New Soliton and Periodic Wave Solutions of Nonlinear Evolution Equations Arising in Wave Interactions

Meetha Lal Meena [1], Ram Dayal Rankaj [2] and Arun Kumar [3]

[1] *Department of Mathematics,*
Research Scholar, Career Point University, Kota
SCRS Government College, Sawai Madhopur India

[2] *Department of Mathematics*
J.N.V. University, Jodhpur, India
drrdpankaj@yahoo.com

[3] *Department of Mathematics*
Government College, Kota, India
arunkr71@gmail.com

Abstract

Several compound phenomenons including interaction amongst waves are well explained by the travelling wave solutions of the nonlinear evolution equations (NLEEs). In this article to exhibit this phenomenon Benjamin-Ono equation is taken as the model equation. New Soliton and periodic wave solutions with arbitrary parameters are obtained in this article. The wave solutions are expressed in terms of the hyperbolic, trigonometric, and rational functions. These solutions can also be classified as solitary and periodic wave solutions. Some of the obtained solutions are graphically sketched.

Subject Classification:[2010]35M10; 49J45
Keywords: Benjamin-Ono Equations, Travelling Wave Solutions, Nonlinear Evolution Equation

1 Introduction

Almost, in all the scientific and engineering fields nonlinear wave phenomena appears in one or other ways. This phenomenon can be well explained by the nonlinear evolution equations (NLEEs). These are of more practical use if one can find the travelling wave solutions for these equations. The wave interactions can also be explained by these solutions. Therefore, the powerful and efficient method to find exact solutions of nonlinear equations is still a challenge in the field of physical sciences.

In the past three decades, several methods explored which includes- the homogeneous balance method [1], the tanh-function method [2], the extended tanh-function method [3, 4], the Exp-function method [5-7], the sine-cosine method [8], the modified Exp-function method [9], the generalized Riccati equation [10], the Jacobi elliptic function expansion method [11, 12], the Hirotas bilinear method [13], the Miura transformation [14], the (G/

G) -expansion method [15-19], the novel (G / G) expansion method [20, 21], the modified simple equation method [22, 23], the improved (G/ G) -expansion method [24,26], the inverse scattering transform [25] and so on.
Here in this article the study, relating to the new expansion method for solving the Benjamin-Ono equation, is presented. Solitary wave propagation is also discussed to explain the wave interaction phenomenon arising in NLEEs.
The outline of this paper is organized as follows: In Section 2, we give the description of the new expansion method. In Section 3, we apply this method to the Benjamin-Ono equation, results and discussions and graphical representation of solutions. Conclusions are given in the last section.

2 Description of the Method:

Let us consider a general nonlinear PDE in the form

$$\Phi(v, v_t, v_x, v_{tt}, v_{xx}, v_{xt}, ...) = 0 \tag{2.1}$$

where $v = v(x, t)$ is an unknown function,Φ is a polynomial in $v(x, t)$ and its derivatives in which highest order derivatives and nonlinear terms are involved and the subscripts stand for the partial derivatives.
Step 1: We combine the real variables x and t by a complex variable

$$v = v(\eta), \qquad \eta = x \pm Vt \tag{2.2}$$

Where v is the speed of the travelling wave. The travelling wave transformation (2.2) converts Eq. (2.1) into an ordinary differential equation (ODE) for $v = v(\eta)$:

$$\Psi(v, v', v'', v''', ...) = 0 \tag{2.3}$$

where Ψ is a polynomial of v and its derivatives and the superscripts indicate the ordinary derivatives with respect to η .
Step 2: According to possibility, Eq. (2.3) can be integrated term by term one or more times, yields constant(s) of integration. The integral constant may be zero, for simplicity.
Step 3: Suppose the travellingg wave solution of Eq. (2.3) can be expressed as follows:

$$n(\eta) = \sum_{i=0}^{N} \alpha_i (d + M)^i + \sum_{i=0}^{N} \beta_i (d + M)^{-i} \tag{2.4}$$

where either a_N or β_N may be zero, but could be zero simultaneously, $\alpha_i(i = 0, 1, 2, ..., N)$ and $\beta_i(i = 1, 2, ..., N)$ and d are arbitrary constant to be determined and $M(\eta)$ is

$$M(\eta) = (G'/G) \tag{2.5}$$

Where $G = G(\eta)$ satisfies the following auxiliary nonlinear ordinary differential equation:

$$AGG'' - BGG' - EG^2 - C(G')^2 = 0 \tag{2.6}$$

where the prime stands for derivative with respect to ; A, B, C and E are real parameters.
Step 4: To determine the positive integer, taking the homogeneous balance between the highest order nonlinear terms and the derivatives of the highest order appearing in Eq. (2.3).
Step 5: Substitute Eq. (2.4) and Eq. (2.6) including Eq. (2.5) into Eq. (2.3) with the value of N obtained in Step 4, we obtain polynomials in $(d+M)^N (N = 0,1,2,.....)$ and $(d+M)^{-N}$, $(N = 0,1,2,.....)$. Subsequently, we collect each coefficient of the resulted polynomials to zero, yields a set of algebraic equations for α_i, $(i = 0,1,2,....,N)$ and $\beta_i (i = 1,2,....,N)$, d and V.
Step 6: Suppose that the value of the constant $\alpha_i, (i = 0,1,2,.....,N), \beta_i (i = 1,2,.....,N)$, d and V can be found by solving the algebraic equations obtained in Step 5. Since the general solutions of Eq. (2.6) are known to us, inserting the values of $\alpha_i (i = 0,1,2,.....,N)$,$\beta_i (i = 1,2,....,N)$, d and V into Eq. (2.4), we obtain more general type and new exact travelling wave solutions of the nonlinear partial differential equation (2.1).
Step 7: Using the solution of Eq. (2.6), one can derive the value of $M(\eta)$of Eq. (2.5):
Family 1: when $B \neq 0, \omega = A - C$ and $\Omega = B^2 + 4E(A-C) > 0$,

$$M(\eta) = \frac{B}{2\omega} + \frac{\sqrt{\Omega}}{2\omega} \frac{C_1 \sinh\left(\frac{\sqrt{\Omega}}{2A}\eta\right) + C_2 \cosh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)}{C_1 \cosh\left(\frac{\sqrt{\Omega}}{2A}\eta\right) + C_2 \sinh\left(\frac{\sqrt{\Omega}}{2A}\eta\right)} \tag{2.7}$$

Family 2: When $B \neq 0, \omega = A - C$ and $\Omega = B^2 + 4E(A-C) < 0$,

$$M(\eta) = \frac{B}{2\omega} + \frac{\sqrt{\Omega}}{2\omega} \frac{-C_1 \sin\left(\frac{\sqrt{-\Omega}}{2A}\eta\right) + C_2 \cosh\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)}{C_1 \cos\left(\frac{\sqrt{-\Omega}}{2A}\eta\right) + C_2 \sinh\left(\frac{\sqrt{-\Omega}}{2A}\eta\right)} \tag{2.8}$$

Family 3:$B \neq 0$, $\omega = A - C$ and $\Omega = B^2 + 4E(A-C) = 0$,

$$M(\eta) = \frac{B}{2\omega} + \frac{C_2}{C_1 + C_2\eta} \tag{2.9}$$

Family 4: $B = 0$, $\omega = A - C$ and $\Delta = \omega E > 0$,

$$M(\eta) = \frac{\sqrt{\Delta}}{\omega} \frac{C_1 \sinh(\frac{\sqrt{\Delta}}{A}\eta) + C_2 \cosh(\frac{\sqrt{\Delta}}{2A}\eta)}{C_1 \cosh(\frac{\sqrt{\Delta}}{A}\eta) + C_2 \sinh(\frac{\sqrt{\Delta}}{A}\eta)} \tag{2.10}$$

Family 5: When $B = 0$,$\omega = A - C$ and $\Delta = \omega E < 0$,

$$M(\eta) = \frac{\sqrt{-\Delta}}{\omega} \frac{C_1 \sin(\frac{\sqrt{-\Delta}}{A}\eta) + C_2 \cos(\frac{\sqrt{\Delta}}{2A}\eta)}{C_1 \cos(\frac{\sqrt{-\Delta}}{A}\eta) + C_2 \sin(\frac{\sqrt{-\Delta}}{A}\eta)} \tag{2.11}$$

3 Application of the Method

In this section, we will present the proposed new generalized expansion method to construct new and more general travelling wave solutions of the Benjamin-Ono equation. Let us consider the Benjamin-Ono equation,

$$v_t + hv_{xx} + vv_x = 0. \tag{3.1}$$

We utilize the travelling wave variable $w(\eta) = v(x,t)$, $\eta = x - Vt$, Eq. (3.1) is carried to an ODE

$$-Vw' + hw'' + \frac{1}{2}(w^2)' = 0. \tag{3.2}$$

Eq. (3.2) is integrable, therefore, integrating with respect to ηonce yields:

$$P - Vw + hw' + \frac{1}{2}w^2 = 0, \tag{3.3}$$

where P is an integration constant which is to be determined. Taking the homogeneous balance between highest order nonlinear term w^2 and linear term of the highest order w' in Eq. (3.3), we obtain $N = 1$. Therefore, the solution of Eq. (3.3) is of the form

$$w(\eta) = \alpha_0 + \alpha_1(d + M) + \beta_1(d + M)^{-1}, \tag{3.4}$$

Where $\alpha_0, \alpha_1, \beta_1$ and d are constants to be determined.
Substituting Eq. (3.4) together with Eqs. (2.5) and (2.6) into Eq. (3.3), the left-hand side is converted into polynomials in $(d + M)^N (N = 0, 1, 2, ...)$and $(d + M)^{-N} (N = 1, 2, ...)$. We collect each coefficient of these resulted polynomials to zero yields a set of simultaneous algebraic equations (for simplicity, the equations are not presented) for $\alpha_0, \alpha_1, \beta_1$ d, P and V. Solving these algebraic equations with the help of computer algebra, we obtain following:
Set-1:-

$$\begin{aligned} -P &= \frac{1}{2A^4}(4h^2d^2\omega^2 + 4h^2Bd\omega - 4h^2E\omega - 2a_0AhB + 4a_0AhCd + a_0^2A^2 - 4a_0A^2hd), \\ \alpha_0 &= \alpha_0, V = -\frac{1}{A}(hB + 2hd\omega - a_0A), d = d, \alpha_1 = 0,\ \beta_1 = -\frac{2h}{A}(d^2\omega + Bd - E), \end{aligned} \tag{3.5}$$

where $\omega = A - C$, $a_0, d,\ A, B, C, E$ are free parameters.
Set-2:-

$$\begin{aligned} P &= \frac{1}{2A^4}(4h^2d^2\omega^2 - 4h^2E\omega + a_0^2A^2 + 2a_0AhB + 4a_0A^2hd - 4a_0AhCd + 4h^2Bd\omega), \\ \alpha_0 &= \alpha_0, V = \frac{1}{A}(a_0A + hB + 2hd\omega), d = d, \beta = 0,\ \alpha_1 = \frac{2h\omega}{A}. \end{aligned} \tag{3.6}$$

where $\omega = A - C, a_0, d,\ A, B, C, E$ are free parameters.
Set-3:-

(3.7)
$$P = \frac{1}{2A^2}(a_0^2A^2 - 16h^2E\omega - 4h^2B^2), V = a_0, d = -\frac{B}{2\omega}\alpha_0 = \alpha_0, \alpha_1 = \frac{2h\omega}{A},\ \beta_1 = \frac{h}{A\omega}(4E\varepsilon + B^2),$$

where $\omega = A - C, V, A, B, C, E$ are free parameters.
For set-1, substituting Eq. (3.5) into Eq. (3.4), along with Eq. (2.7) and simplifying, yields following travelling solutions, if $C_1 = 0$ but $C_2 \neq 0; C_2 = 0$ but $C_1 \neq 0$respectively.

$$w_{11}(\eta) = \alpha_0 - \frac{2h}{A}(d^2\omega + Bd - E) \times \left(d + \frac{B}{2\omega} + \frac{\sqrt{\Omega}}{2\omega}\coth(\frac{\sqrt{\Omega}}{2A}\eta)\right)^{-1}$$
$$w_{12}(\eta) = \alpha_0 - \frac{2h}{A}(d^2\omega + Bd - E) \times \left(d + \frac{B}{2\omega} + \frac{\sqrt{\Omega}}{2\omega}\tanh(\frac{\sqrt{\Omega}}{2A}\eta)\right)^{-1}$$

Substituting Eq. (3.5) into Eq. (3.5), along with Eq. (2.8) and simplifying our exact solutions become, if $C_1 = 0$ but $C_2 = 0$ but $C_1 \neq 0$ respectively.

$$w_{1_3}(\eta) = \alpha_0 - \frac{2h}{A}(d^2\omega + Bd - E) \times \left(d + \frac{B}{2\omega} + \frac{\sqrt{-\Omega}}{2\omega}\coth(\frac{\sqrt{-\Omega}}{2A}\eta)\right)^{-1}$$
$$w_{14}(\eta) = \alpha_0 - \frac{2h}{A}(d^2\omega + Bd - E) \times \left(d + \frac{B}{2\omega} + \frac{\sqrt{-\Omega}}{2\omega}\tan(\frac{\sqrt{\Omega}}{2A}\eta)\right)^{-1}$$

Substituting Eq. (3.5) into Eq. (3.4), together with Eq. (2.9) and simplifying, our obtained solution becomes:

$$w_{15}(\eta) = \alpha_0 - \frac{2h}{A}(d^2\omega + Bd - E) \times \left(d + \frac{B}{2\omega} + \frac{C\&2}{C_1 + C_{2n}\eta}\right)^{-1}$$

Subsituting Eq. (3.5) into Eq. (3.4), along with Eq. (2.10) and simplifying, we obtain following travelling wave solutions, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$ respectively.

$$w_{1_6}(\eta) = \alpha_0 - \frac{2h}{A}(d^2\omega + Bd - E) \times \left(d + \frac{\sqrt{\Delta}}{\omega}\coth(\frac{\sqrt{\Delta}}{A}\eta)\right)^{-1}$$
$$w_{1_7}(\eta) = \alpha_0 - \frac{2h}{A}(d^2\omega + Bd - E) \times \left(d + \frac{\sqrt{\Delta}}{\omega}\tanh(\frac{\sqrt{\Delta}}{A}\eta)\right)^{-1}$$

Substituting Eq. (3.5) into Eq. (3.4), together with Eq. (2.11) and simplifying, our obtained exact solution become, if $C_1 \neq 0$ respectively:

$$w_{1_8}(\eta) = \alpha_0 - \frac{2h}{A}(d^2\omega + Bd - E) \times \left(d + \frac{\sqrt{-\Delta}}{\omega}\cot(\frac{\sqrt{-\Delta}}{A}\eta)\right)^{-1}$$
$$w_{1_9}(\eta) = \alpha_0 - \frac{2h}{A}(d^2\omega + Bd - E) \times \left(d - \frac{\sqrt{-\Delta}}{\omega}\tan(\frac{\sqrt{-\Delta}}{A}\eta)\right)^{-1}$$

where $\eta = x + \frac{1}{A}(hB + 2hd\omega - a_0 A)t$.
Again for set 2, substituting Eq. (3.6) into Eq. (3.4), along with Eq. (2.7) and simplifying, our travelling wave solutions become, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$ respectively:

$$w_{2_1}(\eta) = \alpha_0 + \frac{1}{A}(h(B + 2d\omega) + h\sqrt{\Omega}\coth(\frac{\sqrt{\Omega}}{2A}\eta)),$$
$$w_{2_2}(\eta) = \alpha_0 + \frac{1}{A}(h(B + 2d\omega) + h\sqrt{\Omega}\tanh(\frac{\sqrt{\Omega}}{2A}\eta)),$$

Substituting Eq. (3.6) into Eq. (3.4), along with Eq. (2.8) and simplifying yields exact solution, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$ respectively:

$$w_{2_3}(\eta) = \alpha_0 + \frac{1}{A}(h(B + 2d\omega) + ih\sqrt{\Omega}\cot(\frac{\sqrt{-\Omega}}{2A}\eta)),$$
$$w_{2_4}(\eta) = \alpha_0 + \frac{1}{A}(h(B + 2d\omega) - ih\sqrt{\Omega}\cot(\frac{\sqrt{-\Omega}}{2A}\eta)),$$

Substituting Eq. (3.6) into Eq. (3.4), along with Eq. (2.9) and simplifying, our obtained solution becomes:

$$w_{2_5}(\eta) = \alpha_0 + \frac{1}{A}(h(B + 2d\omega) + 2h\omega(\frac{C_2}{C_1 + C_2\eta})).$$

Subsituting Eq. (3.6) into Eq. (3.4), together with Eq. (2.10) and simplifying, yield following travelig wave solution if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$ respectively:

$$w_{2_6}(\eta) = \alpha_0 + \frac{1}{A}(2h\omega d + 2h\sqrt{\Delta}\coth(\frac{\sqrt{\Delta}}{A}n)),$$
$$w_{2_7}(\eta) = \alpha_0 + \frac{1}{A}(2h\omega d + 2h\sqrt{\Delta}\tanh(\frac{\sqrt{\Delta}}{A}n))$$

Subsituting Eq. (3.6) into Eq. (3.4), along with Eq.(2.11) and simplifying, our exact solutions become, if $C_1 = 0$ but $C_2 \neq 0$; $C_2 = 0$ but $C_1 \neq 0$ respectively:

$$w_{2_8}(\eta) = \alpha_0 + \frac{1}{A}(2h\omega d + 2ih\sqrt{\Delta}\cot(\frac{\sqrt{-\Delta}}{A}n))$$
$$w_{2_9}(\eta) = \alpha_0 + \frac{1}{A}(2h\omega d + 2ih\sqrt{\Delta}\tan(\frac{\sqrt{-\Delta}}{A}n))$$

where $\eta = x + \frac{1}{A}(a_0 A + hB + 2hd\omega)t$.
Similarly, for Set 3, substituting Eq. (3.7) into Eq. (3.4), together with Eq. (2.7) and simplifying, yields following travelling wave solutions, if $C_1 = 0$ but$C_2 \neq 0; C_2 = 0$ but $C_1 \neq 0$ respectively:

$$w_{3_1}(\eta) = a_0 + \frac{1}{A}(h\sqrt{\Omega} \times \coth(\frac{\sqrt{\Omega}}{2A}\eta) + \frac{2h}{\sqrt{\Omega}}(4E\omega + B^2) \times \tanh(\frac{\sqrt{\Omega}}{2A}\eta))$$
$$w_{3_2}(\eta) = a_0 + \frac{1}{A}(h\sqrt{\Omega} \times \tanh(\frac{\sqrt{\Omega}}{2A}\eta) + \frac{2h}{\sqrt{\Omega}}(4E\omega + B^2) \times \coth(\frac{\sqrt{\Omega}}{2A}\eta))$$

Subsituting Eq. (3.7) into Eq. (3.4), along with Eq. (2.8) and simplifying, we obtain following solutions, if $C_1 = 0$ but $C_2 \neq 0; C_2 = 0$ but $C_1 \neq 0$ respectively:

$$w_{3_3}(\eta) = a_0 + \frac{1}{A}(hi\sqrt{\Omega} \times \cot(\frac{\sqrt{-\Omega}}{2A}\eta) + \frac{2h}{i\sqrt{\Omega}}(4E\omega + B^2) \times \tan(\frac{\sqrt{-\Omega}}{2A}\eta))$$

$$w_{3_4}(\eta) = a_0 - \frac{1}{A}(hi\sqrt{\Omega} \times \tan(\frac{\sqrt{-\Omega}}{2A}\eta) + \frac{2h}{i\sqrt{\Omega}}(4E\omega + B^2) \times \cot(\frac{\sqrt{-\Omega}}{2A}\eta))$$

Substituting Eq. (3.7) into Eq. (3.4), along with Eq. (2.9) and simplifying, our obtained solution becomes:

$$w_{3_5}(\eta) = a_0 + \frac{2h\omega}{A} \times (\frac{C_2}{C_1 + C_2\eta}) + \frac{h}{A\omega}(4E\omega + B^2) \times (\frac{C_2}{C_1 + C_2\eta})^{-1}$$

Subsituting Eq. (3.7) into Eq. (3.4), along with Eq. (2.10) and simplifying, yields following travelig wave solutions, if $C_1 = 0$ but $C_2 \neq 0; C_2 = 0$ but $C_1 \neq 0$respectively:

$$w_{3_6}(\eta) = a_0 + \frac{2h\omega}{A} \times (\frac{-B}{2\omega} + \frac{\sqrt{\Delta}}{\omega}\coth(\frac{\sqrt{\Delta}}{A}n)) + \frac{h}{A\omega}(4E\omega + B^2) \times (\frac{-B}{2\omega} + \frac{\sqrt{\Delta}}{\omega}\coth(\frac{\sqrt{\Delta}}{A}\eta))^{-1}$$

$$w_{3_7}(\eta) = a_0 + \frac{2h\omega}{A} \times (\frac{-B}{2\omega} + \frac{\sqrt{\Delta}}{\omega}\tanh(\frac{\sqrt{\Delta}}{A}n)) + \frac{h}{A\omega}(4E\omega + B^2) \times (\frac{-B}{2\omega} + \frac{\sqrt{\Delta}}{\omega}\tanh(\frac{\sqrt{\Delta}}{A}\eta))^{-1}$$

Subsituting Eq. (3.7) into Eq. (3.4), along with Eq. (2.11) and simplifying, our obtained exact solutions become, if $C_1 = 0$ but $C_2 \neq 0; C_2 = 0$ but $C_1 \neq 0$ respectively.

$$w_{3_8}(\eta) = a_0 + \frac{2h\omega}{A} \times (\frac{-B}{2\omega} + \frac{\sqrt{-\Delta}}{\omega}\cot(\frac{\sqrt{-\Delta}}{A}n)) + \frac{h}{A\omega}(4E\omega + B^2) \times (\frac{-B}{2\omega} + \frac{\sqrt{-\Delta}}{\omega}\cot(\frac{\sqrt{\Delta}}{A}\eta))^{-1}$$

$$w_{3_9}(\eta) = a_0 + \frac{2h\omega}{A} \times (\frac{-B}{2\omega} + \frac{\sqrt{-\Delta}}{\omega}\tan(\frac{\sqrt{-\Delta}}{A}n)) + \frac{h}{A\omega}(4E\omega + B^2) \times (\frac{-B}{2\omega} - \frac{\sqrt{-\Delta}}{\omega}\tan(\frac{\sqrt{-\Delta}}{A}\eta))^{-1}$$

where$\eta = x - a_0 t$.

3.1 Results and Discussion

It is worth declaring that some of our obtained solutions are in good agreement with already published results which are presented in the following table.
Table 1. Comparison between Neyrame et al. [27] solutions and our solutions Beside this table, we obtain more new exact travelling wave solutions $w_2, (\eta)$, $w_{2_4}(\eta)$, $w_{2_6}(\eta)$, $-w_{2_9}(\eta)$, $w_{1_1}(\eta) - w_{1_9}(\eta)$, $w_{3_1}(\eta) - w_{3_9}(\eta)$ in this article, which have not been reported in the previous literature.

3.2 Graphic representation of the solutions

The graphical illustrations of the solutions are given below in the figures with the aid of maple

Neyrame et al. [27] solutions	**Obtained solutions**
i. If $C_1=0$ and $u(\xi)=w_{2_1}(\eta)$, solutions Eq. (16) becomes: $w_{2_1}(\eta)=2h(\lambda^2-4\mu)\times\coth^2(\frac{\sqrt{\lambda^2-4\mu}\eta}{2})-\frac{\lambda}{2}+\alpha_0$ ii. If $C_1=0$ and $u(\xi)=w_{2_3}(\eta)$, solutions Eq. (16) becomes: $w_{2_3}(\eta)=2h(4\mu-\lambda^2)\times\coth^2(\frac{\sqrt{4\mu-\lambda^2}\eta}{2})-\frac{\lambda}{2}+\alpha_0$	i. If $A=1, C=0, \Omega=\lambda^2-4\mu, B=1, h=2h$, $d=-\frac{\lambda}{2h}, B=0$ then the solution is $w_{2_1}(\eta)=2h(\lambda^2-4\mu)\times\coth^2(\frac{\sqrt{\lambda^2-4\mu}\eta}{2})-\frac{\lambda}{2}+\alpha_0$ ii. If $A=1, C=0, \Omega=\lambda^2-4\mu, B=1, h=2h$, $d=-\frac{\lambda}{4h}, B=0$ then the solution is $w_{2_3}(\eta)=2h(4\mu-\lambda^2)\times\coth^2(\frac{\sqrt{4\mu-\lambda^2}\eta}{2})-\frac{\lambda}{2}+\alpha_0$
iii. If $u(\xi)=w_{2_5}(\eta)$, solutions Eq. (16) becomes: $w_{2_5}(\eta)=2h(\frac{C_2}{C_1+C_2\eta})$	iii. If $A=1, C=0, B=-2d$ and $\alpha_0=0$ then the solutions is $w_{2_5}(\eta)=2h(\frac{C_2}{C_1+C_2\eta})$

The solutions corresponding to $w_{1_1}(\eta)$, $w_{1_5}(\eta)$, $w_{1_6}(\eta)$, $w_{2_5}(\eta)$, $w_{2_8}(\eta)$, $w_{3_1}(\eta)$, $w_{3_5}(\eta)-w_{3_7}(\eta)$ is identical to the solution $w_{2_5}(\eta)$ the solution corresponding to $w_{1_7}(\eta)$, $w_{2_2}(\eta)$, $w_{2_7}(\eta)$ is identical to the solution $w_{1_2}(\eta)$,the solution corresponding to $w_{1_3}(\eta)$, $w_{1_4}(\eta)w_{1_8}(\eta)$, $w_{1_9}(\eta)$, $w_{2_3}(\eta)$, $w_{2_4}(\eta)$, $w_{2_9}(\eta)$is identical to the solution $w_{1_8}(\eta)$ and the solution corresponding to $w_{3_4}(\eta)$, $w_{3_8}(\eta)$, $w_{3_9}(\eta)$ is identical to the solution $w_{3_3}(\eta)$.

4 Conclusion

A new expansion method is applied to solve the Benjamin-Ono equation. As a result, many exact travelling wave solutions are obtained which include new soliton-like solutions, trigonometric function solutions and rational solutions. The used method is much simpler in comparing to other methods because this method is straightforward and its calculation procedure is very concise. Therefore, the applied method is quite efficient and practically well suited and could be more effectively used to solve various NLEEs which regularly arise in science, engineering and other technical arenas.

References

[1] E. G. Fan, (2000). Two new applications of the homogeneous balance method. Phys. Lett. A. 265 pg. 353-357.

[2] Arun Kumar, Ram Dayal Pankaj (2014) Tanhcoth scheme for traveling wave solutions for Nonlinear Wave Interaction model, Journal of the Egyptian Mathematical Society. 23(2), 282285

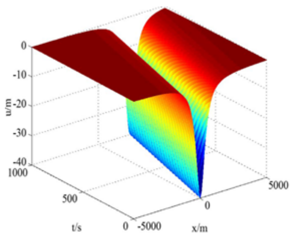

Fig. 1: The one-solitary wave propagation

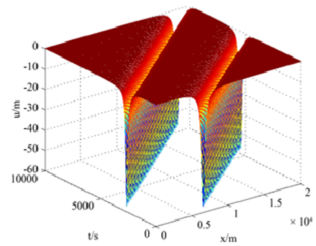

Fig. 2: Two parallel solitons plot for Eqn.(12) with amplitude 40 m.

[3] E. Fan, (2000) Extended tanh-function method and its applications to nonlinear equations. Phys Lett A 277(4- 5) pg. 212-218

[4] S. A. El-Wakil, M. A. Abdou, (2007) New exact travelling wave solutions using modified extended tanh-function method. Chaos Solitons Fractals 31(4) pg. 840-852.

[5] J. H. He and X. H. Wu, (2006) Exp-function method for nonlinear wave equations, Chaos, Solitons Fract.30, pg.700-708

[6] M. A. Akbar and N. H. M. Ali, (2012) New Solitary and Periodic Solutions of Nonlinear Evolution Equation by Exp-function Method, World Appl. Sci. J., 17(12) pg.1603-1610

[7] Pankaj R. D. and. Kumar, A. (2013) Solutions of the Coupled Klein-Gordon Equation by Modified Exp-Function Method, Indian Journal Of Applied Research Volume: 3(3), pg. 276-278

[8] A. M. Wazwaz, (2004) A sine-cosine method for handle nonlinear wave equations, Applied Mathematics and Computer Modeling, 40 , pg.499-508.

[9] M. Usman, A. Nazir, T. Zubair, I. Rashid, Z. Naheed, S.T. Mohyud-Din, (2013) Solitary wave solutions of (3+1)-dimensional Jimbo Miwa and Pochhammer-Chree equations by modified Exp-function method, Int. J. Modern Math. Sci. 5(1) pg.27-36.

[10] Z. Yan and H. Zhang, (2001) New explicit solitary wave solutions and periodic wave solutions for Whitham Broer-Kaup equation in shallow water, Physics Letters. A, vol. 285 no.5-6, pg. 355- 362.

[11] D. Liu, (2005) Jacobi elliptic function solutions for two variant Boussinesq equations, Chaos solitons Fractals, 24, pg. 1373-85.

[12] Ram Dayal Pankaj Bhawani Singh, Arun Kumar (2019). New extended Jacobi elliptic function expansion scheme for wave-wave interaction in ionic media. Nanosystems: physics, chemistry, mathematics, 9(5), Pg.581-585

[13] A. M. Wazwaz, (2012) Multiple-soliton solutions for a (3+1)-dimensional generalized KP equation, Commun Nonlinear Sci Numer Simulat 17, pg.491495.

[14] T. L. Bock and M. D. Kruskal, (1979) A two-parameter Miura transformation of the Benjamin-One equation, Phys. Lett. A ,74, pg.173-176.

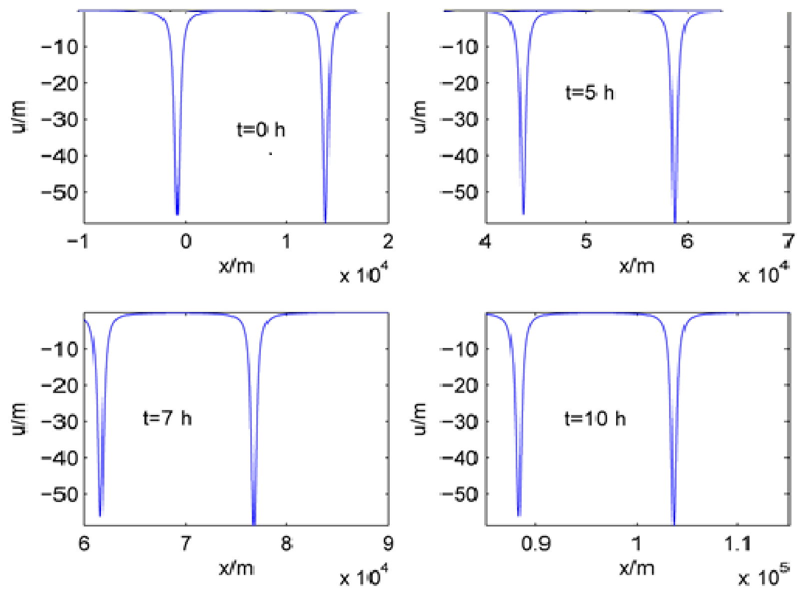

Fig. 3: The sectional plots of two parallel solitons interaction at different times

[15] M. L. Wang, X. Z. Li and J. Zhang, (2008) The (G / G) expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics, Phys. Lett. A, 372, pg. 417-423.

[16] A. Bekir, (2008) Application of the (G / G) -expansion method for nonlinear evolution equations, Phys. Lett. A 372, pg. 3400-3406.

[17] E. M. E. Zayed, (2009) The (G / G) -expansion method and its applications to some nonlinear evolution equations in the mathematical physics, J. Appl. Math. Comput., 30, pg. 89-103.

[18] S. Zhang, J. Tong and W. Wang, (2008) A generalized (G / G) -expansion method for the mKdV equation with variable coefficients, Phys. Lett. A, 372, pg. 2254-2257

[19] M. A. Akbar, N. H. M. Ali and E. M. E. Zayed, (2012) A generalized and improved (G / G) -expansion method for nonlinear evolution equations, Math. Prob. Engr., Vol. 2012 22 pages. doi: 10.1155/2012/459879.

[20] M. N. Alam, M. A. Akbar and S. T. Mohyud-Din, A novel (G / G) -expansion method and its application to the Boussinesq equation, Chin. Phys. B, Article ID 131006 (accepted for publication).

[21] M. N. Alam and M. A. Akbar, Travelling wave solutions of the nonlinear (1+1)-dimensional modified Benjamin-Bona-Mahony equation by using new (G / G)-expansion, Physical Review & Research International (Sciencedomain international), Manuscript number: 2013_$PRRI$_5710, 2013(accepted for publication)

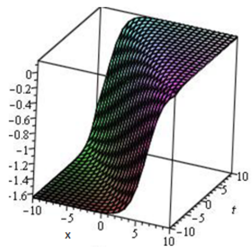

Fig. 4: Modulus plot of Kink wave, Shape of when $A = 4, B = 1, C = 1, 0 = 1, d = 1, h = 1$and $-10 \leq (x, t) \leq 10$.

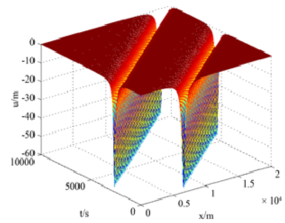

Fig. 5: Modulus plot of Kink wave, Shape of when$A = 1, B = 2, C = 2, E = 1, d = 1, h = 10 = 1, C1 = 2, C2 = 1$ and $-15 \leq (x, t) \leq 15$.

[22] A. J. M. Jawad, M. D. Petkovic and A. Biswas, (2010) Modified simple equation method for nonlinear evolution equations, Appl. Math. Comput., 217, pg. 869-877.

[23] K. Khan, M. A. Akbar and M. N. Alam, Travelling wave solutions of the nonlinear Drinfeld-Sokolov-Wilson equation and modified Benjamin-Bona-Mahony equations, J. Egyptian Math. Soc., DOI org/10.1016/j.joems.2013.04.010 (in press).

[24] J. Zhang, F. Jiang and X. Zhao, (2010) An improved (G / G)-expansion method for solving nonlinear evolution equations, Int. J. Com. Math., 87(8), pg. 1716-1725.

[25] M. J. Ablowitz and P. A. Clarkson, Soliton, nonlinear evolution equations and inverse scattering (Cambridge University Press, New York, 1991).

[26] H. Naher and F. A. Abdullah, (2013) New approach of (G / G)-expansion method and new approach of generalized (G / G) -expansion method for nonlinear evolution equation, AIP Advances 3, 032116 DOI: 10. 1063/1.4794947.

[27] A. Neyrame, A. Roozi, S. S. Hosseini, S. M. Shafiof, (2010) Exact travelling wave solutions for some nonlinear partial differential equations, journal of King Saud University (Science), 22, pg. 275-278.