On semi-generalized W_3 recurrent manifolds

K. Lalnunsiami [1] and Jay Prakash Singh [2]

[1] *Department of Mathematics and Computer Science*
Mizoram University, Aizawl
siamiofficial@gmail.com

[2] *Department of Mathematics and Computer Science*
Mizoram University, Aizawl
jpsmaths@gmail.com

Abstract

In this paper, we study semi-generalized W_3 recurrent manifolds. We obtain a necessary and sufficient condition for the scalar curvature to be constant in such a manifold. Ricci symmetric and decomposable semi-generalized W_3 recurrent manifolds are studied. Also, we obtain a sufficient condition for such a manifold to be quasi Einstein. Finally, we construct two examples of a semi-generalized W_3 recurrent manifold.

Subject Classification:[2010]Primary 53C25; Secondary 53B21, 53C21
Keywords: semi-generalized recurrent manifold, semi-generalized W_3 recurrent manifold, Ricci symmetric manifold, Einstein manifold.

1 Introduction

Symmetric spaces play a significant role in the study of differential geometry. Cartan [19] studied Riemanian symmetric spaces and obtain its classification. A Riemannian manifold is said to be a locally symmetric manifold [19] if $\nabla R = 0$, where ∇ is the operator of covariant differentiation with repect to the metric g. This notion has been diversified in various ways by several authors such as recurrent manifolds by Walker [18], projective symmetric manifolds by Soós [13], weakly symmetric manifolds by Tamassy and Binh [8], pseudosymmetric manifolds by Chaki [10] and so on.

The idea of recurrent manifolds has been generalized to Ricci recurrent manifolds by Patterson [15], 2-recurrent manifolds by Lichnerowicz [17], projective 2-recurrent manifolds by Ghosh [11] and others. A $(0, p)$ tensor field is said to be recurrent [18] if

$$\begin{aligned} &(\nabla_X T)(Y_1, Y_2, ..., Y_p)T(Z_1, Z_2, ..., Z_p) \\ - \ &T(Y_1, Y_2, ..., Y_p)(\nabla_X T)(Z_1, Z_2, ..., Z_p) = 0, \end{aligned} \tag{1.1}$$

holds on (M^n, g).

De and Guha [9] first studied generalized recurrent manifolds denoted by GK_n with non-zero associated 1-forms A and B. If $B = 0$, then the GK_n reduces to a recurrent manifold introduced by Ruse [16].

Prasad [6] initiated the notion of semi-generalized recurrent manifolds. A Riemannian manifold (M^n, g) is called a semi-generalized recurrent manifold if the curvature tensor R satisfies

$$(\nabla_X R)(Y,Z)W = \alpha(X)R(Y,Z)W + \beta(X)g(Z,W)Y, \tag{1.2}$$

where α and β are 1-forms ($\beta \neq 0$) and P, Q are vector fields given by

$$g(X,P) = \alpha(X), \quad g(X,Q) = \beta(X), \tag{1.3}$$

for arbitrary vector field X.

Let L denote the symmetric endomorphism of the tangent space at each point of M^n corresponding to the Ricci tensor S such that

$$S(X,Y) = g(LX,Y) \tag{1.4}$$

for every vector fields X and Y.

Chaki and Maity [5] introduced the notion of quasi Einstein manifold. A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called a quasi Einstein manifold if the Ricci tensor S is not identically zero and satisfies

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{1.5}$$

where a, b ($b \neq 0$) are scalars and η is a non-zero 1-form defined by

$$\eta(X) = g(X,\rho),$$

where ρ is a unit vector field.

In 1973, Pokhariyal [12] defined the W_3 curvature tensor as

$$\begin{aligned} W_3(Y,Z,U,V) &= R(Y,Z,U,V) + \frac{1}{(n-1)}\big[g(Z,U)S(Y,V) \\ &- g(Z,V)S(Y,U)\big]. \end{aligned} \tag{1.6}$$

Hui [3] studied weakly W_3 symmetric manifolds and its decomposability showing that both the decompositions in such a manifold are weakly Ricci symmetric. In 2018, Moindi et al. [1] studied symmetric and semi-symmetric properties of the W_3 curvature tensor in K-contact Riemannian manifolds.

In this paper, we consider a non-flat Riemannian manifold (M^n, g) $(n > 2)$ whose W_3 curvature tensor satisfies

$$(\nabla_X W_3)(Y,Z,U,V) = \alpha(X)W_3(Y,Z,U,V) + \beta(X)g(Z,U)g(Y,V), \tag{1.7}$$

where α, β are 1-forms, β is non-zero. Such a manifold is known as a semi-generalized W_3 recurrent manifold.

Singh and Khan [7] studied generalized recurrent and generalized conformally recurrent manifolds. In 2014, De and Pal [2] studied some geometric properties of generalized m-projectively recurrent manifolds. Generalized concircularly recurrent manifolds have been analysed by De and Gazi [4]. Motivated by these, we study semi-generalized W_3 recurrent manifolds.

This paper is divided into sections as follows: After the introduction, we obtain a necessary and sufficient condition for the scalar curvature to be constant in a semi-generalized W_3 recurrent manifold. In section 3, Ricci symmetric semi-generalized W_3 recurrent manifolds are studied. In the next section, we obtain a sufficient condition for such a manifold to be a quasi Einstein manifold. Section 5 is on the study of decomposable semi-generalized W_3 recurrent manifolds. Finally, two examples of a semi-generalized W_3 recurrent manifold are constructed.

2 Necessary and Sufficient Condition for the scalar curvature to be constant in a semi-generalized W_3 recurrent manifold

From equations (1.6) and (1.7), we have

$$\begin{aligned}(\nabla_X R)(Y,Z,U,V) &= \alpha(X)R(Y,Z,U,V) + \frac{1}{(n-1)}\Big[\alpha(X)\big\{g(Z,U)S(Y,V)\\ &- g(Z,V)S(Y,U)\big\} - \big\{g(Z,U)(\nabla_X S)(Y,V)\\ &- g(Z,V)(\nabla_X S)(Y,U)\big\}\Big] + \beta(X)g(Z,U)g(Y,V). \end{aligned} \tag{2.1}$$

Using Bianchi's second identity and equation (2.1), we have

$$\begin{aligned}&\alpha(X)R(Y,Z,U,V) + \alpha(Y)R(Z,X,U,V) + \alpha(Z)R(X,Y,U,V)\\ &+\beta(X)g(Z,U)g(Y,V) + \beta(Y)g(X,U)g(Z,V) + \beta(Z)g(Y,U)g(X,V)\\ &+\frac{1}{(n-1)}\Big[\alpha(X)\big\{g(Z,U)S(Y,V) - g(Z,V)S(Y,U)\big\}\\ &+\alpha(Y)\big\{g(X,U)S(Z,V) - g(X,V)S(Z,U)\big\}\\ &+\alpha(Z)\big\{g(Y,U)S(X,V) - g(Y,V)S(X,U)\big\}\\ &-\big\{g(Z,U)(\nabla_X S)(Y,V) - g(Z,V)(\nabla_X S)(Y,U)\big\}\\ &-\big\{g(X,U)(\nabla_Y S)(Z,V) - g(X,V)(\nabla_Y S)(Z,U)\big\}\\ &-\big\{g(Y,U)(\nabla_Z S)(X,V) - g(Y,V)(\nabla_Z S)(X,U)\big\}\Big] = 0.\end{aligned} \tag{2.2}$$

Putting $Y = V = e_i$ in equation (2.2) and summing over $i, 1 \le i \le n$, we get

$$\begin{aligned}&\alpha(X)S(Z,U) + \alpha(R(Z,X)U) - \alpha(Z)S(X,U)\\ &+n\beta(X)g(Z,U) + 2\beta(Z)g(X,U)\\ &+\frac{1}{(n-1)}\Big[\alpha(X)\big\{rg(Z,U) - 2S(Z,U)\big\} + \alpha(LZ)g(X,U)\\ &-(n-1)\alpha(Z)S(X,U) + 2(\nabla_Z S)(X,U)\\ &-\big\{g(Z,U)dr(X) + g(X,U)\frac{dr(Z)}{2}\big\}\Big] = 0.\end{aligned} \tag{2.3}$$

Contracting equation (2.3) with Z and U, we obtain

$$\left(\frac{2n-3}{n-1}\right)r\alpha(X) - \left(\frac{3n-4}{n-1}\right)\alpha(LX) + (n^2+2)\beta(X) - \frac{1}{(n-1)}dr(X) = 0, \tag{2.4}$$

which can be written as

$$r\alpha(X) = \left(\frac{3n-4}{2n-3}\right)\alpha(LX) - \frac{(n^2+2)(n-1)}{(2n-3)}\beta(X) + \frac{1}{(2n-3)}dr(X). \tag{2.5}$$

Thus, we can state the following:

Theorem 2.1. *In a semi-generalized W_3 recurrent manifold, the scalar curvature r is constant if and only if*

$$r\alpha(X) = \left(\frac{3n-4}{2n-3}\right)\alpha(LX) - \frac{(n^2+2)(n-1)}{(2n-3)}\beta(X),$$

for all vector fields X.

Suppose r is constant in a semi-generalized W_3 recurrent manifold, i. e., $dr = 0$. Then, equation (2.5) becomes

$$r\alpha(X) = \left(\frac{3n-4}{2n-3}\right)\alpha(LX) - \frac{(n^2+2)(n-1)}{(2n-3)}\beta(X). \tag{2.6}$$

Contraction of equation (2.1) yields

$$\begin{aligned}(\nabla_X S)(Z,U) &= \alpha(X)S(Z,U) + n\beta(X)g(Z,U) + \frac{1}{(n-1)}\Big[\alpha(X)\{rg(Z,U) \\ &- S(Z,U)\} - \{dr(X)g(Z,U) - (\nabla_X S)(Z,U)\}\Big].\end{aligned} \tag{2.7}$$

Making use of (2.6) and $dr = 0$ in (2.7), we obtain

$$\begin{aligned}(\nabla_X S)(Z,U) &= \alpha(X)S(Z,U) + \frac{(n-1)}{(n-2)(2n-3)}\Big[(n^2-3n-2)\beta(X) \\ &+ (3n-4)\alpha(LX)\Big]g(Z,U),\end{aligned}$$

which can be written as

$$(\nabla_X S)(Z,U) = \alpha(X)S(Z,U) + n\gamma(X)g(Z,U),$$

where

$$\gamma(X) = \frac{(n-1)}{n(n-2)(2n-3)}\Big[(n^2-3n-2)\beta(X) + (3n-4)\alpha(LX)\Big].$$

This leads to the theorem:

Theorem 2.2. *A semi-generalized W_3 recurrent manifold with constant scalar curvature is semi-generalized Ricci recurrent.*

3 Ricci symmetric semi-generalized W_3 recurrent manifold

Assume that the semi-generalized W_3 recurrent manifold is Ricci symmetric. Then, $\nabla S = 0$, i. e., $\nabla L = 0$. This implies that r is constant and $dr = 0$. Then, from equation (2.7), we have

$$\left(\frac{n-2}{n-1}\right)\alpha(X)S(Z,U) + \left\{\frac{r}{(n-1)}\alpha(X) + n\beta(X)\right\}g(Z,U) = 0. \tag{3.1}$$

Since r is constant, equation (2.6) holds. Substituting the value of $\beta(X)$ from equation (2.6) in (3.1), we have

$$S(Z,U) = \frac{n}{(n-2)(n^2+2)}\left[r(n^2-3n-2) - (3n-4)\frac{\alpha(LX)}{\alpha(X)}\right]g(Z,U),$$

which can be written as

$$S(Z,U) = \lambda g(Z,U),$$

where $\lambda = \frac{n}{(n-2)(n^2+2)}\left[r(n^2-3n-2) - (3n-4)\frac{\alpha(LX)}{\alpha(X)}\right]$. Thus, we have:

Theorem 3.1. *A Ricci symmetric semi-generalized W_3 recurrent manifold is an Einstein manifold.*

4 Sufficient condition for a semi-generalized W_3 recurrent manifold to be a quasi Einstein manifold

Equation (2.7) yields

$$\begin{aligned}(\nabla_X S)(Y,Z) &= \alpha(X)S(Y,Z) + \frac{n}{(n-1)}\Big[(n-1)\beta(X) \\ &+ \frac{r}{n}\alpha(X) - \frac{dr(X)}{n}\Big]g(Y,Z).\end{aligned} \tag{4.1}$$

A vector field P defined by $g(X,P) = \alpha(X)$ is said to be a concircular vector field if

$$(\nabla_X \alpha)(Y) = \nu g(X,Y) + \omega(X)\alpha(Y), \tag{4.2}$$

where ν is a non-zero scalar and ω is a closed 1-form. If P is unit, then equation (4.2) can be written as

$$(\nabla_X \alpha)(Y) = \nu[g(X,Y) - \alpha(X)\alpha(Y)]. \tag{4.3}$$

Suppose a semi-generalized W_3 recurrent manifold admits a unit concircular vector field P. Using Ricci identity in equation (4.3), we have

$$\alpha(R(X,Y)Z) = -\nu^2[g(X,Z)\alpha(Y) - g(Y,Z)\alpha(X)]. \tag{4.4}$$

Contraction of equation (4.4) with respect to Y and Z gives

$$\alpha(LX) = (n-1)\nu^2\alpha(X), \tag{4.5}$$

where L is the Ricci operator given by

$$S(X,Y) = g(LX,Y).$$

This implies

$$S(X,P) = (n-1)\nu^2\alpha(X). \tag{4.6}$$

We know that

$$(\nabla_X S)(Y,P) = \nabla_X S(Y,P) - S(\nabla_X Y,P) - S(Y,\nabla_X P). \tag{4.7}$$

Using equation (4.6) in (4.7), we have

$$(\nabla_X S)(Y,P) = (n-1)\nu^2\nabla_X\alpha(Y) - (n-1)\nu^2\alpha(\nabla_X Y) - S(Y,\nabla_X P),$$

or

$$(\nabla_X S)(Y,P) = (n-1)\nu^2(\nabla_X\alpha)(Y) - S(Y,\nabla_X P).$$

Applying equation (4.3) in the above equation, we get

$$(\nabla_X S)(Y,P) = (n-1)\nu^3[g(X,Y) - \alpha(X)\alpha(Y)] - S(Y,\nabla_X P). \tag{4.8}$$

Now,

$$\begin{aligned}(\nabla_X\alpha)(Y) &= \nabla_X\alpha(Y) - \alpha(\nabla_X Y) = \nabla_X g(Y,P) - g(\nabla_X Y,P)\\ &= g(Y,\nabla_X P), \quad \text{since} \quad (\nabla_X g)(Y,P) = 0.\end{aligned}$$

By virtue of equation (4.3), this implies

$$\nu[g(X,Y) - \alpha(X)\alpha(Y)] = g(Y,\nabla_X P),$$

$$\Rightarrow g(\nu X,Y) - g(\nu\alpha(X)P,Y) = g(\nabla_X P,Y),$$

$$\text{or,} \;\; \nabla_X P = \nu[X - \alpha(X)P].$$

Therefore,

$$S(Y,\nabla_X P) = S(Y,\nu X) - S(Y,\nu\alpha(X)P),$$

$$\text{which implies} \;\; S(Y,\nabla_X P) = \nu[S(X,Y) - \alpha(X)S(Y,P)]. \tag{4.9}$$

Making use of equation (4.9) in (4.8), we have

$$\begin{aligned}(\nabla_X S)(Y,P) &= (n-1)\nu^3[g(X,Y) - \alpha(X)\alpha(Y)]\\ &- \nu[S(X,Y) - \alpha(X)S(Y,P)].\end{aligned} \tag{4.10}$$

Applying equation (4.6) in (4.10), we obtain

$$(\nabla_X S)(Y,P) = (n-1)\nu^3 g(X,Y) - \nu S(X,Y). \tag{4.11}$$

From equation (4.1), we have

$$\begin{aligned}(\nabla_X S)(Y,P) &= \Big(\frac{n-2}{n}\Big)\alpha(X)S(Y,P) + (n-1)\Big[\beta(X) \\ &- \frac{r}{n(n-2)}\alpha(X) + \frac{dr(X)}{n(n-1)}\Big]g(Y,P).\end{aligned}$$

Using equations (4.8) and (4.11), the above equation becomes

$$\begin{aligned}(n-1)\nu^3 g(X,Y) - \nu S(X,Y) &= \frac{(n-2)}{n(n-1)}\nu^2\alpha(X)\alpha(Y) \\ +(n-1)\Big[\beta(X) &- \frac{r}{n(n-2)}\alpha(X) + \frac{dr(X)}{n(n-1)}\Big]\alpha(Y). \qquad (4.12)\end{aligned}$$

If the scalar curvature is constant, then $dr = 0$. From equation (2.6), we have

$$r\alpha(X) = \Big(\frac{3n-4}{2n-3}\Big)\alpha(LX) - \frac{(n^2+2)(n-1)}{(2n-3)}\beta(X),$$

which can be written as

$$\beta(X) = \frac{1}{(n^2+2)}\Big[(3n-4)\nu^2 - \Big(\frac{2n-3}{n-1}\Big)r\Big]\alpha(X). \tag{4.13}$$

Making use of equation (4.13) and $dr = 0$ in (4.12), we get

$$\begin{aligned}(n-1)\nu^3 g(X,Y) - \nu S(X,Y) &= \Big[(n-1)\nu^2 + \frac{1}{n(n-2)(n^2+2)}\big\{(n-1)(3n-4)\nu^2 \\ &- -r(n^2-3n-2)\big\}\Big]\alpha(X)\alpha(Y).\end{aligned}$$

$$\begin{aligned}\text{i. e., } (n-1)\nu^3 g(X,Y) - \nu S(X,Y) &= \frac{(n-1)}{(n-2)(n^2+2)}\Big[\big\{(n^3+n^2-2n-4)\nu^2 \\ &- r(n^2-3n-2)\big\}\Big]\alpha(X)\alpha(Y).\end{aligned}$$

Thus, we get

$$\begin{aligned}S(X,Y) &= (n-1)\nu^2 g(X,Y) + \frac{(n-1)}{(n-2)(n^2+2)}\Bigg[\Bigg\{\frac{r}{\nu}(n^2-3n-2) \\ &- (n^3+n^2-2n-4)\nu\Bigg\}\Bigg]\alpha(X)\alpha(Y),\end{aligned}$$

$$\text{or,}\quad S(X,Y) = ag(X,Y) + b\alpha(X)\alpha(Y),$$

where $a = (n-1)\nu^2$ and $b = \dfrac{(n-1)}{(n-2)(n^2+2)}\left[\left\{\dfrac{r}{\nu}(n^2-3n-2) - (n^3+n^2-2n-4)\nu\right\}\right]$ are two non-zero constants. Hence, the manifold is a quasi Einstein manifold. Thus, we have the theorem:

Theorem 4.1. *A semi-generalized W_3 recurrent manifold which admits a unit concircular vector field and whose associated scalar is a non-zero constant is a quasi Einstein manifold.*

5 Decomposable semi-generalized W_3 recurrent manifold

A Riemannian manifold $(M^n, g)(n > 2)$ is said to be a decomposable Riemannian manifold [14] if it can be expressed in the form $M^n = M_1^p X M_2^{n-p}$ for some p, $2 \leq p \leq (n-2)$, i. e., in some coordinate neighbourhood of M^n , the metric g can be written as

$$ds^2 = g_{ij}dx^i dx^j = \bar{g}_{ab}dx^a dx^b + g^*_{\alpha\beta}dx^\alpha dx^\beta, \tag{5.1}$$

where $\bar{g}_{ab}$ are functions of $x^1, x^2, ..., x^p$ denoted by $\bar{x}$, $g^*_{\alpha\beta}$ are functions of $x^{p+1}, x^{p+2}, ..., x^n$ denoted by x^*, $a, b, c......$ runs from 1 to p and $\alpha, \beta, \gamma, ...$ runs from $p+1$ to n. M_1^p and M_2^{n-p} are called the components of M^n.

Suppose a semi-generalized W_3 recurrent manifold $(M^n, g)(n > 2)$ is decomposable. Then, $M^n = M_1^p X M_2^{n-p}$ for some p, $2 \leq p \leq (n-2)$. Let $\bar{X}, \bar{Y}, \bar{Z}, \bar{U}, \bar{V} \in \chi(M_1)$, $X^*, Y^*, Z^*, U^*, V^* \in \chi(M_2)$. Since M^n is decomposable, we have

$$S(\bar{X}, \bar{Y}) = \bar{S}(\bar{X}, \bar{Y}),$$

$$S(X^*, Y^*) = S^*(X^*, Y^*),$$

$$(\nabla_{\bar{X}} S)(\bar{Y}, \bar{Z}) = (\bar{\nabla}_{\bar{X}} S)(\bar{Y}, \bar{Z}),$$

$$(\nabla_{X^*} S)(Y^*, Z^*) = (\nabla^*_{X^*} S)(Y^*, Z^*)$$

and $r = \bar{r} + r^*$.

From equation (1.6), we have

$$W_3(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}) = \bar{W}_3(\bar{X}, \bar{Y}, \bar{Z}, \bar{U}), \tag{5.2}$$

$$W_3(X^*, Y^*, Z^*, U^*) = W_3^*(X^*, Y^*, Z^*, U^*),$$

$$W_3(Y^*, \bar{Z}, \bar{U}, \bar{V}) = 0 = W_3(\bar{Y}, Z^*, U^*, V^*) = W_3(\bar{Y}, Z^*, \bar{U}, \bar{V}) = W_3(\bar{Y}, \bar{Z}, U^*, \bar{V}),$$

$$W_3(\bar{Y}, Z^*, U^*, \bar{V}) = \frac{1}{(n-1)} g(Z^*, U^*) S(\bar{Y}, \bar{V}), \tag{5.3}$$

$$W_3(Y^*, \bar{Z}, \bar{U}, V^*) = \frac{1}{(n-1)} g(\bar{Z}, \bar{U}) S(Y^*, V^*), \tag{5.4}$$

$$W_3(Y^*, \bar{Z}, U^*, \bar{V}) = -\frac{1}{(n-1)} g(\bar{Z}, \bar{V}) S(Y^*, U^*),$$

$$W_3(\bar{Y}, Z^*, \bar{U}, V^*) = -\frac{1}{(n-1)} g(Z^*, V^*) S(\bar{Y}, \bar{U}),$$

$$(\nabla_{X^*} W_3)(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = 0 = (\nabla_{\bar{X}} W_3)(Y^*, Z^*, U^*, V^*).$$

From equation (1.7), we get

$$(\nabla_{\bar{X}} W_3)(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = \alpha(\bar{X}) W_3(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) + \beta(\bar{X}) g(\bar{Z}, \bar{U}) g(\bar{Y}, \bar{V}),$$

$$\text{and} \quad \alpha(X^*) W_3(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) + \beta(X^*) g(\bar{Z}, \bar{U}) g(\bar{Y}, \bar{V}) = 0. \tag{5.5}$$

Also,

$$\beta_{(\bar{p}, p^*)}(0 \oplus v) = 0,$$

where $\bar{p} \in M_1, p^* \in M_2$ and $v \in T_{p^*}(M_2)$. Also, for every $(\bar{p}, p^*) \in M^n$, we have from equation (1.7),

$$(\nabla_{X^*} W_3)_{(\bar{p}, p^*)}(Y^*, Z^*, U^*, V^*) = (\nabla^*_{X^*} W_3)_{p^*}(Y^*, Z^*, U^*, V^*) \tag{5.6}$$

and the R. H. S does not depend on $\bar{p} \in M_1$.

Suppose $\beta(X^*) = 0$, $\forall X^* \in \chi(M_2)$, then equation (5.5) yields

$$\alpha(X^*) W_3(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = 0, \tag{5.7}$$

and

$$\alpha(X^*) \bar{W}_3(\bar{Y}, \bar{Z}, \bar{U}, \bar{V}) = 0. \tag{5.8}$$

If M_1 is not W_3 flat, i. e., $(\bar{W}_3)_{\bar{p}_0} \neq 0$, for some $\bar{p}_0 \in M_1$, then equations (5.7) and (5.8) gives

$$\alpha_{(\bar{p}, p^*)}(0 \oplus v) = 0. \tag{5.9}$$

Then, equation (1.7) yields

$$(\nabla_{X^*} W_3)_{(\bar{p}, p^*)}(Y^*, Z^*, U^*, V^*) = 0,$$

for every $\bar{p} \in M_1, p^* \in M_2$ and $v \in T_{p^*}(M_2)$. It follows that if M_1 is not W_3 flat, then

$$\alpha_{(\bar{p}, p^*)}(W_3^*)_{p^*}(Y^*, Z^*, U^*, V^*) = 0, \tag{5.10}$$

for all $\bar{p} \in M_1, p^* \in M_2$.

Assume that

$$(\nabla_X W_3)(Y, Z, U, V) = \bar{\alpha}(X) W_3(Y, Z, U, V) + \bar{\beta}(X) g(Z, U) g(Y, V), \tag{5.11}$$

where $\bar{\alpha}$ and $\bar{\beta}$ are 1-forms.

Using (5.11) in equation (1.7), we obtain

$$[\alpha(X) - \bar{\alpha}(X)]W_3(Y, Z, U, V) + [\beta(X) - \bar{\beta}(X)]g(Z, U)g(Y, V) = 0. \tag{5.12}$$

Contraction of equation (5.12) over Y and V gives

$$\begin{aligned}[\alpha(X) - \bar{\alpha}(X)]\Big[S(Z, U) \quad &- \quad \frac{1}{(n-1)}\{rg(Z, U) - S(Z, U)\}\Big] \\ &+ \quad [\beta(X) - \bar{\beta}(X)]g(Z, U) = 0.\end{aligned} \tag{5.13}$$

Again contracting equation (5.13) over Z and U, we have

$$\beta(X) = \bar{\beta}(X),$$

which implies, from equation (5.12)

$$\alpha(X) = \bar{\alpha}(X),$$

for all $X \in M^n$ provided $W_3 \neq 0$, i. e., the manifold is not W_3 flat. Thus, the 1-forms α and β are uniquely determined provided that the manifold is not W_3 flat. So, from equation (5.9), we have

$$\alpha_{(\bar{p}, p^*)}(X^*) = 0, \tag{5.14}$$

for all $\bar{p} \in M_1, p^* \in M_2$.

Hence, from equation (5.7), we can conclude that either

1. $\alpha(X^*) = 0$, or
2. M_1 is W_3 flat.

Also, from equation (1.7), we get

$$\begin{aligned}(\nabla_{X^*} W_3)(Y^*, \bar{Z}, \bar{U}, V^*) \quad &= \quad \alpha(X^*)W_3(Y^*, \bar{Z}, \bar{U}, V^*) \\ &+ \quad \beta(X^*)g(\bar{Z}, \bar{U})g(Y^*, V^*).\end{aligned} \tag{5.15}$$

Consider case (1). From equation (5.15), we have

$$(\nabla_{X^*} W_3)(Y^*, \bar{Z}, \bar{U}, V^*) = 0,$$

which by virtue of (5.3) gives

$$(\nabla_{X^*} S)(Y^*, V^*) = 0, \tag{5.16}$$

i. e., the component M_2 is Ricci symmetric. Using equations (5.4), (5.6), (5.9), (5.10) and (5.14), and $\alpha(X^*) = 0$, $\beta(X^*) = 0$, for all $X^* \in M_2$, we have from (1.7),

$$(\nabla_{X^*} W_3)(Y^*, Z^*, U^*, V^*) = 0,$$

and hence

$$\begin{aligned}(\nabla_{X^*}R)(Y^*,Z^*,U^*,V^*) &+ \frac{1}{(n-1)}\big[g(Z^*,U^*)(\nabla_{X^*}S)(Y^*,V^*)\\ &- g(Z^*,V^*)(\nabla_{X^*}S)(Y^*,U^*)\big]=0,\end{aligned}$$

which by virtue of equation (5.16) yields

$$(\nabla_{X^*}R)(Y^*,Z^*,U^*,V^*)=0.$$

Hence, M_2 is locally symmetric. Similarly, we can prove for M_1. Thus, we can state the theorem:

Theorem 5.1. *Let M^n be a decomposable semi-generalized W_3 recurrent manifold which is not W_3 flat such that $M^n = M_1^p X M_2^{n-p}$, $2 \le p \le (n-2)$. If $\beta(X^*) = 0$ for all $X^* \in M_2$,(respectively $\beta(\bar{X}) = 0$ for all $\bar{X} \in M_1$), then either (1) or (2) holds.*

1. *$\alpha(X^*) = 0$, $\forall X^* \in \chi(M_2)$,(respectively $\alpha(\bar{X}) = 0$, $\forall \bar{X} \in \chi(M_1)$), and hence M_2(respectively M_1) is Ricci symmetric as well as locally symmetric.*
2. *M_2(respectively M_1) is W_3 flat.*

Also, from equation (1.7), we have

$$\begin{aligned}(\nabla_{\bar{X}}W_3)(\bar{Y},Z^*,U^*,\bar{V}) &= \alpha(\bar{X})W_3(\bar{Y},Z^*,U^*,\bar{V})\\ &+ \beta(\bar{X})g(Z^*,U^*)g(\bar{Y},\bar{V}).\end{aligned}\tag{5.17}$$

Using equation (5.3) in (5.17), we get

$$\begin{aligned}\frac{1}{(n-1)}g(Z^*,U^*)(\nabla_{\bar{X}}S)(\bar{Y},\bar{V}) &= \frac{\alpha(\bar{X})}{n-1}g(Z^*,U^*)S(\bar{Y},\bar{V})\\ &+ \beta(\bar{X})g(Z^*,U^*)g(\bar{Y},\bar{V}).\end{aligned}\tag{5.18}$$

Assume $g(Z^*,U^*) \neq 0$, then equation (5.18) becomes

$$(\nabla_{\bar{X}}S)(\bar{Y},\bar{V}) = \alpha(\bar{X})S(\bar{Y},\bar{V}) + (n-1)\beta(\bar{X})g(\bar{Y},\bar{V}),$$

$$\Rightarrow (\nabla_{\bar{X}}S)(\bar{Y},\bar{V}) = A(\bar{X})S(\bar{Y},\bar{V}) + nB(\bar{X})g(\bar{Y},\bar{V}),$$

where $A(\bar{X}) = \alpha(\bar{X})$ and $B(\bar{X}) = \frac{(n-1)}{n}\beta(\bar{X})$ are two non-zero 1-forms. This leads to the theorem:

Theorem 5.2. *Let M^n be a decomposable semi-generalized W_3 recurrent manifold which is not W_3 flat such that $M^n = M_1^p X M_2^{n-p}$, $2 \le p \le (n-2)$. Then M_1(respectively M_2) is semi-generalized Ricci recurrent.*

6 Example of a semi-generalized W_3 recurrent manifold

Example 1: Consider $\mathbb{R}^4$ with the Riemannian metric defined by

$$ds^2 = g_{ij}dx^i dx^j = (1-4q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2], \tag{6.1}$$

where $q = \frac{e^{x^1}}{k^2}$, for a non-zero constant k and $x^1 \neq 0$. The non-vanishing components of the Christoffel's symbols, the Riemannian curvature tensors and the Ricci tensors are

$$\Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = \frac{2q}{1-4q},$$

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = -\frac{2q}{1-4q},$$

$$R_{1221} = R_{1331} = R_{1441} = -\frac{2q}{1-4q},$$

$$S_{11} = \frac{6q}{(1-4q)^2}, \quad S_{22} = S_{33} = \frac{2q}{(1-4q)^2}$$

and the components which can be obtained by symmetry properties. Using

$$r = g^{ij}S_{ij}, \tag{6.2}$$

we get $r = \dfrac{12q}{(1-4q)^3}$, which is non-zero. By virtue of equation (1.6), we get the non-zero components of the W_3 curvature tensor as

$$(W_3)_{1221} = -\frac{4q}{1-4q}, \quad (W_3)_{1331} = (W_3)_{1441} = -\frac{8q}{3(1-4q)},$$

whose non-zero covariant derivatives are

$$(W_3)_{1221,1} = -\frac{4q}{(1-4q)^2}, \quad (W_3)_{1331,1} = (W_3)_{1441,1} = -\frac{8q}{3(1-4q)^2}$$

and their symmetric components. Here "," denotes the operator of covariant differentiation with respect to the metric g. To show that $(\mathbb{R}^4, g)$ is a semi-generalized W_3 recurrent manifold, we choose the 1-forms α and β as

$$\alpha_i = \begin{cases} \dfrac{1}{1-4q}, & i = 1, \\ 0, & otherwise, \end{cases}$$

$$\beta_i = 0, \text{for } i = 1, 2, 3, 4.$$

Then, equation equation (1.7) reduces to

$$(W_3)_{1221,1} = \alpha_1 (W_3)_{1221}, \tag{6.3}$$

$$(W_3)_{1331,1} = \alpha_1 (W_3)_{1331}, \tag{6.4}$$

$$(W_3)_{1441,1} = \alpha_1 (W_3)_{1441} \tag{6.5}$$

and the other cases hold trivially.

$$\begin{aligned} \text{R. H. S of (6.3)} &= \alpha_1 (W_3)_{1221} \\ &= \Big(\frac{1}{1-4q}\Big).\Big(-\frac{4q}{1-4q}\Big) \\ &= -\frac{4q}{(1-4q)^2} = (W_3)_{1221,1} \\ &= \text{L. H. S of (6.3).} \end{aligned}$$

Similarly, equations (6.4) and (6.5) can be proved. Therefore, $(\mathbb{R}^4, g)$ is a semi-generalized W_3 recurrent manifold.
Example 2: Define a Riemannian metric g on $\mathbb{R}^4$ by

$$ds^2 = g_{ij} dx^i dx^j = (x^1)^{\frac{1}{3}} [(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2, \text{ where } x^1 \neq 0. \tag{6.6}$$

We obtain the non-vanishing components of the Christoffel's symbols, the curvature tensors and the Ricci tensors as

$$\Gamma^1_{22} = \Gamma^1_{33} = \frac{1}{6x^1},$$

$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = -\frac{1}{6x^1},$$

$$R_{1221} = -\frac{5}{36(x^1)^{\frac{5}{3}}} = R_{1331}, \quad R_{2332} = -\frac{1}{36(x^1)^{\frac{5}{3}}},$$

$$S_{11} = -\frac{5}{18(x^1)^2}, \quad S_{22} = S_{33} = \frac{1}{9(x^1)^2}$$

and their symmetric components.

Using (6.2), we get $r = \dfrac{1}{2(x^1)^{\frac{7}{3}}}$, which is non-zero and non constant. From equation (1.6), we obtain

$$(W_3)_{1221} = -\frac{5}{108(x^1)^{\frac{5}{3}}} = (W_3)_{1331}, \; (W_3)_{2332} = -\frac{1}{36(x^1)^{\frac{5}{3}}},$$

and the components obtained by symmetric properties. Using these, we get the covariant derivatives of the W_3 curvature tensors as

$$(W_3)_{1221,1} = \frac{25}{324(x^1)^{\frac{8}{3}}} = (W_3)_{1331,1}, \quad (W_3)_{2332,1} = \frac{5}{108(x^1)^{\frac{8}{3}}}.$$

To show that the manifold under consideration is semi-generalized W_3 recurrent, we choose the 1-forms α and β as

$$\alpha_i = \begin{cases} -\frac{5}{3(x^1)}, & i = 1, \\ 0, & otherwise, \end{cases}$$

$$\beta_i = 0, \text{for } i = 1, 2, 3, 4.$$

From equation (1.6), we have

$$(W_3)_{1221,1} = \alpha_1 (W_3)_{1221}, \tag{6.7}$$

$$(W_3)_{1331,1} = \alpha_1 (W_3)_{1331}, \tag{6.8}$$

$$(W_3)_{2332,1} = \alpha_1 (W_3)_{2332}, \tag{6.9}$$

and all other cases hold trivially. Now,

$$\begin{aligned} \text{R. H. S of (6.7)} &= \alpha_1 (W_3)_{1221} \\ &= \Big(-\frac{5}{3(x^1)}\Big).\Big(-\frac{5}{108(x^1)^{\frac{5}{3}}}\Big) \\ &= \frac{25}{324(x^1)^{\frac{8}{3}}} = (W_3)_{1221,1} \\ &= \text{L. H. S of (6.7)} \end{aligned}$$

and equations (6.8) and (6.9) can be proved in a similar manner. Therefore, $\mathbb{R}^4$ with the given metric is a semi-generalized W_3 recurrent manifold.

References

[1] Moindi, S. K., Njui, F. and Pokhariyal, G. P. (2018): A study of W_3-symmetric K-contact Riemannian manifold, Int. J. Innov. Sci. Math., 6(3), 102-104.

[2] De, U. C. and Pal, P. (2014): On generalized m-projectively recurrent manifolds, Ann. Univ. Paedagog. Crac. Stud. Math., 13, 77-101.

[3] Hui, S. K. (2011): On weakly W_3 symmetric manifolds, Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica, 50(1), 53-71.

[4] De, U. C. and Gazi, A. K. (2009): On generalized concircularly recurrent manifolds, Studia Sci. Math. Hungar., 46(2), 287-296.

[5] Chaki, M. C. and Maity, R. K. (2000): On quasi Einstein manifolds, Publ. Math. Debrecen, 57, 297-306.

[6] Prasad, B. (2000): On semi-generalized recurrent manifold, Math. Balkanica, N. S, 14, 77-82.

[7] Singh, H. and Khan, Q. (2000): On generalized recurrent Riemannian manifolds, Publ. Math. Debrecen, 56, 87–95.

[8] Tamassy, L. and Binh, T. Q. (1993): On weak symmetrics of Einstein and Sasakian manifolds, Tensor, N. S., 53, 140-158.

[9] De, U. C. and Guha, N. (1991): On generalized recurrent manifold, J. National Academy of Math. India, 9, 85-92.

[10] Chaki, M. C. (1987): On pseudo symmetric manifolds, An. Ştiinţ. Univ. Al. I. Cuza Iaşi, 33(1), 53-58.

[11] Ghosh, D. (1970): On projective recurrent spaces of second order, Acad. Roy. Belg. Bull. Cl. Sci., 56(5), 1093-1099.

[12] Pokhariyal, G. P. (1973): Curvature tensors and their relativistic significance III, Yokohama Math. J., 21, 115-119.

[13] Soós, G. (1958): Uber die geodatischen Abbildungen von Riemannschen Raumen auf projektiv symmetrische Riemannsche Raume, Acta Math. Acad. Sci. Hungar., 9, 359-361.

[14] Schouten, J. A. (1954): Ricci-calculus, an introduction to tensor analysis and its geometrical applications, 2d. ed. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Bercksichtigung der Anwendungsgebiete, Bd X, Springer-Verlag, Berlin-Gttingen-Heidelberg.

[15] Patterson, E. M. (1952): Some theorems on Ricci-recurrent spaces, J. London Math. Soc., 27, 287-295.

[16] Ruse, H. S. (1951): A classification of K^*-spaces, Proc. London Math. Soc., 53(2), 212-229.

[17] Lichnerowicz, A. (1950): Courbure, nombres de Betti, et espaces symtriques (French), Proc. Int. Cong. Math., Cambridge, Mass., 2, 216-223.

[18] Walker, A. G. (1950): On Ruses spaces of recurrent curvature, Proc. London Math. Soc., 52(2), 36-64.

[19] Cartan, E. (1926): Sur une classe remarquable despaces de Riemann, Bull. Soc. Math. France, 54, 214-264.