Subclass of Analytic Functions Associated with Gegenbaur Polynomials

Pasunoori Srinivasulu [1], Rajkumar N. Ingle [2] and P.Thirupathi Reddy [3]

[1] *Department of Mathematics*
Rajeev Gandhi University of KnowledgeTechnologies, IIIT,Basar,Nirmal-504107. T.S
Srinivaspasupuleti12@gmail.com1

[2] *Department of Mathematics*
Bahirji Smarak Mahavidyalay, Bashmathnagar - 431 512, Hingoli Dist., Maharashtra, India.
ingleraju11@gmail.com

[3] *Department of Mathematics*
Department of Mathematics, Kakatiya University, Warangal -506009
reddypt2@gmail.com

Abstract

In this work, we introduce and study a new subclass of analytic functions defined by a geganbaur polynomial and obtained coefficient estimates, growth and distortion theorems, radii of starlikeness, convexity and close-to-convexity are obtained. Furthermore, we obtained integral means inequalities for the function .

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1 Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $E = \{z \in C : |z| < 1\}$.

A function f in the class A is said to be in the class $ST(\alpha)$ of starlike functions of orderα in E , if it satisfy the inequality

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \tag{1.2}$$

$0 \leq \alpha < 1$, and $z \in E$.

Note that $ST(0) = ST$ is the class of starlike functions.
Denote by T the subclass of A consisting of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0). \tag{1.3}$$

This subclass was introduced and extensively studied by Silverman [6].

The class $\mathcal{T}(\lambda), \lambda \geq 0$ were introduced and investigated by Szynal [10] as the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = \int_{-1}^{1} k(z,m) d\mu(m), \tag{1.4}$$

where

$$k(z,m) = \frac{z}{(1 - 2mz + z^2)^{\lambda}}, \quad (z \in U), m \in [-1,1] \tag{1.5}$$

and μ is a probability measure on the interval $[-1,1]$. The collection of such measures on $[a,b]$ is denoted by $P_{[a,b]}$.
The Taylor series expansion of the function in (1.5) gives

$$k(z,m) = z + c_1^{\lambda}(m) z^2 + c_2^{\lambda}(m) z^3 + \cdots \tag{1.6}$$

and the coefficients for (1.6) were given below:

$$\begin{aligned} &c_0^{\lambda}(m) = 1;\; c_1^{\lambda}(m) = 2\lambda m; c_2^{\lambda}(m) = 2\lambda(\lambda+1)m^2 - \lambda; \\ &c_3^{\lambda}(m) = \frac{4}{3}\lambda(\lambda+1)(\lambda+2)m^3 - 2\lambda(\lambda+1)m \quad \cdots \end{aligned} \tag{1.7}$$

where $c_{\eta}^{\lambda}(m)$ denotes the Gegenbauer polynomial of degree η. Varying the parameter λ in (1.6), we obtain the class of typically real functions studied by [1][4],[5],[9], [11] and [12].
For $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, the Hadamard product of f and g

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$$

.
Let $G_{\lambda}^{m} : A \to A$ defined in terms of the convolution by

$$G_{\lambda}^{m} f(z) = k(z,m) * f(z)$$

We have

$$G_\lambda^m f(z) = z + \sum_{n=2}^{\infty} \phi_n(\lambda, m) a_n z^n \tag{1.8}$$

where $\phi_n(\lambda, m) = c_{n-1}^{\lambda}(m)$.

In this paper , using the operator $G_\lambda^m f(z)$, we define the following new class motivated by Murugusunderamoorthy and Magesh [3].

Definition 1. The function $f(z)$ of the form (1.1) is in the class $S_\lambda^m(\gamma, \varsigma)$ if it satisfies the inequality

$$\mathrm{Re}\left\{\frac{z(G_\lambda^m f(z))'}{G_\lambda^m f(z)} - \gamma\right\} > \varsigma \left|\frac{z(G_\lambda^m f(z))'}{G_\lambda^m f(z)} - 1\right|$$

for $0 \le \gamma \le 1$ and $\varsigma \ge 0$.

Further we define $TS_\lambda^m(\gamma, \varsigma) = S_\lambda^m(\gamma, \varsigma) \cap T$.

The aim of this paper is to study the coefficient bounds , radii of close-to-convex and starlikeness convex linear combinations for the class $TS_\lambda^m(\gamma, \varsigma)$ .Furthermore, we obtained integral means inequalities for the functions in this class.

2 Coefficient estimate

Theorem 1: A function $f(z)$ of the form (1.1) is in $S_\lambda^m(\gamma, \varsigma)$

$$\sum_{n=2}^{\infty} [n(1+\varsigma) - (\gamma+\varsigma)] \phi_n(\lambda, m) |a_n| \le 1 - \gamma. \tag{2.1}$$

where $0 \le \gamma < 1$, $\varsigma \ge 0$ and $\phi_n(\lambda, m)$ is givn by (1.8).

Proof. It suffices to show that

$$\varsigma \left|\frac{z(G_\lambda^m f(z))'}{G_\lambda^m f(z)} - 1\right| - \mathrm{Re}\left\{\frac{z(G_\lambda^m f(z))'}{G_\lambda^m f(z)} - 1\right\} \le 1 - \gamma$$

We have

$$\begin{aligned}
&\varsigma \left|\frac{z\left(G_\lambda^m f(z)\right)'}{G_\lambda^m f(z)} - 1\right| - \mathrm{Re}\left\{\frac{z\left(G_\lambda^m f(z)\right)'}{G_\lambda^m f(z)} - 1\right\} \\
&\le (1+\varsigma)\left|\frac{z\left(G_\lambda^m f(z)\right)'}{G_\lambda^m f(z)} - 1\right| \\
&\le (1+\varsigma)\frac{\sum_{n=2}^{\infty} n\phi_n(\lambda,m)|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty} \phi_n(\lambda,m)|a_n||z|^{n-1}} \\
&\le (1+\varsigma)\frac{\sum_{n=2}^{\infty} (n-\mu)\phi_n(\lambda,m)|a_n|}{1 - \sum_{n=2}^{\infty} \mu\phi_n(\lambda,m)|a_n|}
\end{aligned}$$

The last expression is bounded $(1-\gamma)$ if

$$\sum_{n=2}^{\infty}[n(1+\varsigma)-(\gamma+\varsigma)]\phi_n(\lambda,m)\,|a_n|\leq 1-\gamma.$$

Theorem 2: Let $0\leq\gamma<1$and $\varsigma\geq 0$ then a function f of the form (1.3) to be in the class $TS_{\lambda}^{m}(\gamma,\varsigma)$ if and only if

$$\sum_{n=2}^{\infty}[n(1+\varsigma)-(\gamma+\varsigma)]\phi_n\,(\lambda,m)\leq 1-\gamma \tag{2.2}$$

where $\phi_n\,(\lambda,m)$ is given by (1.8).

Proof: In view of Theorem 1, we need only to prove the necessity. If $f\in TS_{\lambda}^{m}(\gamma,\varsigma)$ and z is real then

$$\Re\left\{\frac{1-\sum\limits_{n=2}^{\infty}n\phi_n\,(\lambda,m)\,a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}\phi_n\,(\lambda,m)\,a_n z^{n-1}}-\gamma\right\}>\varsigma\left|\frac{\sum\limits_{n=2}^{\infty}(n-1)\phi_n\,(\lambda,m)\,a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}\phi_n\,(\lambda,m)\,a_n z^{n-1}}\right|.$$

Letting $z\to 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}[n(1+\varsigma)-(\gamma+\varsigma)]\phi_n\,(\lambda,m)\,|a_n|\leq 1-\gamma,$$

where $0\leq\gamma<1$, $\varsigma\geq 0$ and $\phi_n(\lambda,m)$ is given by (1.8).

Corollary 1. If *f(z)* $\in TS_{\lambda}^{m}(\gamma,\varsigma)$, then

$$|a_n|\leq\frac{1-\gamma}{[n(1+\varsigma)-(\gamma+\varsigma)]\phi_n\,(\lambda,m)} \tag{2.3}$$

where $0\leq\gamma<1$, $\varsigma\geq 0$ and $\phi_n(\lambda,m)$ are given by (1.8). Equality holds for the function

$$f(z)=z-\frac{1-\gamma}{[n(1+\varsigma)-(\gamma+\varsigma)]\phi_n\,(\lambda,m)}z^n. \tag{2.4}$$

3 Extreme points

Theorem 3. Let $f_1(z)=z$ and

$$f_n(z)=z-\frac{1-\gamma}{[n(1+\varsigma)-(\gamma+\varsigma)]\phi_n\,(\lambda,m)}z^n, n\geq 2. \tag{3.1}$$

Then *f(z)* $\in TS_{\lambda}^{m}(\gamma,\varsigma)$, if and only if it can be expressed in the form

$$f(z)=\sum_{n=1}^{\infty}w_n f_n(z), w_n\geq 0, \sum_{n=1}^{\infty}w_n=1. \tag{3.2}$$

Proof. Suppose *f(z)* can be written as in (3.2). Then

$$f(z) = z - \sum_{n=2}^{\infty} w_n \frac{1-\gamma}{[n(1+\varsigma)-(\gamma+\varsigma)]\phi_n(\lambda,m)} z^n.$$

Now,

$$\sum_{n=2}^{\infty} w_n \frac{(1-\gamma)[n(1+\varsigma)-(\gamma+1)]\phi_n(\lambda,\upsilon,\tau)}{(1-\gamma)[n(1+\varsigma)-(\gamma+1)]\phi_n(\lambda,m)} = \sum_{n=2}^{\infty} w_n = 1 - w_1 \leq 1.$$

Thus $f(z) \in TS_{\lambda}^{m}(\gamma,\varsigma)$.

Conversely , let us have *f(z)* $\in TS_{\lambda}^{m}(\gamma,\varsigma)$. Then by using (2.3), we get

$$w_n = \frac{[n(1+\varsigma)-(\gamma+1)]\phi_n(\lambda,m)}{(1-\gamma)} a_n, n \geq 2$$

and $w_1 = 1 - \sum_{n=2}^{\infty} w_n$. Then we have $f(z) = \sum_{n=1}^{\infty} w_n f_n(z)$ and hence this completes the proof of Theorem.

Theorem 4. The class $TS_{\lambda}^{m}(\gamma,\varsigma)$ is a convex set.

Proof. Let $f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, a_{n,j} \geq 0$, j =1,2 be in the class $TS_{\lambda}^{m}(\gamma,\varsigma)$. It sufficient to show that the function $h(z)$ defined by

$$h(z) = \xi f_1(z) + (1-\xi) f_2(z), 0 \leq \xi < 1$$

is in the class $TS_{\lambda}^{m}(\gamma,\varsigma)$. Since

$$h(z) = z - \sum_{n=2}^{\infty} \left[\xi a_{n,1} + (1-\xi) a_{n,2}\right] z^n.$$

An easy compution with the aid of of Theorem 2, gives

$$\sum_{n=2}^{\infty}[n(1+\varsigma)-(\gamma+\varsigma)]\, \xi\phi_n(\lambda,m)\, a_{n,1} \quad + \sum_{n=2}^{\infty}[n(1+\varsigma)-(\gamma+\varsigma)]\, (1-\xi)\phi_n(\lambda,m)\, a_{n,2}$$

$$\leq \xi(1-\gamma) + (1-\xi)(1-\gamma)$$

$$\leq (1-\gamma),$$

which implies that $h \in TS_{\lambda}^{m}(\gamma,\varsigma)$.

Hence $TS_{\lambda}^{m}(\gamma,\varsigma)$ is convex.

Next we obtain the radii of close-to-convexity and starlikeness for the class$TS_{\lambda}^{m}(\gamma,\varsigma)$.

4 radii of close-to-convexity and starlikeness

Theorem 5. Let the function $f(z)$defined by (1.3) belong to the class $TS_{\lambda}^{m}(\gamma,\varsigma)$. Then $f(z)$ is close-to-convex of order δ $(0 \leq \delta < 1)$ in the disc $|z| < r_1$, where

$$r_1 = \inf_{n\geq 2}\left[\frac{(1-\delta)\sum_{n=2}^{\infty}[n(1+\varsigma)-(\gamma+\varsigma)]\phi_n\left(\lambda, \upsilon, \tau\right)}{n(1-\gamma)}\right]^{1/n-1}. \tag{4.1}$$

The estimate is sharp, with the extremal function $f(z)$ is given by (3.1).
Proof. Given $f \in T$, and f is close-to-convex of order δ, we have

$$\left|f'(z)-1\right| < 1-\delta. \tag{4.2}$$

For the left hand side of (4.2) we have

$$\left|f'(z)-1\right| \leq \sum_{n=2}^{\infty} na_n \left|z\right|^{n-1}.$$

The last expression is less than $1-\delta$

$$\sum_{n=2}^{\infty} \frac{n}{1-\delta} a_n \left|z\right|^{n-1} \leq 1.$$

Using the fact, that *f(z)* $\in TS_{\lambda}^{m}(\gamma,\varsigma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[n(1+\varsigma)-(\gamma+\varsigma)]\phi_n\left(\lambda, m\right)}{(1-\gamma)} a_n \leq 1.$$

We can (4.2) is true if

$$\frac{n}{1-\delta}\left|z\right|^{n-1} \leq \frac{[n(1+\varsigma)-(\gamma+\varsigma)]\phi_n\left(\lambda m\right)}{(1-\gamma)}$$

or, equivalently,

$$|z| \leq \left\{\frac{(1-\delta)[n(1+\varsigma)-(\gamma+\varsigma)]\phi_n\left(\lambda, m\right)}{n(1-\gamma)}\right\}^{1/n-1}$$

which completes the proof.
Theorem 6. Let the function $f(z)$ defined by (5) belong to the class $TS_{\lambda}^{m}(,\gamma,\varsigma)$.Then $f(z)$ is starlike of order of order δ $(0 \leq \delta < 1)$ in the disc $|z| < r_2$, where

$$r_2 = \inf_{n\geq 2}\left[\frac{(1-\delta)\sum_{n=2}^{\infty}[n(1+\varsigma)-(\gamma+\varsigma)]\phi_n\left(\lambda, m\right)}{(n-\delta)(1-\gamma)}\right]^{1/n-1}. \tag{4.3}$$

The estimate is sharp,with extremal function $f(z)$ is given by (3.1).
Proof. Given $f \in T$, and f is starlike of order δ, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \delta. \tag{4.4}$$

For the left hand side of (4.3) we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \sum_{n=2}^{\infty} \frac{(n-1)a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}.$$

The last expression is less than $1 - \delta$ if

$$\sum_{n=2}^{\infty} \frac{n-\delta}{1-\delta} a_n |z|^{n-1} < 1.$$

Using the fact that *f(z)* $\in TS_{\lambda}^{m}(\gamma, \varsigma)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[n(1+\varsigma) - (\gamma+\varsigma)]\phi_n(\lambda, m)}{(1-\gamma)} a_n \leq 1.$$

We can say (4.3) is true if

$$\sum_{n=2}^{\infty} \frac{n-\delta}{1-\delta} |z|^{n-1} \leq \frac{[n(1+\varsigma) - (\gamma+\varsigma)]\phi_n(\lambda, m)}{(1-\gamma)}$$

or equilently

$$|z|^{n-1} \leq \frac{(1-\delta)[n(1+\varsigma) - (\gamma+\varsigma)]\phi_n(\lambda, m)}{(n-\delta)(1-\gamma)}$$

which yields the starlikeness of the family.

5 Integral means inequalities

In [6], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. This function was used to solve his integral means inequality, which was conjectured [7] and settled in [8], that

$$\int_0^{2\pi} \left|f(re^{i\varphi})\right|^n d\varphi \leq \int_0^{2\pi} \left|f_2(re^{i\varphi})^n\right| d\varphi,$$

for all $f \in T$, $n > 0$ and $0 < r < 1$. In [6] , he also proved his conjuncture for the subclasses $T^*(\alpha)$ and $C(\alpha)$ of T.

For the class of functions $TS_{\lambda}^{m}(\gamma, \varsigma)$, we now prove Silverman's conjecture.

Littlewood's subordination theorem [2] and the concept of subordination between analytic functions are required.

Two functions f and g, which are analytic in U, the function f is said to be subordinate to g in U if there exists a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$, $(z \in U)$ Such that $f(z) = g(w(z))$, $(z \in U)$. We denote this subordination by $f(z) \prec g(z)$ ($\prec$ denotes subordination).

Lemma 1. If the functions $f$and $g$are analytic in U with $f(z) \prec g(z)$, then for $n > 0$ and $z = re^{i\varphi}$ $0 < r < 1$,

$$\int_0^{2\pi} \left|f(re^{i\varphi})\right|^n d\varphi \leq \int_0^{2\pi} \left|g(re^{i\varphi})\right|^n d\varphi.$$

Now, we discuss the integral means inequalities for functions f in $TS_\lambda^m(\gamma, \varsigma)$.

Theorem 7. Let $f \in TS_\lambda^m(\gamma, \varsigma)$, $0 \leq \gamma < 1$, and $f_2(z)$be specified by

$$f_2(z) = z - \frac{1-\gamma}{\varphi_2(\lambda, m, \varsigma, \gamma)} z^2. \tag{5.1}$$

Then for $z = re^{i\varphi}$ we have, $\int_0^{2\pi} |f(z)|^n d\varphi \leq \int_0^{2\pi} |f_2(z)|^n d\varphi$.

Proof. For $f(z) = z - \sum\limits_{n=2}^{\infty} a_n z^n$, (20) is equivalent to

$$\int_0^{2\pi} \left|1 - \sum_{n=2}^{\infty} a_n z^{n-1}\right|^n d\varphi \leq \int_0^{2\pi} \left|1 - \frac{1-\gamma}{\varphi_2(\lambda, m, \varsigma, \gamma)} z\right|^n d\varphi.$$

By Lemma 1, it is enough to prove that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{1-\gamma}{\varphi_2(\lambda, m, \varsigma, \gamma)} z.$$

Assuming

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{1-\gamma}{\varphi_2(\lambda, m, \varsigma, \gamma)} w(z),$$

and using (2.3) we obtain

$$|w(z)| = \left|\sum_{n=2}^{\infty} \frac{\varphi_2(\lambda, m, \varsigma, \gamma)}{1-\gamma} a_n z^{n-1}\right| \leq |z| \sum_{n=2}^{\infty} \frac{\varphi_2(\lambda, m, \varsigma, \gamma)}{1-\gamma} a_n \leq |z|$$

where $\varphi_n(\lambda, m, \varsigma, \gamma) = [n(1+\varsigma) - (\gamma + \varsigma)]\phi_n(\lambda, m)$.

This completes the proof.

6 Conclusion

This study looked at some basic properties of geometric function theory and introduced a new subclass of univalent functions defined by Geganbaur polynomial with negative coefficients. As a result, some coefficient estimates, radii of starlike and close-to-convexity, extreme points, and other findings have been taken into account, paving the way for further research in this area.

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Competing Interests

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