Operational Matrix Technique for Initial Value Problems of Nonlinear Fractional Order Differential Equation

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Abstract

In this article the shifted Legendre operational matrix of fractional derivatives is derived to obtain the solution of the nonlinear fractional order differential equations (FDEs) by using the collocation method. The main characteristic behind this technique is that it reduces the problems to those of solving a system of algebraic equations which simply can be solved by the Newton iterative method using software like Mathematica or Matlab. Some test problems are considered to explain the validity and applicability of the proposed technique.

Subject Classification:[2020]Primary 76S05
Keywords: Fractional Calculus; Multi-term FDE; Nonlinear FDE; Operational matrix; Shifted Legendre polynomial; Newton iterative method.

1 Introduction

During the last few decades fractional calculus has evolved and grown in pure mathematics as well as in scientific applications. In fact, the origin of fractional calculus lies nearly as far back as classical calculus itself. On the other hand todays mathematical topics which fall under the class of fractional calculus are far from being the calculus of fractions as one might suspect by the notation itself. Instead, integration and differentiation to an arbitrary order would be a better notion for the field of fractional calculus as it is understood today. Both the age of fractional calculus and the misconception of fractional calculus in its use today can be explained by surveying some aspects of the history of this mathematical field. Therefore the beginning stage of this work is concerned with a short summarization of the history of fractional calculus. The history and the comparative treatment of fractional order have been given by Oldham and Spanier [1], Miller and Ross [2], Podlubny [3] and Hilfer [4]. In fact, many scientific areas are currently working on fractional calculus concepts and we can introduce its adoption in visco elasticity and damping, chaos and fractals, diffusion and wave propagation, electromagnetism, biology, heat transfer, electronics, signal processing, robotics, system identification, traffic systems, genetic algorithms, percolation, modelling and identification, tele-communications, irreversibility, physical sciences, control systems as well as economy, and finance [5, 6, 7, 8, 9, 10, 11, 12]. Fractional differential equations cannot be solved analytically. Thus to analyse the solutions of fractional order differential equations accurate and efficient numerical methods are needed. Various numerical techniques have been proposed for approximate solutions of the fractional order differential equations, For example, the Adomian decomposition method [13, 14], Variational iteration method [15, 16, 17], Homotopy perturbation method [18, 19], Predictorcorrector Method [20], mstep Methods [21], Generalized differential transform method [22], Fractional convolution quadrature based on generalized Adams methods [23], Fractional linear multi-step method [24] and so on. Researchers used fractional calculus to model and analyse the real world problems. Fractional calculus has greater degree of freedom and it helps to analysis the solution of nonlinear problems and with the use of fractional derivative interdisciplinary applications can be studied. The nonlinear oscillation of earthquake can be studied with the help of fractional order derivative.

The Legendre equation is the special type of polynomial equation known as the Diophantine equation. The Legendre polynomials are the most general solution to the Legendre equation. In mathematics these functions are the solution of Legendre differential equation

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{d\,P_n(x)}{dx}\right)+n\,(n+1)\,P_n\,(x)=0 \tag{1.1}$$

This ordinary differential equation is frequently used in physics and the technical fields. It has regular singular points at $x=\pm 1$ and the series solution of this equation will converge for $|x|<1$.For n, $n\in N\cup\{0\}$the series solution $P_n\,(x)$ that is regular at $x=\pm 1$ forms a polynomial sequence of orthogonal polynomials called the Legendre polynomials. Legendre polynomials were introduced by the French Mathematician A. M. Legendre in the year 1784. These are special cases of the Legendre functions. The applications of Legendre functions are important for the problems involving spherical coordinates. Due to their orthogonality properties they are also useful in numerical analysis. Orthogonal functions belong to a vector space over the field $\mathbb{R}$(real numbers) that has the property of bi-linearity that is linearity in both coordinates. When the function space has an interval as the domain, the bilinear form may be the integral of the product of two functions over the interval. The main idea behind the approach using this method is that it reduces the problems to those of solving a system of algebraic equations.

The operational matrix of fractional order derivatives has been determined for some special types of orthogonal polynomials, such as Chebyshev polynomials, Legendre polynomials. Saadatmandi and Dehghan [25] introduced the shifted Legendre operational matrix for fractional derivatives and applied spectral methods for numerical solution of the multi-term nonlinear fractional differential equation with initial conditions. Many articles concerned with the application of shifted Legendre polynomials have appeared in the literature [26, 27, 28, 29].

The article is organized as follows: In the section 1, some basic definitions and properties of fractional order derivatives and Legendre polynomials are discussed. In Section 2 the application of the method is shown. Section 3 contains the numerical results with discussion. Finally, we conclude the paper with some remarks in Section 4.

2 Preliminaries and notation

In this section, we give some definitions and properties of the fractional calculus.

label=() The fractional-order derivative in the Caputo sense [30] is defined as

$$D^{\alpha}\psi\left(\xi\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\int_{0}^{\xi}\frac{\psi^{(n)}\left(t\right)}{\left(\xi-t\right)^{\alpha+n-1}}dt,\quad n-1<\alpha\leq n,\ n\in N,$$

where α, the order of the derivative, is a non-negative real number.

lbbel=()

$$D^{\alpha}\,\xi^{\lambda}=\begin{cases}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)}\,\xi^{\lambda-\alpha}, & for\,\lambda\in N\bigcup\{0\},\lambda\geq\lceil\alpha\rceil\ or\,\lambda\notin N,\,\lambda>\lfloor\alpha\rfloor\,,\\ 0, & for\,\lambda\in N\bigcup\{0\},\,\lambda<\lceil\alpha\rceil\,.\end{cases}$$

where $\lceil\alpha\rceil$ is the ceiling function denotes the smallest integer greater than or equal toα and $\lfloor\alpha\rfloor$ is the floor function which is largest integer less than or equal to α.

lcbel=() The Caputo fractional differentiation is a linear operation

$$D^{\alpha}(a_1\psi_1(\xi)+a_2\psi_2(\xi))=a_1D^{\alpha}\psi_1(\xi)+a_2D^{\alpha}\psi_2(\xi),$$

where a_1 and a_2 are constants.

2.1 Analysis of numerical methods

The Legendre polynomials of degree n are defined on the closed interval $[-1,1]$ and can be formulated through the following recurrence formula

$$(n+1)L_{n+1}(y)=(2n+1)yL_n(y)-nL_{n-1}(y),\ n\in\mathbb{N}, \tag{2.1}$$

with $L_n\left(y\right)=1,\ n=0$

$$L_n(y)=\begin{cases}1, & n=0\\ y, & n=1.\end{cases}$$

In order to use these Legendre polynomials on the interval $[0,1]$, we define the shifted Legendre polynomials by using the change of variable $y=2x-1$. Denoting the shifted Legendre polynomials $L_n(2x-1)$ as $P_n(x)$, we obtain the recurrence relation

$$(n+1)P_{n+1}(x)=(2n+1)(2x-1)P_n(x)-nP_{n-1}(x),\quad n\in\mathbb{N} \tag{2.2}$$

where for and the polynomials of degree zero and one are 1 and $2x-1$ respectively. The closed form of the shifted Legendre polynomials $P_n(x)$ is

$$P_n\left(x\right)=\sum_{i=0}^{n}\frac{\left(-1\right)^{n+i}\left(n+i\right)!\,x^i}{\left(n-i\right)\,\left(i!\right)^2}$$

and

$$P_n(x)=\begin{cases}(-1)^n, & x=0,\\ 1, & x=1.\end{cases}$$

Legendre polynomials satisfy the orthogonal relation in $[0,1]$ as

$$\int_0^1 P_m(x)P_n(x)dx=\begin{cases}\frac{1}{2n+1}, & m=n,\\ 0, & m\neq n.\end{cases}$$

Any function which is square integrable in$[0,1]$ may be expressed in terms of shifted Legendre polynomials as

(2.3) $$y(x)=\Sigma_{m=0}^{\infty}c_mP_m(x),$$

The coefficients c_m are given by

(2.4) $$c_m=(2m+1)\int_0^1 y(x)P_m(x)dx,\quad m\in\mathbb{N}.$$

Let us consider the first $(k+1)$- terms shifted Legendre polynomials as

(2.5) $$y(\xi)=C^t\phi(\xi),$$

where

(2.6) $$C^t=[c_0,c_1,\cdots,c_k],$$

and

(2.7) $$\phi(\xi)=[P_0,P_1,\cdots,P_k]^t.$$

The first derivative of the shifted Legendre vector $\phi(x)$ is $D^{(1)}\phi(x)$, where $D^{(1)}$ is the $(k+1)^2$ operational matrix of derivative is given by

(2.8)

$$D^{(1)}=(a_{ij})=\begin{cases}4m-2, \quad for\, m=n-i, & \begin{cases} i=1,\,3,.....\,k,\ \text{when } k \text{ is odd number},\\ i=1,\,3,.....\,(k-1),\ \text{when } k \text{ is even number},\end{cases}\\ 0, \quad \text{otherwise}\end{cases}$$

The p-th derivative is given by $\left(D^{(1)}\right)^p$ where

(2.9) $$D^{(p)}=\left(D^{(1)}\right)^p,$$

which will help to generalize the operational matrix of derivative of shifted Legendre polynomials from integer order to fractional order.

Theorem 2.1. *If $P_i(x)$ is a shifted Legendre polynomial, then its fractional order derivative is*

$$D^{\alpha}P_i(\xi)=0,\qquad n=0,\,1,\,........,\lceil\alpha\rceil-1,\;\;\alpha>0. \tag{2.10}$$

Theorem 2.2. *Let $\phi(\xi)$ be shifted Legendre vector and suppose $\alpha>0$ then*

$$D^{\alpha}\phi(\xi)\cong \mathrm{D}^{(\alpha)}\phi(\xi) \tag{2.11}$$

where $\mathrm{D}^{(\alpha)}$is the $(m+1)^2$ operational matrix of fractional derivative of order α in the Caputo sense and is defined as

$$\mathrm{D}^{(\alpha)}=\begin{pmatrix} 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 0\\ \sum_{i=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\omega_{\lceil\alpha\rceil,0,i} & \sum_{i=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\omega_{\lceil\alpha\rceil,1,i} & \cdots & \sum_{i=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\omega_{\lceil\alpha\rceil,k,i}\\ \vdots & \vdots & \cdots & \vdots\\ \sum_{i=\lceil\alpha\rceil}^{n}\omega_{n,0,i} & \sum_{i=\lceil\alpha\rceil}^{n}\omega_{n,1,i} & \cdots & \sum_{i=\lceil\alpha\rceil}^{n}\omega_{n,m,i}\\ \vdots & \vdots & \cdots & \vdots\\ \sum_{i=\lceil\alpha\rceil}^{k}\omega_{k,0,i} & \sum_{i=\lceil\alpha\rceil}^{k}\omega_{k,1,i} & \cdots & \sum_{i=\lceil\alpha\rceil}^{k}\omega_{k,k,i} \end{pmatrix} \tag{2.12}$$

where $\omega_{n,m,i}$ is given by

$$\omega_{n,m,i}=(2m+1)\sum_{l=0}^{m}\frac{(-1)^{\lambda}(n+i)!\,(l+m)!}{(n-i)!\,i!\,\Gamma(i-\alpha+1)\,(m-l)!\,(l!)^2\,(i+l-\alpha+1)},$$

and $\lambda=n+m+i+l$and in $\mathrm{D}^{(\alpha)}$the first $\lceil\alpha\rceil$ rows are all zero.

In particular, for $m=5$, the shifted Legendre operational matrices of fractional order derivative $D^{(\alpha)}$ for $\alpha=0.9$ and $\alpha=1.8$ are given by

$$D^{(0.9)}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 1.9111 & 0.2730 & -0.1321 & 0.0857 & -0.0626 & 0.0489\\ -0.2730 & 4.9936 & 0.6326 & -0.3155 & 0.2113 & -0.1585\\ 1.7790 & -0.3595 & 7.7555 & 1.02944 & -0.5292 & 0.3625\\ -0.3587 & 4.6781 & -0.3111 & 10.3539 & 1.4487 & -0.7639\\ 1.7163 & -0.5709 & 7.2889 & -0.2078 & 12.8440 & 1.8834 \end{pmatrix}$$

$$D^{(1.8)}=\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 10.8912 & 2.9703 & -1.2376 & 0.7425 & -0.5140 & 0.3851\\ -4.9505 & 40.8422 & 10.6084 & -4.6649 & 2.9258 & -2.0944\\ 33.4163 & -4.9505 & 81.3987 & 23.7148 & -11.0436 & 7.1856\\ -16.0893 & 105.9610 & 2.3954 & 130.619 & 42.9601 & -23.7259 \end{pmatrix}$$

2.2 Applications of the operational matrix of fractional derivative

Consider the multi-order nonlinear fractional order differential equation with initial condition as

$$D^{\alpha} y\left(\xi\right) = \phi\left(\xi,\, y\left(\xi\right),\, D^{\beta_1} y\left(\xi\right),\, ...D^{\beta_j} y\left(\xi\right)\right), \tag{2.13}$$

$$y^{j}\left(\xi_0\right) = a_j, \qquad j = 0,\, 1,\, ...,\, n, \tag{2.14}$$

where $n < \alpha \le n+1$ and β_j are positive monotonically increasing and bounded by 0 and αi.e.,

$$0 < \beta_1 < \beta_2 < ..... < \beta_j < \alpha.$$

To apply the of shifted Legendre polynomials to equation (2.13), we first approximate $y\left(\xi\right)$with $D^{\alpha} y\left(\xi\right)$and $D^{\beta_k} y\left(\xi\right)$for $k = 0,\, 1,\, ...,\, j$ as

$$\left.\begin{array}{ll} y\left(\xi\right) = \sum_{i=0}^{m} c_i P_i\left(\xi\right) = C^t \phi\left(\xi\right), & \\ D^{\alpha} y\left(\xi\right) = C^t D^{\alpha} \phi\left(\xi\right) = C^t \mathrm{D}^{(\alpha)} \phi\left(\xi\right), & \\ D^{\beta} y\left(\xi\right) = C^t D^{\beta_k} \phi\left(\xi\right) = C^t \mathrm{D}^{(\beta_k)} \phi\left(\xi\right), & k = 1,\, ...j \end{array}\right\} \tag{2.15}$$

Substituting these equations in (2.14), we get

$$C^t\, D^{\alpha} \phi\left(\xi\right) = f\left(\xi,\, C^t \phi\left(\xi\right),\, C^t D^{(\beta_1)} \phi\left(\xi\right),\, ...C^t D^{(\beta_j)} \phi\left(\xi\right)\right) \tag{2.16}$$

Also using the equations (2.9) and (2.3) in the initial conditions (2.14), we get

$$\begin{array}{l} y^{j}\left(\xi_0\right) = C^t \phi\left(\xi_0\right) = a_0, \\ y^{(j)}\left(\xi_0\right) = C^t D^{(j)} \phi\left(\xi_0\right) = d_j, \quad j = 1, ..., n. \end{array} \tag{2.17}$$

To find the solution, we first collocate equation (2.16) at $(m-n)$ points by using $(m-n)$ shifted Legendre roots of $P_{m+1}\left(\xi\right)$. These equations together with equation (2.17) generate $(m+1)$ nonlinear equations which can be solved using Newton's iterative method.

3 Application of the Method

In this section, three numerical examples are considered for the above technique to illustrate the applicability and accuracy of the proposed method. All the numerical computations are carried out using the software Mathematica. Comparisons of the results obtained by the present technique with those obtained by other methods reveal that the present method is very effective and convenient.

Consider the nonlinear multi-order fractional Vander Pol differential equation [31]

$$\frac{d^{\alpha} y}{dt^{\alpha}} - \gamma\left(1 - y^2\right) \frac{d^{\beta} y}{dt^{\beta}} + y = f\left(t\right), \quad t > 0, \qquad 0 < \alpha \le 2,\; 0 < \beta \le 1. \tag{3.1}$$

Subject to

$$y(0) = 0 \ \text{ and } y^{'}(0) = 0. \tag{3.2}$$

Substituting $y(t) = \sum_{i=0}^{m} c_i\, P_i(t) = C^t \phi(t)$, the equations (3.1) and (3.2) are converted to

$$C^t\, \mathrm{D}^{(\alpha)} \phi(t) - \gamma \left(1 - \left(C^t \phi(t)\right)^2\right) C^t\, \mathrm{D}^{(\beta)} \phi(t) + C^t\, \phi(t) = t \tag{3.3}$$

with

$$C^t \phi(0) = 0 \text{ and } C^t D^{(1)} \phi(0) = 0 \tag{3.4}$$

To get the solution of the problem we first collocate equations (3.3) using suitable collocation points of the first $(m - n)$ shifted Legendre roots of $P_{m+1}(t)$. Equation (3.3) together with equation (3.4) generate $(m + 1)$ non-linear equations which can be solved by the Newton's Iterative method using Mathematica or Matlab.

The absolute error for different fractional order α and β and different values of m are shown in Table 1. A good approximation can be achieved through increasing the terms of the shifted Legendre polynomials. The numerical results for $y(t)$ for m=*2, 3, 5* and $\alpha = 1.8$, $\beta = 0.8$; $\alpha = 1.9$, $\beta = 0.9$ and for standard order $\alpha = 2.0$, $\beta = 1.0$ are plotted in Figures 1(a), 1(b) and 1(c). These plots also imply that the numerical solutions are bounded and our numerical technique is stable.

Here the nonlinear fractional Emden-Fowler equation [32] which has received a great deal of attention for its physical importance and mathematical significance, is described as

$$\frac{d^\alpha y}{dx^\alpha} + \frac{2}{x}\frac{dy}{dx} + a\, \varphi_1(x)\, \varphi_2(y) = 0, \quad 0 < \alpha \leq 2 \tag{3.5}$$

subject to

$$y(0) = \psi(x) \ \text{ and } y'(0) = 0, \tag{3.6}$$

where $\varphi_1(x) = x^2$and $\varphi_2(y) = y^2$

Let

$$y(x) = \sum_{i=0}^{m} c_i\, P_i(x) = C^t \phi(x) \tag{3.7}$$

Choosing the same expression of $y(x)$ as in problem 1, we get

$$x\, C^t \mathrm{D}^{(\alpha)} \phi(x) + 2\, C^t \mathrm{D}^{(1)} \phi(x) + a\, x^2 \left(C^t \phi(x)\right)^2 = 0 \tag{3.8}$$

with the initial conditions

$$C^t\phi(0) = \psi(0) \text{ and } C^t D^{(1)}\phi(0) = 0, \tag{3.9}$$

Here we have used Mathematica to solve equations (3.8) and (3.9) for $a = 1$ and $\psi(x) = 1$ and also for $\alpha = 1.8,\ 1.9$ and standard order $\alpha = 2.0$ for $m = 2, 3, 5$. The plots are depicted through Fig. 2(a), 2(b) and 2(c) respectively. The errors of $y(x)$ at given points for different values of m are shown in Table 2.

The Ricatti equation plays a big role in the wide fields of applied and engineering sciences such as the transmission-line phenomena, theory of random processes, optimal control theory and diffusion problems etc. The Riccati equation refers to matrix equations with an analogous quadratic term, which occurs in both continuous and discrete-time linear-quadratic-Gaussian control.

The fractional order Riccati equation [33] considered is given as

$$\frac{d^\alpha y(x)}{dx^\alpha} = a_0(x) + a_1(x)\, y(x) + a_2(x)\, y(x)^2, \qquad 0 < \alpha \le 1, \tag{3.10}$$

subject to initial condition

$$y(x_0) = a, \tag{3.11}$$

where $a_0(x) \neq 0$ and $a_2(x) \neq 0$.

Applying the method, we get

$$C^t \mathrm{D}^{(\alpha)}\phi(x) - a_0(x) - a_1(x)\, C^t\phi(x) - a_2(x)\, \left(C^t\phi(x)\right)^2 = 0 \tag{3.12}$$

with

$$C^t\phi(x_0) = a \tag{3.13}$$

In this Example $a_0(x) = 1, a_1(x) = 2,\ a_2(x) = -1$,and $x_0 = a = 0$ are consider during numerical computation. After using the method which is described in section 2, the differential equation is converted to a system of algebraic equations where the unknowns can be found by using the Newton iterative method. The plots are shown in Figures 3(a), 3(b) and 3(c) for *m=2, 3* and *5* and for different values of fractional order $\alpha = 0.8$, 0.9 and standard order $\alpha = 1$. The errors of $y(x)$ at given points for different values of m are shown in Table 3.

Variations of solution function y for three considered example are plotted and displayed through Figures 4(a), 4(b) and 4(c) respectively for the values of $m = 2$, 3 and 5 for different values of fractional orders. It is seen from the figures that as the orders of the operational matrices increase, the solution profiles coincide for each example considered even for arbitrary choices of fractional order derivatives.

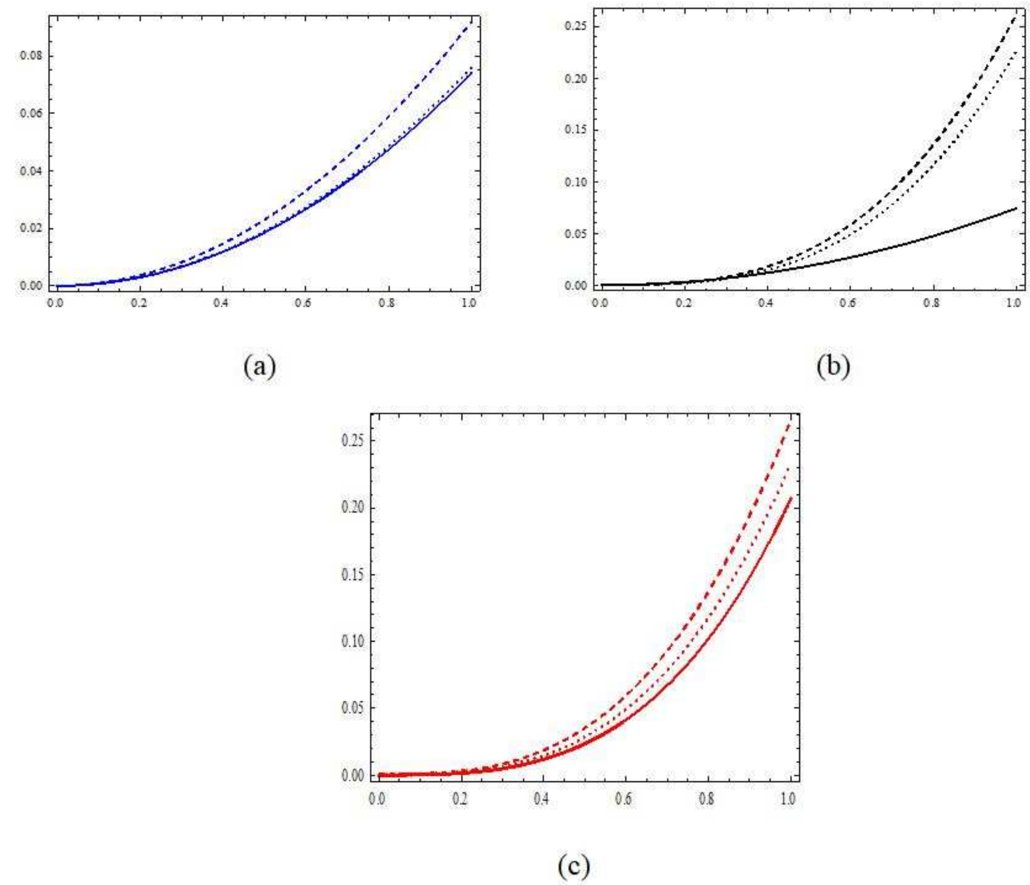

Fig. 1: Plots of $y(t)$ vs. t for Example 3: (a) $m = 2$; (b) $m = 3$; (c) $m = 5$. where in each figure the dashed, dotted and solid lines are representing $\alpha = 1.8, \beta = 0.8; \alpha = 1.9, \beta = 0.8; \alpha = 2.0, \beta = 0.8$ respectively.

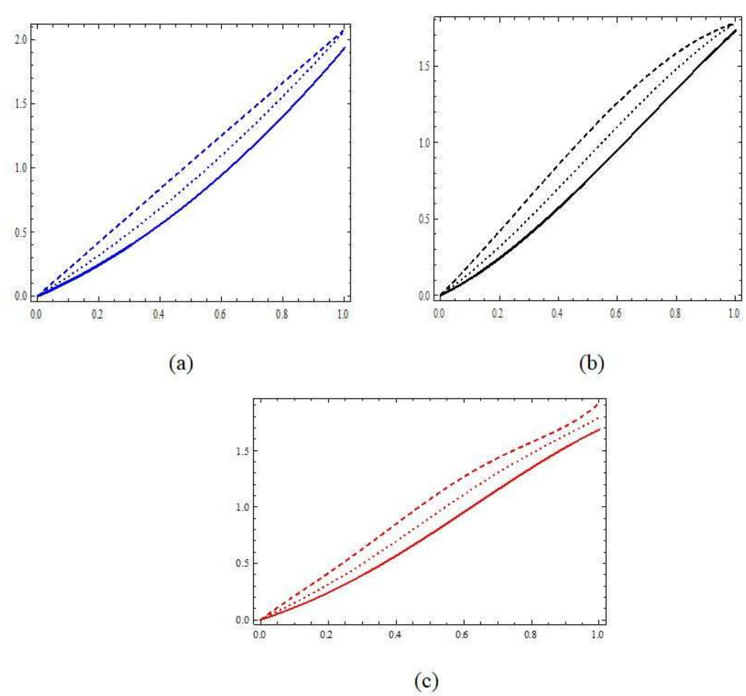

Fig. 3: Plots of $y\,(x)$ vs. x. For Example 3: (a) $m = 2$; (b) $m = 3$; (c) $m = 5$, where $\alpha = 0.8$, $\alpha = 0.9$, $\alpha = 1.0$ is represented by dashed, dotted and solid line respectively.

Tab. 1: Absolute error for different fractional order and different values of m for Example 3

	$m=2$			$m=3$			$m=5$		
x	$\alpha=1.8$ $\beta=0.8$	$\alpha=1.9$ $\beta=0.9$	$\alpha=2.0$ $\beta=1.0$	$\alpha=1.8$ $\beta=0.8$	$\alpha=1.9$ $\beta=0.9$	$\alpha=2.0$ $\beta=1.0$	$\alpha=1.8$ $\beta=0.8$	$\alpha=1.9$ $\beta=0.9$	$\alpha=2.0$ $\beta=1.0$
0.1	9.1e-4	7.5e-4	7.4e-4	3.7e-4	2.4e-4	1.4e-4	3.9e-4	2.4e-4	1.7e-4
0.2	3.6e-3	3.0e-3	2.9e-3	2.4e-3	1.8e-3	1.3e-3	2.6e-3	1.8e-3	1.3e-3
0.3	8.2e-3	6.8e-3	6.6e-3	7.8e-3	6.1e-3	4.8e-3	8.1e-3	6.1e-3	4.8e-3
0.4	1.4e-2	1.2e-2	1.1e-2	1.7e-2	1.4e-2	1.1e-2	1.8e-2	1.4e-2	1.1e-2
0.5	2.2e-2	1.8e-2	1.8e-2	3.4e-2	2.8e-2	2.3e-2	3.5e-2	2.8e-2	2.3e-2
0.6	3.3e-2	2.7e-2	2.6e-2	5.8e-2	4.9e-2	4.1e-2	5.9e-2	4.9e-2	4.1e-2
0.7	4.5e-2	3.7e-2	3.6e-2	9.1e-2	7.7e-2	6.5e-2	9.2e-2	7.8e-2	6.6e-2
0.8	5.8e-2	4.8e-2	4.7e-2	1.3e-1	1.1e-1	9.8e-2	1.3e-1	1.1e-1	1.0e-1
0.9	7.4e-2	6.1e-2	6.0e-2	1.9e-1	1.6e-1	1.4e-1	1.9e-1	1.6e-1	1.4e-1
1.0	9.1e-2	7.5e-2	7.4e-2	2.6e-1	2.2e-1	1.9e-1	2.6e-1	2.3e-1	2.0e-1

Tab. 2: Absolute error for different fractional order and different values of m for Example 3

	$m=2$			$m=3$			$m=5$		
x	$\alpha=1.8$	$\alpha=1.9$	$\alpha=2.0$	$\alpha=1.8$	$\alpha=1.9$	$\alpha=2.0$	$\alpha=1.8$	$\alpha=1.9$	$\alpha=2.0$
0.1	9.1e-1	9.1e-1	9.1e-1	1.0e-0	1.0e-0	1.0e-0	9.9e-1	9.9e-1	9.9e-1
0.2	9.1e-1	9.1e-1	9.1e-1	9.9e-1	9.9e-1	9.9e-1	9.9e-1	9.9e-1	9.9e-1
0.3	9.1e-1	9.1e-1	9.1e-1	9.9e-1	9.9e-1	9.9e-1	9.9e-1	9.9e-1	9.9e-1
0.4	9.1e-1	9.1e-1	9.1e-1	9.9e-1	9.9e-1	9.9e-1	9.9e-1	9.9e-1	9.9e-1
0.5	9.1e-1	9.1e-1	9.1e-1	9.9e-1	9.9e-1	9.9e-1	9.9e-1	9.9e-1	9.9e-1
0.6	9.1e-1	9.1e-1	9.1e-1	9.9e-1	9.9e-1	9.9e-1	9.9e-1	9.9e-1	9.9e-1
0.7	9.1e-1	9.1e-1	9.1e-1	9.8e-1	9.8e-1	9.9e-1	9.8e-1	9.8e-1	9.8e-1
0.8	9.1e-1	9.1e-1	9.1e-1	9.8e-1	9.8e-1	9.9e-1	9.7e-1	9.7e-1	9.7e-1
0.9	9.1e-1	9.1e-1	9.1e-1	9.7e-1	9.7e-1	9.7e-1	9.6e-1	9.6e-1	9.6e-1
1.0	9.1e-1	9.1e-1	9.1e-1	9.6e-1	9.6e-1	9.7e-1	9.4e-1	9.4e-1	9.5e-1

Tab. 3: Absolute error for different fractional order and different values of m for Example 3

	$m=2$			$m=3$			$m=5$		
x	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1.0$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1.0$	$\alpha=0.8$	$\alpha=0.9$	$\alpha=1.0$
0.1	2.1e-1	1.5e-1	1.1e-1	2.0e-1	1.4e-1	1.0e-1	2.1e-1	1.5e-1	1.1e-1
0.2	4.2e-1	3.1e-1	2.4e-1	4.1e-1	3.1e-1	2.4e-1	4.1e-1	3.1e-1	2.4e-1
0.3	6.2e-1	4.9e-1	3.9e-1	6.3e-1	5.0e-1	3.9e-1	6.2e-1	4.9e-1	3.9e-1
0.4	8.3e-1	6.8e-1	5.5e-1	8.5e-1	7.0e-1	5.7e-1	8.5e-1	6.9e-1	5.6e-1
0.5	1.0e-0	8.8e-1	7.4e-1	1.0e-0	9.0e-1	7.5e-1	1.0e-0	9.0e-1	7.5e-1
0.6	1.2e-0	1.0e-0	9.4e-1	1.2e-0	1.1e-0	9.4e-1	1.2e-0	1.1e-0	9.5e-1
0.7	1.4e-0	1.3e-1	1.1e-1	1.4e-1	1.2e-1	1.1e-1	1.4e-0	1.3e-1	1.1e-1
0.8	1.6e-0	1.5e-0	1.4e-0	1.5e-0	1.4e-0	1.3e-0	1.5e-0	1.4e-0	1.3e-0
0.9	1.8e-1	1.8e-1	1.6e-1	1.6e-0	1.6e-0	1.5e-0	1.7e-0	1.6e-0	1.5e-0
1.0	2.0e-1	2.0e-1	1.9e-1	1.7e-0	1.7e-0	1.7e-0	1.9e-0	1.7e-0	1.6e-0

4 Conclusion

In the present scientific contribution a general formulation for the shifted Legendre operational matrix of fractional derivatives is used to approximate numerical solutions of a class of fractional order differential equations. The approach was based on the collocation methods. It is shown that based on our proposed approach few terms of the shifted Legendre polynomials are needed to obtain accurate results. The most important part of the study is the graphical exhibition of accuracy of the solution profiles through increase in the orders of the operational matrices.

Acknowledgement
The present research work is carried out with the financial support provided by TEQIP Collaborative Research Scheme, Government of India.

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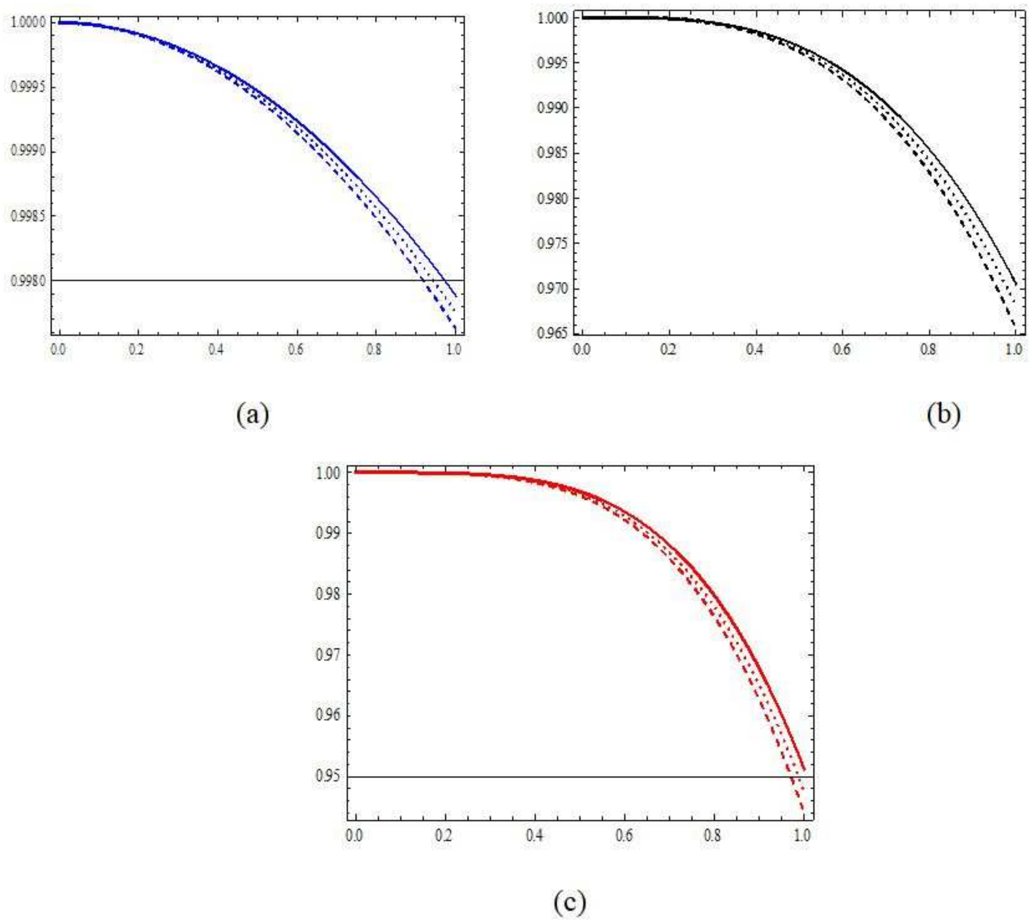

Fig. 2: Plots of $y\,(x)$ vs. x. For Example 2: (a) $m = 2$; (b) $m = 3$; (c) $m = 5$, where in each figure the dashed, dotted and solid lines are representing $\alpha = 1.8$, $\alpha = 1.9$, $\alpha = 2.0$ respectively.

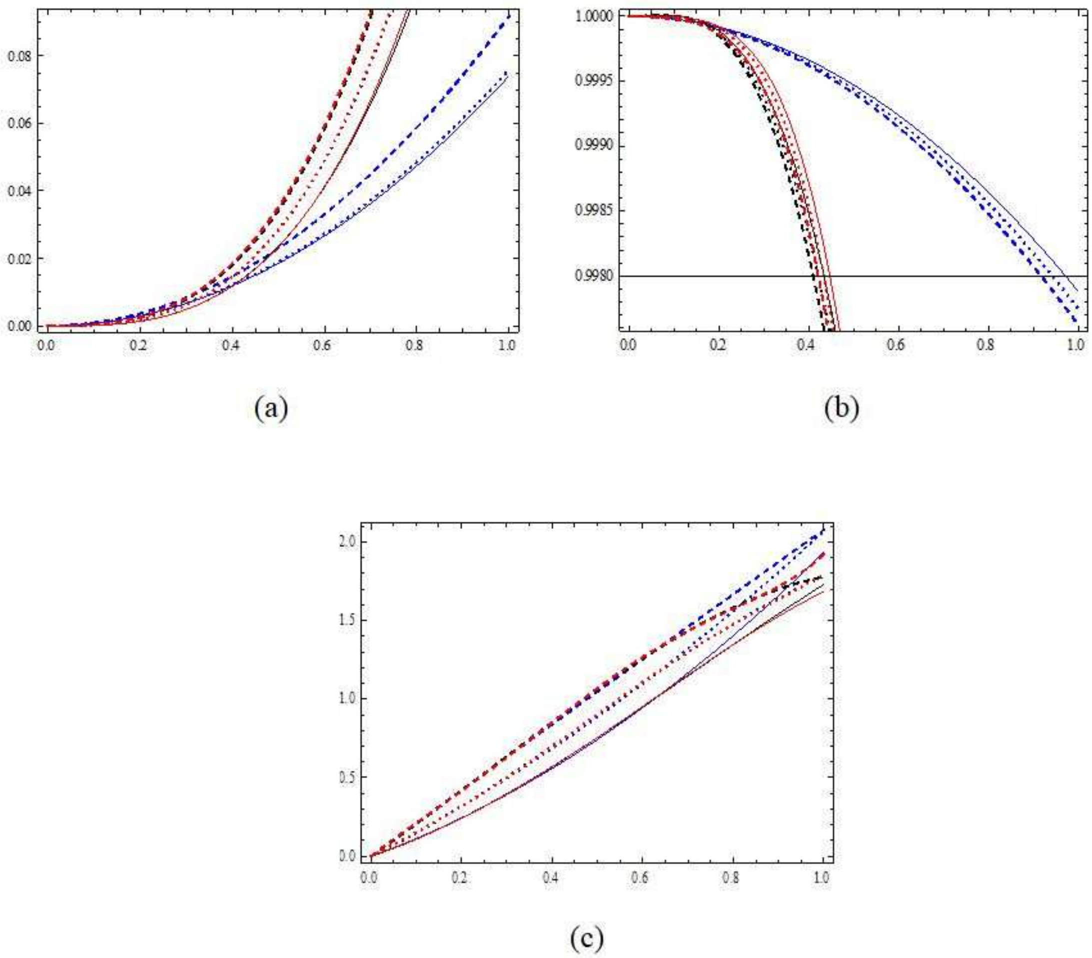

Fig. 4: (a). Comparisons of $y(t)$ for $m = 2, 3, 5$ with dashed (α, β) =(1.8, 0.8) ,dotted (α, β) =(1.9, 0.9) , thick (α, β) =(2.0, 1.0) for Example 3;
(b). Comparisons of $y(x)$ for $m = 2, 3, 5$ with dashed $\alpha = 1.8$,dotted $\alpha = 1.9$, thick $\alpha = 2.0$, for Example 3;
(c). Comparisons of $y(x)$ for $m = 2, 3, 5$ with dashed $\alpha = 0.8$,dotted $\alpha = 0.9$, thick $\alpha = 1.0$ for Example 3.