimization, there are two components, namely objective functions and constraints. Practically, objective functions rarely hold real number as coefficients. Most of the time, they have uncertainty. These values may not be accurate also. The disadvantages of uncertainty or inaccuracy can be tackled by the use of fuzzy programming approach. Works of Rommelfanger [1] and Delgado et al. [2] viewed this from 90s onwards. Lodwick [3] gives a detailed literature review on this topic. Slowinski and Teghem [4] compare optimization problems with multiple objectives. Inuiguchi [5] has done a similar comparison but for problems to solve portfolio selection.
Generalization of Hukuhara differentiability(HD) of set valued functions will give HD of fuzzy valued functions where the differentiability is based upon Hukuhara difference. Hukuhara [15] developed the subtraction of two sets. Hukuhara derivatives introduced in [15] is widely used by researchers in the field of set and fuzzy valued functions due to its

importance in fuzzy differential equations as well as optimization problems. From [10], [11] and [12], compared to H differentiable functions gH differentiable fuzzy functions are relatively general. [6], [7], [8] and [9] studied convex fuzzy mapping, duality theorem, KKT optimality conditions in single and multiobjective fuzzy valued functions.
Here we propose a new idea known as generalized Hukuhara symmetrically(gHs) differentiable fuzzy functions. We can see that gHs derivative of fuzzy functions is more general than gH derivative.
Section 2 contains preliminaries. We define our main definition, gHs differentiable fuzzy function and some theorems related to it in section 3. Section 4 deals with fuzzy optimization of gHs differentiable functions. In last section, we obtain the optimality conditions of non-dominated solution applying gHs derivative to fuzzy optimization.

2 Preliminaries

Assume $\mathcal{I}_{\mathcal{C}}$ represents the family of all intervals belongs to R which are bounded space. i.e., $\mathcal{I}_{\mathcal{C}} = \{[\underline{k}, \overline{k}] | \underline{k}, \overline{k} \in \mathbb{R}$ and $\underline{k} \leq \overline{k}\}$. Suppose two intervals $A = [\underline{a}, \overline{a}]$ and $B = [\underline{b}, \overline{b}]$ we define the Hausdorff- Pompeiu distance between A and B by
$H_p(A, B) = \max\{|\underline{a} - \underline{b}|, |\overline{a} - \overline{b}|\}$
Clearly, $(\mathcal{I}_{\mathcal{C}}, H_p)$ denotes a complete metric space.

A fuzzy set on $\mathbb{R}^n$ is a mapping $l : \mathbb{R}^n \to [0,1]$ and for each fuzzy set l we denote the α level set, $[l]^\alpha = \{t \in \mathbb{R}^n | l(t) \geq \alpha \}$ for any $\alpha \in (0,1]$.
Now, we recall the definition of support as: supp$(l) = \{t \in \mathbb{R}^n | l(t) > 0\}$.

Definition 1. *A fuzzy set l on $\mathbb{R}$ is said to be a fuzzy interval if:*
1. l is normal and upper semi continuous.
2. The value of $l(\lambda x + (1-\lambda)y)$ should be greater than or equal to $\min\{l(x), l(y)\}, x, y \in \mathbb{R}, \lambda \in [0, 1]$
3. $[l]^0$ should be compact.

Assume $\mathsf{F_C}$ denotes the family of all fuzzy intervals. The α levels of fuzzy intervals are defined as, $[l]^\alpha = [\underline{l}_\alpha, \overline{l}_\alpha]$, where $\underline{l}_\alpha, \overline{l}_\alpha \in \mathbb{R}$, $\forall\ \alpha \in [0, 1]$ and $[l]^\alpha \in \mathcal{I}_{\mathcal{C}}$, $\forall\ \alpha \in [0, 1]$.
Now, it is time to define the arithmetic operations such as addition and scalar multiplication of fuzzy intervals $l, m \in \mathsf{F_C}$ as follows:

$$(l + m)(t) = \sup_{y+z=t} \min\{l(y), m(z)\}$$

$$(\lambda l)(t) = \begin{cases} l(\frac{t}{\lambda}) & if \lambda \neq 0 \\ 0 & if \lambda = 0 \end{cases}$$

Clearly, $\forall\ \alpha \in [0,1]$,

$$[l + m]^\alpha = [(\underline{l + m})_\alpha, (\overline{l + m})_\alpha] = [\underline{l}_\alpha + \underline{m}_\alpha, \overline{l}_\alpha + \overline{m}_\alpha] \tag{2.1}$$

and

$$[(\lambda l)]^\alpha = [(\underline{\lambda l})_\alpha, (\overline{\lambda l})_\alpha] = [\min\{\lambda \underline{l}_\alpha, \lambda \overline{l}_\alpha\}, \max\{\lambda \underline{l}_\alpha, \lambda \overline{l}_\alpha\}] \tag{2.2}$$

Definition 2. *(Stefanini 2010)* $l \ominus_{gH} m = p \Leftrightarrow \begin{cases} (1) l = m + p, \\ or (2) m = l + (-1)p. \end{cases}$

$[l \ominus_{gH} m]^\alpha = [l]^\alpha \ominus_{gH} [m]^\alpha = [\min\{\underline{l}_\alpha - \underline{m}_\alpha, \bar{l}_\alpha - \overline{m}_\alpha\}, \max\{\underline{l}_\alpha - \underline{m}_\alpha, \bar{l}_\alpha - \overline{m}_\alpha\}]$, $\forall\ \alpha \in$[0,1], where $[l]^\alpha \ominus_{gH} [m]^\alpha$ represents gH- difference from l to m. Let $l, m \in \mathsf{F_C}$, we can define distance from l to m by

$$\mathsf{D}(l,m) = \sup_{\alpha\in[0,1]} H([l]^\alpha, [m]^\alpha) = \sup_{\alpha\in[0,1]} \max\{|\underline{l}_\alpha - \underline{m}_\alpha|, |\bar{l}_\alpha - \overline{m}_\alpha|\}.$$

So $(\mathsf{F_C}, \mathsf{D})$ is a complete metric space.

Proposition 1. *Suppose the length of the α cuts of l amd m be*

$$[l]^\alpha \ominus_{gH} [m]^\alpha = \begin{cases} [\underline{l}_\alpha - \underline{m}_\alpha, \bar{l}_\alpha - \overline{m}_\alpha], if \quad len[l]^\alpha \leq len[m]^\alpha \\ [\bar{l}_\alpha - \overline{m}_\alpha, \underline{l}_\alpha - \underline{m}_\alpha], if \quad len[l]^\alpha \geq len[m]^\alpha \end{cases} \tag{2.3}$$

where $len[l]^\alpha = \bar{l}_\alpha - \underline{l}_\alpha$ is the length of alpha cut of l. Similarly we can define the length of alpha cut of m

2.1 gH derivative of fuzzy functions

Let E be an open subset of $\mathbb{R}^n$ and define $\mathsf{F} : E \to \mathsf{F_C}$, $\forall\ \alpha \in [0,1]$. The collection of all interval-valued fuzzy functions represented by, $\mathsf{F}_\alpha : E \to \mathcal{I_C}$ and is given by $\mathsf{F}_\alpha(t) = [\mathsf{F}(t)]^\alpha$. For any $\alpha \in [0,1]$, with lower function $\underline{\mathsf{f}}_\alpha(t)$ and upper function $\bar{\mathsf{f}}_\alpha(t)$ we denote $\mathsf{F}_\alpha(t) = [\underline{\mathsf{f}}_\alpha(t), \bar{\mathsf{f}}_\alpha(t)]$

Definition 3. *Let $E \subset \mathbb{R}$ with F : $E \to \mathcal{I_C}$ a fuzzy function and $t_0 \in E$ and h be such that $t_0 + h \in E$. Then the generalized Hukuhara derivative (gH-derivative) of F at t_o is defined as*

$$\mathsf{F}'(t_0) = \lim_{h\to 0} \frac{\mathsf{F}(t_0 + h) \ominus_{gH} \mathsf{F}(t_0)}{h} \tag{2.4}$$

If $\mathsf{F}'(t_0) \in \mathcal{I_C}$ satisfying (2.4) exists, then it is clear that F is gH differentiable at t_0.

Definition 4. *Let F : $E \to \mathbb{R}$. Then F is symmetrically differentiable at $t_o \in E$ if $\exists\ S \in \mathbb{R}$ and*

$$\lim_{h\to 0} \frac{[\mathsf{F}(t_o + h) - \mathsf{F}(t_o - h)]}{2h} = S$$

3 Main Result

Definition 5. *Let F : $E \to \mathsf{F_C}$ and $t_o \in E$ and $h \in \mathbb{R}$ such that $t_o + h \in E$. Then F is gH symmetrically continuous at t_o if*

$$\lim_{h\to 0} \mathsf{F}(t_o + h) \ominus_{gH} \mathsf{F}(t_o - h) = 0 \tag{3.1}$$

Definition 6. *Let $E \subset \mathbb{R}$ with $\mathsf{F} : E \to \mathsf{F_C}$ be a fuzzy function and $t_0 \in E$ and $t_0 + h, t_0 - h \in E$. Then the gHs-derivative of F at t_o is defined as*

$$\mathsf{F}^s(t_0) = \lim_{h\to 0} \frac{\mathsf{F}(t_0 + h) \ominus_{gH} \mathsf{F}(t_0 - h)}{2h} \tag{3.2}$$

If $\mathsf{F}^s(t_0) \in \mathcal{I_C}$ satisfying (3.2) exists, we say that F is gHs differentiable at t_0.

Theorem 3.1. *If* $\mathsf{F} : E \to \mathsf{F_C}$ *is gHs differentiable then the fuzzy function defined on an interval* $\mathsf{F}_\alpha : E \to \mathcal{I_C}$ *is gHs differentiable uniformly* $\forall\ \alpha \in [0,1]$. *Furthermore* $[\mathsf{F}^s(t)]^\alpha = \mathsf{F}^s_\alpha(t)$.

Proof. Obvious from the definition of gHs differentiability. □

Theorem 3.2. *Let* $\mathsf{F} : E \to \mathsf{F_C}$. *If* F *is gHs differentiable at* $t_0 \in E$ *uniformly* $\forall$ $\alpha \in [0,1]$, *then one of the following conditions satisfied:*

1. $\bar{\mathsf{f}}_\alpha$ *and* $\underline{\mathsf{f}}_\alpha$ *are symmetrically differentiable at* t_0. *Also*

$$[\mathsf{F}^s(t_0)]_\alpha = [\min\{(\underline{\mathsf{f}}_\alpha)^s(t_0), (\bar{\mathsf{f}}^s_\alpha(t_0)\}, \max\{(\underline{\mathsf{f}}_\alpha)^s(t_0), (\bar{\mathsf{f}}_\alpha)^s(t_0)\}] \tag{3.3}$$

2. $(\underline{\mathsf{f}}_\alpha)^s_-(t_0), (\underline{\mathsf{f}}_\alpha)^s_+(t_0), (\bar{\mathsf{f}}_\alpha)^s_-(t_0), (\bar{\mathsf{f}}_\alpha)^s_+(t_0)$ *exist and satisfy* $(\underline{\mathsf{f}}_\alpha)^s_-(t_0) = (\bar{\mathsf{f}}_\alpha)^s_+(t_0)$ *and* $(\underline{\mathsf{f}}_\alpha)^s_+(t_0) = (\bar{\mathsf{f}}_\alpha)^s_-(t_0)$. *Moreover* $[\mathsf{F}^s(t_0)]_\alpha = [\min\{(\underline{\mathsf{f}}_\alpha)^s_-(t_0), (\bar{\mathsf{f}}_\alpha)^s_-(t_0)\}, \max\{(\underline{\mathsf{f}}_\alpha)^s_-(t_0), (\bar{\mathsf{f}}_\alpha)^s_-(t_0)\}]$ $= [\min\{(\underline{\mathsf{f}}_\alpha)^s_+(t_0), (\bar{\mathsf{f}}_\alpha)^s_+(t_0)\}, \max\{(\underline{\mathsf{f}}_\alpha)^s_+(t_0), (\bar{\mathsf{f}}_\alpha)^s_+(t_0)\}]$

Proof. (1) Assume F is gHs differentiable at t_0.
i.e.,

$$(\mathsf{F}^s(t_0))_\alpha = \lim_{h\to 0} \frac{\mathsf{F}_\alpha(t_0+h) \ominus_{gH} \mathsf{F}_\alpha(t_0-h)}{2h}$$

$$= \lim_{h\to 0} \frac{(\underline{\mathsf{f}}_\alpha(t_0+h), \bar{\mathsf{f}}_\alpha(t_0+h) \ominus_{gH} (\underline{\mathsf{f}}_\alpha(t_0-h), \bar{\mathsf{f}}_\alpha(t_0-h))}{2h}$$

$$= \lim_{h\to 0}[\min\{\frac{\underline{\mathsf{f}}_\alpha(t_0+h) - \underline{\mathsf{f}}_\alpha(t_0-h)}{2h}, \frac{\bar{\mathsf{f}}_\alpha(t_0+h) - \bar{\mathsf{f}}_\alpha(t_0-h)}{2h}\},$$
$$\max\{\frac{\underline{\mathsf{f}}_\alpha(t_0+h) - \underline{\mathsf{f}}_\alpha(t_0-h)}{2h}, \frac{\bar{\mathsf{f}}_\alpha(t_0+h) - \bar{\mathsf{f}}_\alpha(t_0-h)}{2h}\}]$$

Thus we can say that $\underline{\mathsf{f}}_\alpha$ and $\bar{\mathsf{f}}_\alpha$ are symmetrically differentiable.
(2) Consider the following function to illustrate the second instance.

$$\mathsf{F}_\alpha(t) = \begin{cases} [(1+\alpha)t, 2(3-\alpha)t], & if \quad t \geq 0 \\ [2(3-\alpha)t, (1+\alpha)t], & if \quad t < 0 \end{cases}$$

Here we can see that $(\underline{\mathsf{f}}_\alpha)^s_-(t_0), (\underline{\mathsf{f}}_\alpha)^s_+(t_0), (\bar{\mathsf{f}}_\alpha)^s_-(t_0), (\bar{\mathsf{f}}_\alpha)^s_+(t_0)$ exist and satisfy $(\underline{\mathsf{f}}_\alpha)^s_-(t_0) = (\bar{\mathsf{f}}_\alpha)^s_+(t_0)$ and $(\underline{\mathsf{f}}_\alpha)^s_+(t_0) = (\bar{\mathsf{f}}_\alpha)^s_-(t_0)$.
Now we prove the converse part.
Assume that $\underline{\mathsf{f}}_\alpha$ and $\bar{\mathsf{f}}_\alpha$ are symmetrically differentiable.
i.e.,

$$\lim_{h\to 0}\Big(\frac{\underline{\mathsf{f}}_\alpha(t_0+h) - \underline{\mathsf{f}}_\alpha(t_0-h)}{2h}\Big), \lim_{h\to 0}\Big(\frac{\bar{\mathsf{f}}_\alpha(t_0+h) - \bar{\mathsf{f}}_\alpha(t_0-h)}{2h}\Big)$$

exists.

$$= \lim_{h\to 0}[\min\{\frac{\underline{\mathsf{f}}_\alpha(t_0+h) - \underline{\mathsf{f}}_\alpha(t_0-h)}{2h}, \frac{\bar{\mathsf{f}}_\alpha(t_0+h) - \bar{\mathsf{f}}_\alpha(t_0-h)}{2h}\},$$

$$\max\{\frac{\underline{f}_\alpha(t_0+h)-\underline{f}_\alpha(t_0-h)}{2h},\frac{\overline{f}_\alpha(t_0+h)-\overline{f}_\alpha(t_0-h)}{2h}\}]$$

$$=\lim_{h\to 0}\frac{(\underline{f}_\alpha(t_0+h),\overline{f}_\alpha(t_0+h))\ominus_{gH}(\underline{f}_\alpha(t_0-h),\overline{f}_\alpha(t_0-h))}{2h}$$

$$=\lim_{h\to 0}\frac{F_\alpha(t_0+h)\ominus_{gH}F_\alpha(t_0-h)}{2h}$$

$$=(F^s(t_0))_\alpha$$

. Now we assume that (2) exists and consider the following 3 cases.
(i) $(\underline{f}_\alpha)^s_+(t_0)<(\overline{f}_\alpha)^s_+(t_0)$ (ii) $(\underline{f}_\alpha)^s_+(t_0)=(\overline{f}_\alpha)^s_+(t_0)$ (iii) $(\underline{f}_\alpha)^s_+(t_0)>(\overline{f}_\alpha)^s_+(t_0)$
Case (i): Since $(\underline{f}_\alpha)^s_+(t_0)<(\overline{f}_\alpha)^s_+(t_0)$ then

$$\lim_{h\to 0^+}\frac{F_\alpha(t_0+h)\ominus_{gH}F_\alpha(t_0-h)}{2h}$$

$$=\lim_{h\to 0^+}[\min\{\frac{\underline{f}_\alpha(t_0+h)-\underline{f}_\alpha(t_0-h)}{2h},\frac{\overline{f}_\alpha(t_0+h)-\overline{f}_\alpha(t_0-h)}{2h}\},$$

$$\max\{\frac{\underline{f}_\alpha(t_0+h)-\underline{f}_\alpha(t_0-h)}{2h},\frac{\overline{f}_\alpha(t_0+h)-\overline{f}_\alpha(t_0-h)}{2h}\}]$$

$$=\lim_{h\to 0^+}[\{\frac{\underline{f}_\alpha(t_0+h)-\underline{f}_\alpha(t_0-h)}{2h},\frac{\overline{f}_\alpha(t_0+h)-\overline{f}_\alpha(t_0-h)}{2h}\}],$$

$$=[(\underline{f}_\alpha)^s_+(t_0),(\overline{f}_\alpha)^s_+(t_0)]$$

since $(\underline{f}_\alpha)^s_-(t_0)=(\overline{f}_\alpha)^s_+(t_0)$ and $(\underline{f}_\alpha)^s_+(t_0)=(\overline{f}_\alpha)^s_-(t_0)$ then $(\overline{f}_\alpha)^s_-(t_0)<(\underline{f}_\alpha)^s_-(t_0)$
Thus

$$\lim_{h\to 0^-}\frac{F_\alpha(t_0+h)\ominus_{gH}F_\alpha(t_0-h)}{2h}$$

$$=\lim_{h\to 0^-}[\{\frac{\overline{f}_\alpha(t_0+h)-\overline{f}_\alpha(t_0-h)}{2h},\frac{\underline{f}_\alpha(t_0+h)-\underline{f}_\alpha(t_0-h)}{2h}\}]$$

$$=[(\overline{f}_\alpha)^s_-(t_0),(\underline{f}_\alpha)^s_-(t_0)]$$

Moreover, since $(\underline{f}_\alpha)^s_-(t_0)=(\overline{f}_\alpha)^s_+(t_0)$ and $(\underline{f}_\alpha)^s_+(t_0)=(\overline{f}_\alpha)^s_-(t_0)$
Thus we have

$$\lim_{h\to 0^-}\frac{F_\alpha(t_0+h)\ominus_{gH}F_\alpha(t_0-h)}{2h}$$

$$=\lim_{h\to 0^+}\frac{F_\alpha(t_0+h)\ominus_{gH}F_\alpha(t_0-h)}{2h}$$

Therefore F is gHs differentiable at t_0. Similarly we can prove (ii) and (iii). □

The partial derivative of a fuzzy function F defined on $E \subset \mathbb{R}^n$ will now be defined. Consider the fuzzy function $\mathsf{F} : E \to \mathsf{F_C}$ defined on E. i.e For each $\mathbf{t} = (t_1, t_2, ..., t_n) \in E$, $\mathsf{F}(\mathbf{t}) = \mathsf{F}(t_1, ..., t_n)$ is a fuzzy number. The corresponding real valued function on E $\forall$ $\alpha \in [0, 1]$is as follows

$$\underline{\mathsf{f}}_\alpha(\mathbf{t}) = \underline{\mathsf{f}}_\alpha(t_1, ..., t_n) = (\underline{\mathsf{f}}(t_1, ..., t_n))_\alpha \text{ and}$$
$$\overline{\mathsf{f}}_\alpha(\mathbf{t}) = \overline{\mathsf{f}}_\alpha(t_1, ..., t_n) = \overline{\mathsf{f}}(t_1, ..., t_n))_\alpha$$

Definition 7. *Let $E \subset \mathbb{R}^n$ and suppose $t_0 = (t_1^{(0)}, ..., t_n^{(0)})$ be a fixed element of E. $k_i(t_i) = \mathsf{F}(t_1^{(0)}, ..., t_i - 1^{(0)}, t_i, t_i + 1^{(0)}, ..., t_n^{(0)})$. If k_i is gHs differentiable at $t_i^{(0)}$, then clearly F has the i^{th} partial gHs derivative at t_0 $\Big($ represented as $\left(\frac{\partial^s \mathsf{F}}{\partial t_i}\right)(t_0)$ $\Big)$ and*

$$\left(\frac{\partial^s \mathsf{F}}{\partial t_i}\right)(t_0) = (k_i)^s(t_i^{(0)})$$

Definition 8. *Suppose F is defined on E and assume that $t_0 \in E$ be fixed such that $t_0 = (t_1^{(0)}, ..., t_n^{(0)})$. Then F is gHs differentiable at t_0 if the entire partial gHs derivatives $\left(\frac{\partial^s \mathsf{F}}{\partial t_1}\right)(t_0), ...,\left(\frac{\partial^s \mathsf{F}}{\partial t_n}\right)(t_0)$ exist on some neighbourhood of t_0. Also they are continuous at t_0.*

If F is gHs differentiable at t_0, then $\left(\frac{\partial^s \mathsf{F}}{\partial t_i}\right)(t_0)$ is a fuzzy interval. Now we define,

$$\left[\frac{\partial^s \mathsf{F}}{\partial t_i}(t_0)\right]^\alpha = \frac{\partial^s \mathsf{F}}{\partial t_i}(t_0) = \left[\frac{\partial^s \underline{\mathsf{F}}_\alpha}{\partial t_i}(t_0), \frac{\partial^s \overline{\mathsf{F}}_\alpha}{\partial t_i}(t_0)\right], \forall\ \alpha \in [0, 1].$$

Using(3.2), we get $\left(\frac{\partial^s \underline{\mathsf{F}}_\alpha}{\partial t_i}\right)(t_0)$ and $\left(\frac{\partial^s \overline{\mathsf{F}}_\alpha}{\partial t_i}\right)(t_0)$

Consider $\mathsf{F} : \mathbb{R}^2 \to \mathsf{F_C}$ defined by

$$\Big[\mathsf{F}(t_1, t_2)\Big]^\alpha = \mathsf{F}_\alpha(t_1, t_2) = [t_1^2, t_1^2 + (1 - \alpha)t_2^2].$$

Now we have

$$\left[\frac{\partial^s \mathsf{F}}{\partial t_1}(t_1, t_2)\right]^\alpha = \frac{\partial^s \mathsf{F}_\alpha}{\partial t_1}(t_1, t_2) = [2t_1, 2t_1],$$

for any $(t_1, t_2) \in \mathbb{R}^2$ and $\alpha \in [0, 1]$. Also

$$\frac{\partial^s \underline{\mathsf{F}}_\alpha}{\partial t_2}(t) = \min\left[\frac{\partial^s \underline{\mathsf{f}}_\alpha}{\partial t_2}(t), \frac{\partial^s \overline{\mathsf{f}}_\alpha}{\partial t_2}(t)\right]$$

and

$$\frac{\partial^s \overline{\mathsf{F}}_\alpha}{\partial t_2}(t) = \max\left[\frac{\partial^s \underline{\mathsf{f}}_\alpha}{\partial t_2}(t), \frac{\partial^s \overline{\mathsf{f}}_\alpha}{\partial t_2}(t)\right]$$

Applying this we have

$$\frac{\partial^s \mathsf{F}_\alpha}{\partial t_2}(t_1, t_2) = \begin{cases} [0, 2(1 - \alpha)t_2] & if \quad t_1 \in \mathbb{R}, \quad t_2 >= 0 \\ [2(1 - \alpha)t_2, 0] & if \quad t_1 \in \mathbb{R}, \quad t_2 < 0 \end{cases}$$

$\forall\ \alpha \in [0, 1]$. Since both $(\frac{\partial^s \mathsf{F}}{\partial t_1})(t_1, t_2)$ and $(\frac{\partial^s \mathsf{F}}{\partial t_2})(t_1, t_2)$ are symmetrically continuous then F is gHs differentiable.

Proposition 2. *If $\mathsf{F} : E \to \mathsf{F_C}$ is gHs differentiable at $t_0 \in E$ then, $\forall\ \alpha \in [0,1]$, $\underline{\mathsf{f}}_\alpha + \overline{\mathsf{f}}_\alpha : E \to \mathbb{R}$ is symmetrically differentiable at t_0. Moreover*

$$\frac{\partial^s \underline{\mathsf{F}_\alpha}}{\partial t_i}(t_0) + \frac{\partial^s \overline{\mathsf{F}_\alpha}}{\partial t_i}(t_0) = \frac{\partial^s (\underline{\mathsf{f}}_\alpha + \overline{\mathsf{f}}_\alpha)}{\partial t_i}(t_0) \tag{3.4}$$

Proof. The proof follows directly from Theorem 3.2 □

Definition 9. *The symmetric gradient of $\mathsf{F} : E \to \mathsf{F_C}$ at t_0, $\nabla^s \mathsf{F}(t_0)$, is defined as*

$$\nabla^s \mathsf{F}(t_0) = \Big(\Big(\frac{\partial^s \mathsf{F}}{\partial t_1}\Big)(t_0), ..., \Big(\frac{\partial^s \mathsf{F}}{\partial t_n}\Big)(t_0)\Big) \tag{3.5}$$

where $(\frac{\partial^s \mathsf{F}}{\partial t_j})(t_0)$ denotes the j^{th} partial gHs derivative of F at t_0

4 gHs differentiable functions in fuzzy optimization

Results below shows the imbalance between intervals P and Q where $\mathsf{P} =[\underline{\mathsf{p}}, \overline{\mathsf{p}}]$ and $\mathsf{Q} = [\underline{\mathsf{q}}, \overline{\mathsf{q}}] \in \mathcal{I_C}$ will be utilized all through this paper.

$$\begin{aligned} \mathsf{Q} \preceq \mathsf{P} \quad & iff \quad \underline{\mathsf{q}} \preceq \underline{\mathsf{p}}, \quad and \quad \overline{\mathsf{q}} \preceq \overline{\mathsf{p}}; \\ \mathsf{Q} \prec \mathsf{P} \quad & iff \quad \underline{\mathsf{q}} \preceq \underline{\mathsf{p}}, \quad and \quad \overline{\mathsf{q}} \neq \overline{\mathsf{p}}; \end{aligned}$$

Let l and m be two fuzzy intervals such that $[l]^\alpha = [\underline{l}_\alpha, \bar{l}_\alpha]$ and $[m]^\alpha = [\underline{m}_\alpha, \overline{m}_\alpha]$ are two elements of $\mathcal{I_C}$, $\forall\ \alpha \in [0,1]$, then,

$$l \preceq m \quad iff \quad [l]^\alpha \preceq [m]^\alpha, \ \ \forall\ \ \alpha \in [0,1]; \tag{4.1}$$

which is equal to $\underline{l}_\alpha \preceq \underline{m}_\alpha$ and $\bar{l}_\alpha \preceq \overline{m}_\alpha$, $\forall\ \alpha \in [0,1]$.

$$l \preceq m \quad iff \quad l \preceq m \ \ and \ \ l \neq m \tag{4.2}$$

which is equal to $[l]^\alpha \preceq [m]^\alpha$, $\forall\ \alpha \in [0,1]$ and there exists $\alpha^* \in [0,1]$
such that $\underline{l}_{\alpha*} \prec \underline{m}_{\alpha*}$ or $\bar{l}_{\alpha*} \prec \overline{m}_{\alpha*}$.

$$l \prec m \quad iff \quad l \preceq m \ \ and \ \ [l]^\alpha \neq [m]^\alpha, \ \ \forall\ \ \alpha \in [0,1] \tag{4.3}$$

which is equal to $\underline{l}_\alpha \prec \underline{m}_\alpha$ and $\bar{l}_\alpha \prec \overline{m}_\alpha$, $\forall\ \alpha \in [0,1]$.
Now we write the mathematical expression for the fuzzy valued optimization problem

$$\begin{aligned} \min \ \ \mathsf{F}(t) \\ subject\ \ to\ \ g_i(t) \leq 0, i = 1, ..., m; \\ t \in E \subset \mathbb{R}^n \end{aligned} \tag{4.4}$$

where $\mathsf{F} : E \to \mathsf{F_C}$, g_i represents a real valued function on E. Thus the feasible result of (4.4) is attained as follows:

$$T = \{t \in E | g_j(t) \leq 0 j = 1, ..., m\}$$

Definition 10. *Let t^* be a solution of fuzzy optimization problem (4.4) which is feasible, i.e, $t^* \in T$,it is clear that t^* becomes a non- dominated solution of (4.4) $\nexists\ t \in T/\{t^*\}$ such that $F(t) \preceq F(t^*)$*

Now, consider the multiobjective optimization problems defined for each $\alpha \in [0,1]$

$$\min \Big(\underline{\mathsf{f}}_\alpha(t), \overline{\mathsf{f}}_\alpha(t)\Big)$$
$$subject\ \ to\ \ t \in T \tag{4.5}$$

Lemma 1. *If t^* is a Pareto efficient solution for (4.5) for each $\alpha \in [0,1]$, then t^* is a non-dominated solution for (4.4)*

Proof. Suppose that t^* is not a non-dominated solution, then there exists $\tilde{t} \in T$ such that $\mathsf{F}(\tilde{t}) \preceq \mathsf{F}(t^*)$. In particular, there exists α_* such that

$$\underline{\mathsf{f}}_{\alpha*}(\tilde{t}) \leq \underline{\mathsf{f}}_{\alpha*}(t^*)$$
$$\overline{\mathsf{f}}_{\alpha*}(\tilde{t}) \leq \overline{\mathsf{f}}_{\alpha*}(t^*)$$

with a strict inequality, which contradicts the fact that t^* is a Pareto efficient solution for (4.5) with α_*

□

5 KKT conditions for the non-dominated solutions of fuzzy optimization problem

In this section we discuss about the KKT conditions of non-dominated results of fuzzy optimization problem.

5.1 KKT conditions using gHs derivative

Theorem 5.1. *Suppose that g_j is a symmetrically differentiable function on E. Also assume that the constraint is convex. Let $\mathsf{F} : E \to \mathsf{F_C}$ is gHs differentiable and $\underline{\mathsf{f}}_\alpha + \overline{\mathsf{f}}_\alpha$ is convex on $E\ \forall\ \alpha \in [0,1]$.If $\exists$ non negative real valued numbers $\mu_j(\alpha)$ (non negative Lagrange multiplier $\forall\ \alpha$) for j= 1,...,m which satisfies the conditions mentioned below:*

1. $\nabla^s\Big(\underline{\mathsf{f}}_\alpha + \overline{\mathsf{f}}_\alpha\Big)(t^*) + \sum_{j=1}^{m} \mu_j(\alpha)\nabla^s g_j(t^*) = 0\ \ \forall\ \ \alpha \in [0,1]$
2. $\mu_j(\alpha) g_j(t^*) = 0\ \ \forall\ \ j = 1, ..., m,$

Then t^* is a non- dominated result of the following problem

$$\begin{aligned} \min\ \ \mathsf{F}(t) \\ subject\ \ to\ \ g_i(t) \leq 0, i = 1, ..., m; \\ t \in E \subset \mathbb{R}^n \end{aligned}$$

where $\mathsf{F} : E \to \mathsf{F_C}$, g_i represents a real valued function on E.

Proof. Using Prposition 3.1, it is obvious that if F is gHs differentiable then $\underline{\mathsf{f}}_\alpha + \overline{\mathsf{f}}_\alpha$ is symmetrically differentiable on T $\forall\ \alpha \in [0,1]$. Conditions 1 and 2 imply that t^* is a KKT condition for

$$\begin{aligned} &\min \left(\underline{\mathsf{f}}_\alpha + \overline{\mathsf{f}}_\alpha\right)(t) \\ &subject\ \ to\ \ t \in T \end{aligned} \tag{5.1}$$

for all $\alpha \in [0,1]$. Since g_j's are convex on E, $\underline{\mathsf{f}}_\alpha + \overline{\mathsf{f}}_\alpha$ is also convex. Thus we can say that t^* is an optimal result 0f (5.1) $\forall\ \alpha \in [0,1]$. Using results from [13], it is obvious that t^* is a pareto efficient result for (4.5) $\forall\ \alpha \in [0,1]$. Using lemma (4.1), we can easily conclude that t^* is a non- dominated result for (4.4) □

Definition 11. *Let $E \subset \mathbb{R}^n$ and assume that E is convex. Consider the fuzzy function F on E. F becomes convex when*

$$\mathsf{F}(\lambda t^* + (1-\lambda)x) \preceq \lambda \mathsf{F}(t^*) + (1-\lambda)\mathsf{F}(t)$$

$\forall\ \lambda \in (0,1)$ *and each $t, t^* \in E$.*

Corollary 1. *Assume that g_j is symmetrically differentiable on E. Also suppose that the constraint is convex. Assume the convex function, $\mathsf{F} : E \to \mathsf{F}_\mathsf{C}$, is gHs differentiable. If $\exists$ non negative real numbers $\mu_j(\alpha)$ (non negative Lagrange multipliers $\forall\ \alpha$) for j = 1,...,m which satisfies 1 and 2 mentioned below:*

1. $\nabla^s\left(\underline{\mathsf{f}}_\alpha + \overline{\mathsf{f}}_\alpha\right)(t^*) + \sum_{j=1}^{m} \mu_j(\alpha)\nabla^s g_j(t^*) = 0\ \ \forall\ \ \alpha \in [0,1]$
2. $\mu_j(\alpha) g_j(t^*) = 0\ \ \forall\ \ j = 1, ..., m,$

then t^* is a non dominated result of

$$\begin{aligned} \min\ \ &\mathsf{F}(t) \\ subject\ \ to\ \ &g_i(t) \le 0, i = 1, ..., m; \\ &t \in E \subset \mathbb{R}^n \end{aligned}$$

where $\mathsf{F} : E \to \mathsf{F}_\mathsf{C}$, g_i represents a real valued function on E.

Proof. The proof is immediate if apply Theorem 5.1 and Definition 5.1. □

Theorem 6.2 in [15] considered F as level-wise differentiable and convex function. But in corollary 5.1, we assume that F is gHs differentiable. Also in Theorem 5.1 we assume that real valued function $\underline{\mathsf{f}}_\alpha + \overline{\mathsf{f}}_\alpha$ is convex. Example 5.1 clearly explains the fact that Corollary 5.1 generalises Theorem 6.2 in [15].

$$\begin{aligned} \min\ \ &\tilde{3}t_1 + \tilde{2}t_2^2 \\ subject\ \ to\ \ &(t_1 - 2)^2 + t_2^2 \le 4 \end{aligned} \tag{5.2}$$

such that $\tilde{3}$ =(2, 3, 5) and $\tilde{2}$ = (1, 2, 4) be two fuzzy numbers which are triangular. Thus we have

$$\mathsf{F}(t_1, t_2) = \tilde{3}t_1 + \tilde{2}t_2^2, \;\; g_1(t_1, t_2) = (t_1 - 2)^2 + t_2^2 - 4.$$

By 2.2 and 2.3, we get

$$[\mathsf{F}(t_1, t_2)]^\alpha = \begin{cases} [(2+\alpha)t_1 + (1+\alpha)t_2^2, (5-2\alpha)t_1 + (4-2\alpha)t_2^2], & if \quad t_1 \geq 0, t_2 \in \mathbb{R}; \\ [(5-2\alpha)t_1 + (1+\alpha)t_2^2, (2+\alpha)t_1 + (4-2\alpha)t_2^2], & if \quad t_1 < 0, t_2 \in \mathbb{R} \end{cases} \quad \forall \quad \alpha \in [0,1].$$

Here we can see that $\underline{\mathsf{f}}_\alpha$ and $\bar{\mathsf{f}}_\alpha$ are not convex and are not differentiable at (0,0). Thus F is not convex and is not level-wise differentiable at (0,0). Therefore, in this example we cannot apply Theorem 6.2 in [15].
We have the convex function, $\underline{\mathsf{f}}_\alpha(t) + \bar{\mathsf{f}}_\alpha(t)] = (7-\alpha)t_1 + (5-\alpha)t_2^2, \;\; \forall \;\alpha \in [0,1]$. F is gHs differentiable at(0,0). Moreover (1) and (2) of Theorem 5.1 are also satisfied. Furthermore, $\mu(\alpha) = \dfrac{7-\alpha}{4}$.Thus $t^* = (0,0)$ becomes non dominated result of 5.2

Definition 12. *We assume that the constraint function of 4.4 be fuzzy. Then 4.4 becomes a fuzzy pseudoinvex 2 problem if it satisfies following conditions:*

1. F *is gHs differentiable.*
2. g *is symmetrically differentiable on E.*

Furthermore $\forall$ t,$t^* \in T$, $\exists$ $\eta(t^*, t) \in \mathbb{R}^n$ *such that*

$$\begin{aligned} \mathsf{F}(t) \preceq \mathsf{F}(t^*) &\Rightarrow \tilde{\nabla}^s \mathsf{F}(t^*).\eta(t, t^*) \preceq 0, \\ -\nabla^s g_i(t^*)\eta(t, t^*) &\leq 0 \;\; i \in I(t^*), \end{aligned}$$

where $I(t^*)$ *represents index set of constraints.*

Theorem 5.2. *Suppose that optimization problem 4.4 be a fuzzy pseudoinvex 2 on E. For all* α, *let* $\exists$ *non negative numbers* $\mu_j(\alpha), j = 1, ..., m$, *which satisfies the following condition* $\forall$ $\alpha \in [0,1]$

1. $\nabla^s \Big(\underline{\mathsf{f}}_\alpha + \bar{\mathsf{f}}_\alpha\Big)(t^*) + \sum_{j=1}^{m} \mu_j(\alpha) \nabla^s g_j(t^*) = 0 \;\; \forall \alpha \in [0,1]$
2. $\mu_j(\alpha) g_j(t^*) = 0 \;\; \forall \alpha \;\; j = 1, ..., m,$

Then t^* *becomes non dominated result of optimization problem*

$$\begin{aligned} \min \;\; & \mathsf{F}(t) \\ subject \;\; to \;\; & g_i(t) \leq 0, i = 1, ..., m; \\ & t \in E \subset \mathbb{R}^n \end{aligned}$$

where $\mathsf{F} : E \rightarrow \mathsf{F_C}$, g_i *represents a real valued function on E.*

Proof. Assume t^* is a dominated solution. Then $\exists$ an $\hat{t} \in T \mid \mathsf{F}(\hat{t}) \preceq \mathsf{F}(t*)$. Since the fuzzy optimization problem (4.4) is fuzzy pseudoinvex-2, $\exists\ \eta(\hat{t}, t*)$ in which $\tilde{\nabla}^s\mathsf{F}(t^*).\eta(t,t^*) \preceq 0$. Then $\forall \alpha \in [0,1]$ we have

$$\Big(\Big[\frac{\partial^s \underline{F}\mathsf{F}_\alpha}{\partial t_1}(t^*), \frac{\partial^s \overline{\mathsf{F}}_\alpha}{\partial t_1}(t^*)\Big], ...\Big[\frac{\partial^s \underline{\mathsf{F}}_\alpha}{\partial t_n}(t^*), \frac{\partial^s \overline{\mathsf{F}}_\alpha}{\partial t_n}(t^*)\Big]\Big).\eta(t,t^*) \preceq 0$$

i.e.,

$$\Big(\Big[\frac{\partial^s \underline{\mathsf{F}}_\alpha}{\partial t_1}(t^*), \frac{\partial^s \overline{\mathsf{F}}_\alpha}{\partial t_1}(t^*)\Big].\eta(t,t^*) + ... + \Big[\frac{\partial^s \underline{\mathsf{F}}_\alpha}{\partial t_n}(t^*), \frac{\partial^s \overline{\mathsf{F}}_\alpha}{\partial t_n}(t^*)\Big]\Big).\eta(t,t^*) \preceq 0$$

There exists, $\alpha^* \in [0,1]$ such that

$$\min\Big\{\frac{\partial^s \underline{\mathsf{F}}_{\alpha^*}}{\partial t_1}(t^*).\eta_1(t,t^*), \frac{\partial^s \overline{\mathsf{F}}_{\alpha^*}}{\partial t_1}(t^*).\eta_1(t,t^*)\Big\} + ... + \\ \min\Big\{\frac{\partial^s \underline{\mathsf{F}}_{\alpha^*}}{\partial t_n}(t^*).\eta_n(t,t^*), \frac{\partial^s \overline{\mathsf{F}}_{\alpha^*}}{\partial t_n}(t^*).\eta_n(t,t^*)\Big\} \leq 0$$

and

$$\max\Big\{\frac{\partial^s \underline{\mathsf{F}}_{\alpha^*}}{\partial t_1}(t^*).\eta_1(t,t^*), \frac{\partial^s \overline{\mathsf{F}}_{\alpha^*}}{\partial t_1}(t^*).\eta_1(t,t^*)\Big\} + ... + \\ \max\Big\{\frac{\partial^s \underline{\mathsf{F}}_{\alpha^*}}{\partial t_n}(t^*).\eta_n(t,t^*), \frac{\partial^s \overline{\mathsf{F}}_{\alpha^*}}{\partial t_n}(t^*).\eta_n(t,t^*)\Big\} \leq 0$$

with strict inequality. Thus using proposition (3.1), $\underline{\mathsf{f}}_{\alpha^*} + \overline{\mathsf{f}}_{\alpha^*}$ is symmetrically differentiable at t^*.

$$\frac{\partial^s\Big(\underline{\mathsf{f}}_{\alpha^*} + \overline{\mathsf{f}}_{\alpha^*}\Big)}{\partial t_1}(t^*).\eta_1(t,t^*) + ... + \frac{\partial^s\Big(\underline{\mathsf{f}}_{\alpha^*} + \overline{\mathsf{f}}_{\alpha^*}\Big)}{\partial t_n}(t^*).\eta_n(t,t^*) < 0$$

Therefore,

$$\nabla^s\Big(\underline{\mathsf{f}}_{\alpha^*} + \overline{\mathsf{f}}_{\alpha^*}\Big)(t^*)^T\eta(\hat{t},t^*) < 0 \tag{5.3}$$

By pseudoinvexity hypothesis for fuzzy optimization problem

$$\nabla^s g_j(t^*)\eta(t^*,t) \leq 0, \ \ \forall j \in I(t^*) \tag{5.4}$$

Applying Motzkin's alternative Theorem, $\nexists\ 0 < \lambda_0 \in \mathbb{R}$ and $0 \leq \lambda_j \in \mathbb{R}, j \in I(t^*)$ in which

$$\lambda_0\nabla^s\Big(\underline{\mathsf{f}}_{\alpha^*} + \overline{\mathsf{f}}_{\alpha^*}\Big)(t^*) + \sum_{j\in I(t^*)} \lambda_j.\nabla^s g_j(t^*) = 0;$$

or $\nexists$ multipliers $\mu_j(\alpha^*) \in \mathbb{R}, j \in I(t^*)$ in which

$$\nabla^s\Big(\underline{\mathsf{f}}_{\alpha^*} + \overline{\mathsf{f}}_{\alpha^*}\Big)(t^*) + \sum_{j \in I} \mu_j(\alpha^*).\nabla^s g_j(t^*) = 0; \tag{5.5}$$

where $\mu_j(\alpha^*) = \dfrac{\lambda_j}{\lambda_0}$. Since $I(t^*)$ represents index of active constraints, we have $g_j(t^*) < 0$ for $j \notin I(t^*)$. Thus if $j \notin I(t^*)$ and from condition (2) we can say that $\mu_j(\alpha^*) = 0$.
Also from equation(5.5) $\nexists$ multipliers $0 \leq \mu_j(\alpha^*) \in \mathbb{R}$ which satisfies condition (1) and (2). Thus the contradiction. □

$$\begin{aligned} &\min \tilde{2}t \\ \textit{subject to } & -1 \leq t \leq 2 \end{aligned} \tag{5.6}$$

such that $\tilde{2} = (1,\ 2,\ 4)$ be a triangular fuzzy number. Thus we have $\forall \alpha \in [0,1]$,

$$[\mathsf{F}(t)]^\alpha = \begin{cases} [(1+\alpha)t, (4-2\alpha)t], & if \quad t \geq 0; \\ [(4-2\alpha)t, (1+\alpha)t], & if \quad t < 0. \end{cases}$$

$$\underline{\mathsf{f}}_\alpha(t) = \begin{cases} (1+\alpha)t, & if \quad t \geq 0; \\ (4-2\alpha)t, & if \quad t < 0. \end{cases} \quad \text{and } \overline{\mathsf{f}}_\alpha(t) = \begin{cases} (4-2\alpha)t, & if \quad t \geq 0; \\ (1+\alpha)t, & if \quad t < 0. \end{cases}$$

Here we can see that $\underline{\mathsf{f}}_\alpha$ and $\overline{\mathsf{f}}_\alpha$ are not convex and are not differentiable at t=0. Thus F is not convex and is not level-wise differentiable at t=0. It is clear that F satisfies fuzzy pseudoinvex 2 and also it is gHs differentiable at t=0.
Now we have the convex function $(\underline{\mathsf{f}}_\alpha + \overline{\mathsf{f}}_\alpha)(t) = (5-\alpha)t$. Clearly F is gHs differentiable at t=0. Thus by applying Theorem 5.2, we have t=0 as the non dominated solution.

6 Conclusion

In this paper, we have presented a new concept called gHs derivative of fuzzy valued functions. We apply KKT conditions using gHs derivatives and fuzzy pseudoinvex 2 problems and we have shown that pseudoinvex 2 problems are more general than the fuzzy level wise pseudoconvex problem. Also, we can see that Theorem 5.1 and Theorem 5.2 generalises the concept of level wise pseudo convexity obtained in [15]

References

[1] Rommelfanger. H and Slowinski.R. (1998): Fuzzy linear programming with single or multiple objective functions, In R. Slowinski (Ed.), Fuzzy sets in decision analysis, operations research and statistics. Handbook fuzzy sets series, Vol. 1, pp. 179213.

[2] Delgado. M, Kacprzyk.J, Verdegay.J. L and Vila. M. A. (Eds.). (1994): Fuzzy optimization, Recent advances, New York: Physica-Verlag.

[3] Lodwick. W. A. and Kacprzyk. J. (Eds.). (2010): Fuzzy optimization, Recent advances and applications, Berlin: Springer

[4] Slowinski.R. and Teghem. J. (1990): A comparison study of STRANGE and FLIP Stochastic versus fuzzy approaches to multiobjective mathematical programming under uncertainty, Dordrecht: Kluwer Academic Publisher.

[5] Inuiguchi.M. and Ramk. J. (2000): Possibilistic linear programming: A brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem, Fuzzy Sets and Systems,No. 111, pp. 328.

[6] Wu.Z. and Xu.J. (2009): Generalized convex fuzzy mappings and fuzzy variational-like inequality, Fuzzy Sets and Systems, No.160, pp.15901619.

[7] Wu, H.-C. (2003): Duality theory in fuzzy linear programming problems with fuzzy coefficients, Fuzzy Optimization and Decision Making, Vol. 2, pp.6173.

[8] Wu, H.-C. (2007): The KarushKuhnTucker optimality conditions for the optimization problem with fuzzy-valued objective function, Mathematical Methods of Operations Research,No. 66, pp.203224.

[9] Wu, H.-C. (2009): The KarushKuhnTucker optimality conditions for multi-objective programming problems with fuzzy-valued objective functions, Fuzzy Optimization and Decision Making,Vol. 8, pp.128.

[10] Bede.B. and Gal.S. G. (2005): Generalizations of the differentiability of fuzzy number valued functions with applications to fuzzy differential equations, Fuzzy Sets and Systems, No. 151, pp.581599.

[11] Bede.B. and Stefanini. L. (2013): Generalized differentiability of fuzzy-valued functions, Fuzzy Sets and Systems, No.230, pp.119141.

[12] Chalco-Cano.Y., Romn-Flores, H. and Jimnez-Gamero, M. D. (2011): Generalized derivative and π derivative for set-valued functions, Information Sciences, No.181, pp.21772188.

[13] Chankong.V and Haimes. Y. Y, (1981): Multiobjective decision making: Theory and methodology. Amsterdam: North-Holland.

[14] Hukuhara M.(1967): Integration des applications mesurables dont la valeur est un compact con- vexe, Funkc. Ekvac., No. 10, pp.205-223 .

[15] Wu, H.-C.,(2007): The KarushKuhnTucker optimality conditions for the optimization problem with fuzzy-valued objective function, Mathematical Methods of Operations Research, No.66, pp.203224.