Two Hypergeometric Reduction Formulas Involving The Gauss and Clausen Functions

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Abstract

In this paper, by using the series rearrangement technique, we derive closed forms of two reduction formulas for the following Gauss and Clausen hypergeometric functions:

$$ {}_2F_1\left[\begin{array}{c} \frac{a}{2},\ \frac{a+1}{2}; \\ \\ a; \end{array} \frac{4z^3}{27(1-z)^2}\right] $$

and

$$ {}_3F_2\left[\begin{array}{c} a,\ \frac{3a-2}{2},\ \frac{3a-3}{2}; \\ \\ a-1,\ 3a-2; \end{array} \frac{z}{3}\right]. $$

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1 Introduction and Preliminaries

In our investigations, we shall use the following standard notations:
$\mathbb{N} := \{1, 2, 3, \cdots\}$, $\mathbb{N}_0 := \mathbb{N} \bigcup \{0\}$ and $\mathbb{Z}_0^- := \mathbb{Z}^- \bigcup \{0\} = \{0, -1, -2, -3, \cdots\}$.
Also, the symbols $\mathbb{C}, \mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers respectively.
The well-known Pochhammer symbol (or the *shifted* factorial) $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$[3, p.22 eq(1), p.32 Q.N.(8) and Q.N.(9)], see also [5, p.23, eq(22) and eq(23)], is defined by

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$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C}\backslash\{0\}) \\ \prod_{j=0}^{n-1}(\lambda+j) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}) \\ \frac{(-1)^k n!}{(n-k)!} & (\lambda = -n;\ \nu = k;\ n, k \in \mathbb{N}_0;\ 0 \leqq k \leqq n) \\ 0 & (\lambda = -n;\ \nu = k;\ n, k \in \mathbb{N}_0;\ k > n) \\ \frac{(-1)^k}{(1-\lambda)_k} & (\nu = -k;\ k \in \mathbb{N};\ \lambda \in \mathbb{C}\backslash\mathbb{Z}), \end{cases}$$

it being understood conventionally that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.
A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha, \beta; \gamma; z]$, is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$$(1.1)\qquad {}_pF_q\left[\begin{array}{ll} (\alpha_p); & \\ & z \\ (\beta_q); & \end{array}\right] = {}_pF_q\left[\begin{array}{ll} \alpha_1, \alpha_2, \ldots, \alpha_p; & \\ & z \\ \beta_1, \beta_2, \ldots, \beta_q; & \end{array}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_p)_n}{(\beta_1)_n(\beta_2)_n \ldots (\beta_q)_n} \, \frac{z^n}{n!} \ ,$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and the denominator parameters $\beta_1, \beta_2, \ldots, \beta_q$ take on complex values, provided that

$$\beta_j \ \neq \ 0, -1, -2, \ldots \ ; \quad j = 1, 2, \ldots, q.$$

Supposing that none of the numerator and denominator parameters is zero or a negative integer, we note that the ${}_pF_q$ series defined by equation (1.1):

(i) converges for $|z| < \infty$, if $p \leq q$,

(ii) converges for $|z| < 1$, if $p = q + 1$,

(iii) diverges for all z, $z \neq 0$, if $p > q + 1$,

(iv) converges absolutely for $|z| = 1$, if $p = q + 1$ and $\Re(\omega) > 0$,

(v) converges conditionally for $|z| = 1 (z \neq 1)$, if $p = q + 1$ and $-1 < \Re(\omega) \leqq 0$,

(vi) diverges for $|z| = 1$, if $p = q + 1$ and $\Re(\omega) \leqq -1$,

where by convention, a product over an empty set is interpreted as 1 and

$$(1.2)\qquad \omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j,$$

$\Re(\omega)$ being the real part of complex number ω.
Each of the following results will be needed in our present study.
Two Closed Forms [3, p.70, Q.N.(10)] see also[1, p.185, Q.N.(39)],[4, p.19, Eq.(1.5.20)]:

$$\left(\frac{2}{1+\sqrt{(1-z)}}\right)^{2\lambda-1} = {}_2F_1\left[\begin{matrix}\lambda,\ \lambda-\frac{1}{2}; \\ 2\lambda;\end{matrix}\ z\right], \tag{1.3}$$

and

$$\frac{1}{\sqrt{(1-z)}}\left(\frac{2}{1+\sqrt{(1-z)}}\right)^{2\lambda-1} = {}_2F_1\left[\begin{matrix}\lambda,\ \lambda+\frac{1}{2}; \\ 2\lambda;\end{matrix}\ z\right], \tag{1.4}$$

where $(2\lambda \in \mathbb{C}\backslash\mathbb{Z}_0^-)$ and $|arg(1-z)| < \pi$.
Remark 1. For simplification purposes, the algebraic identity has been used:

$$\sqrt{(-4z^3+27z^2-54z+27)} = (3-z)\sqrt{(3-4z)}\ . \tag{1.5}$$

Motivated by the work of Joshi and Vyas [2], we mention the closed forms of some reduction formulas in section 2. In section 3, we give the short proofs of these reduction formulas by using the series rearrangement technique.

2 Some Closed Forms

Any values of parameters and arguments leading to the results which do not make sense, are tacitly excluded.

$$ {}_2F_1\left[\begin{matrix}\frac{a}{2},\ \frac{a+1}{2}; \\ a;\end{matrix}\ \frac{4z^3}{27(1-z)^2}\right] \tag{2.1}$$

$$= \frac{(6\sqrt{3})^a\ \ (1-z)^a}{2\ (3-z)\sqrt{3-4z}\ \ \left[3\sqrt{3}(1-z)+(3-z)\sqrt{3-4z}\ \right]^{a-1}}\ .$$

$${}_3F_2\left[\begin{matrix}a,\ \frac{3a-2}{2},\ \frac{3a-3}{2}; \\ a-1,\ 3a-2;\end{matrix}\ z\right] = \left(\frac{2}{1+\sqrt{1-z}}\right)^{3a-3}\left[1+\frac{3z}{2\left\{1-z+\sqrt{1-z}\ \right\}}\right]. \tag{2.2}$$

Remark 2. The results (2.1) and (2.2) have been verified numerically.

3 Demonstration of Closed Forms

Short proof of the result (2.1):
Suppose the left hand side of the result (2.1) is denoted by

$$\Xi(z) :=\ {}_2F_1\left[\begin{matrix}\frac{a}{2},\ \frac{a+1}{2}; \\ a;\end{matrix}\ \frac{4z^3}{27(1-z)^2}\right]. \tag{3.1}$$

Now applying the closed form (1.4) with $z \longrightarrow \frac{4z^3}{27(1-z)^2}$ and $\lambda \longrightarrow \frac{a}{2}$ in the above equation (3.1) and also using the identity (1.5), we get

$$\Xi(z) = \left[\frac{3\sqrt{3}\ (1-z)}{(3-z)\sqrt{3-4z}} \left(\frac{2}{1+\frac{(3-z)\sqrt{3-4z}}{3\sqrt{3}\ (1-z)}} \right)^{a-1} \right]. \tag{3.2}$$

On further simplification, we obtain the right hand side of equation (2.1).
Short proof of the result (2.2):
Consider the left hand side of equation (2.2) denoted by $\Omega(z)$ and the following identity:

$$\frac{(a)_r}{(a-1)_r} = 1 + \frac{r}{(a-1)}\ ; \quad (r \in \mathbb{N}_0),$$

we have

$$\Omega(z) := {}_3F_2\left[\begin{array}{c} a,\ \frac{3a-2}{2},\ \frac{3a-3}{2}; \\ a-1,\ 3a-2; \end{array} z \right] \tag{3.3}$$

$$= \sum_{r=0}^{\infty} \frac{(\frac{3a-2}{2})_r\ (\frac{3a-2}{2}-\frac{1}{2})_r\ (z)^r}{(3a-2)_r\ r!} + \frac{1}{(a-1)} \sum_{r=1}^{\infty} \frac{(\frac{3a-2}{2})_r\ (\frac{3a-2}{2}-\frac{1}{2})_r\ (z)^r}{(3a-2)_r\ (r-1)!}. \tag{3.4}$$

Now replacing r by $r+1$ in the second member of equation (3.4), we get

$$\Omega(z) = {}_2F_1\left[\begin{array}{c} \frac{3a-2}{2},\ \frac{3a-3}{2}; \\ 3a-2; \end{array} z \right] + \frac{(3a-3)\ z}{4\ (a-1)} \sum_{r=0}^{\infty} \frac{(\frac{3a-2}{2}+1)_r\ (\frac{3a-2}{2}+\frac{1}{2})_r\ (z)^r}{(3a-1)_r\ r!}$$

$$= {}_2F_1\left[\begin{array}{c} \frac{3a-2}{2},\ \frac{3a-3}{2}; \\ 3a-2; \end{array} z \right] + \frac{3z}{4}\ {}_2F_1\left[\begin{array}{c} \frac{3a-1}{2},\ \frac{3a}{2}; \\ 3a-1; \end{array} z \right]. \tag{3.5}$$

Now using the closed forms (1.3) and (1.4) in the equation (3.5), after some simplification, we get the right hand side of equation (2.2).

4 Concluding remarks

We conclude our present investigation by observing that the several other interesting reduction formulas in closed forms, can be derived in an analogous manner. Moreover, the results in closed forms, which we have derived in this paper are potentially useful in a wide range of problems in the mathematical, physical, statistical and engineering sciences.
Conflicts of Interest: The authors declare that there have no conflicts of interest.

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