Singular Maps with Respect to an Ideal

Rohit Gaur [1] **and Anjali Srivastava** [2]

[1] *School of Studies in Mathematics*
Vikram University, Ujjain M.P.
r.gaur010@gmail.com

[2] *School of Studies in Mathematics*
Vikram University, Ujjain M.P.
anjalivuu@gmail.com

Abstract

An ideal is a nonempty collection of subsets closed under heredity and finite additivity. In this paper we introduce the concept of singular maps with respect to an ideal denoted *I-Singular Map*. Further we prove many properties of *I-Singular Map*.

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1 Introduction

Ideals in topological spaces have been considered since 1930. These have been studied by Kuratowski[6] in 1933 and Vaidyanathaswamy[13] in 1946. An ideal I on a topological space (X,τ) is a non-empty collection of subsets of X, which satisfies:
(i) $A \in I$ and $B \subseteq A \Rightarrow B \in I$ $(heridity)$
(ii) $A \in I$ and $B \in I \Rightarrow A \cup B \in I$ $(finite\ additivity)$.
We denote a topological space (X,τ) with an ideal I defined on X by (X,τ,I).
In literature a topological space together with an ideal is popularly known as an ideal topological space or simply an ideal space.
An ideal I is said to be codense or a boundary ideal [5], if $\tau \cap I = \{\emptyset\}$. If $A \subset X$, $cl(A)$ will denote the closure of A in (X,τ).
Let (X,τ,I) be an ideal topological space, (Y,σ) be another topological space. If $f : (X,\tau,I) \to (Y,\sigma)$ is a function. Then $f(I) = \{f(A) : A \in I\}$ is an *ideal* of Y.
If I is an ideal of subsets of X and Y is subset of X, then $I_Y = \{Y \cap A : A \in I\}$ is an ideal subset of Y [5].
Let (X,τ,I) be an ideal topological space. Then $A^*(I,\tau) = \{x \in X : A \cap U \notin I for\ every\ U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$.
When there is no ambiguity, we will write A^* fot $A^*(I,\tau)$ and call it the "*local function* of A ".
The simplest ideals are $\{\emptyset\}$ and $A^* = \{A : A \subseteq X\}$. Obserbe that $A^*(\{\emptyset\}) = cl(A)$ and $A^*(\wp(x)) = \emptyset$ for every $A \subseteq X$.
The concept of compactness with respect to an ideal was defined by R.L.Newcomb[7] and

had been studied by D.V. Rancin[8]. This concept has been further investigated by Hamlett and Jancovic[5] and many others.

The notion of singular maps and singular sets was introduced by G.T. Whyburn[14] and the topic was further pursued by G.L. Cain [1], R.E. Chandler [3] etc.. Later this concept lead to the notion of singular compactifications, infact, the simple observation by Cain, Chandler and Faulkner in [2] that for a map f from a locally compact space X to a locally compact space Y, the remainder induced by f and the singular set of f are the same has led to a fruitful combination of the independent studies of these concepts.

2 Preliminaries:

Lemma 2.1. *[5]Let (X, τ, I) ba an ideal topological space and A be a subset of X. Then:*
(i) $A^* = cl(A^*) \subseteq cl(A)$;
(ii) A is $\tau^* -$ closed if and only if $A^* \subseteq A$.

Definition 2.2. *[5]A subset A of a space (X, τ, I) is said to be $I -$ compact if for every open cover $\{U_\alpha : \alpha \in \nabla\}$ of A, there exists a finite sub-collection $\{U_{\alpha i} : i = 1, 2, 3...n\} \in I$ such that $A - \cup\{U_{\alpha i} : i = 1, 2, 3...n\} \in I$.*
The space (X, τ, I) is said to be $I -$ compact if X is $I -$ compact as a subset.

Theorem 2.3. *[4]Every g-closed subset of I-compact space is I-compact.*

Theorem 2.4. *[4]Every I-compact subset of a Hausdroff ideal space is $\tau^* -$ closed.*

Theorem 2.5. *[4]Continuous image of I-compact space is I-compact.*

Theorem 2.6. *[4]Let (X, τ, I) be any ideal topological space and let A be a subset if X such that for each open set U containing A, there is I-compact set B with $A \subset B \subset U$. Then A is I-compact.*

Definition 2.7. *[4]A space (X, τ, I) is said to be locally $I -$ compact if and only if every point in X has an $I -$ compact nbd. The space is said to be strongly locally I-compact if every point in X has a nbd base of $I -$ compact sets. A subset A of X is said to be (strongly) $I -$ compact if $(A, \tau/A, I/A)$ is strongly locally $I/A - I$compact where τ/A is the usually subspace topology and I/A=$\{I \cap A$:$I \in I\}$.*

Remark 2.8. *It is readily seen that an I-compact space is locally I-compact and that a strongly locally I-compact space is locally I-compact.*

Definition 2.9. *[12]Let X and Y be two locally compact spaces. Let $f : X \to Y$ be a map and let $y \in Y$.Then y is called a singular point of f if for every open set U of Y containing y, $f^{-1}(U)$ is not contained in a compact set of X, equivalently $clf^{-1}(U)$ is not a compact set of X.*

Definition 2.10. *[12]The set of all singular points of a map $f : X \to Y$ is called the singular set of the map f. It is denoted by $S(f)$.*
If $f : X \to Y$ is a map and $S(f) = Y$, then f is called a singular map.

Some examples of singular maps are listed below:-
(i) Let $f : X \to Y$ be a constant map where X is a non compact space then f is a singular map.
(ii) The projection maps π_1 and $\pi_2 : R^2 \to R$ are singular maps.
(iii) The map $f : R^2 \to R$ defined by $f(x, y) = x + y$ is a singular map.
(iv) The maps sin and cos from R to R are singular.
However the identity map or in general any homeomorphism from R to R is not singular. It is interesting to note that the restriction of a singular map is not always singular. However there are subspaces on which the restriction of singular map remains singular.

3 I-Singular Maps

In this section we introduce the notion of $I - singular\ maps$ between $locally\ I - compact\ spaces$ and obtain various properties of $I - singular\ maps$.

Definition 3.1. *Let $f : X \to Y$ be a continuous map, where (X, τ, I) is an ideal topological space. A point $y \in Y$ is called an I-singular point of f if for each open U of Y containing y, the closure $cl\ f^{-1}(U)$ is not I-compact set of X.*
Equivalently $f^{-1}(U)$ is not contained in any I-compact set of X.
The set of all $I - singular$ point of f is called the $I - singular$ set of f and it is denoted by $S_I(f)$.
If for a mapping $f : X \to Y$, $S_I(f) = Y$, then f is called an $I - Singular\ Map$.

Theorem 3.2. *The product of two $I - singular$ maps is $I - singular$.*

Proof. Let $f : X \to Y$ and $g : X' \to Y'$ two I-singular maps.
To show that the product map $f \times g : X \times X' \to Y \times Y'$ is I-singular.
Take a point $(y, y') \in Y \times Y'$ and a basic open set $U \times U'$ of $Y \times Y'$ with $(y, y') \in U \times U'$.
Then $cl(f \times g)^{-1}(U \times U') = clf^{-1}(U) \times clg^{-1}(U')$.
which is not I-compact.
Since $f : X \to Y$ and $g : X' \to Y'$ are I-singular map, then by definition of I-singular map $clf^{-1}(U)$ and $clg^{-1}(U')$ are not I-compact setof X.
Hence product of two I-singular maps is I-singular.

□

Remark 3.3. *The above results extends to an arbitrary product as well i.e. if $f_\alpha : X_\alpha \to Y_\alpha$ is a family of I-singular maps then $\prod f_\alpha : \prod X_\alpha \to \prod Y_\alpha$ is also I-singular map.*

Theorem 3.4. *The composition of two I-singular maps is I-singular.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ two I-singular maps.
To show that the composition (gof) is again an I-singular map, take a nonempty open set U of Z. Then $cl(gof)^{-1}(U) = clf^{-1}(g^{-1}(U))$
which is not I-compact.

□

Remark 3.5. *In fact if $f : X \to Y$ is an I-singular map and $g : Y \to Z$ is any onto map. Then $gof : X \to Z$ is always I-singular.*

Remark 3.6. *The comma map of two I-singilar maps need not be I-singular.*

Consider the maps $\pi_1 : R^2 \to R$ and $\pi_2 : R^2 \to R$ defined by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ respectively.
Then both π_1, π_2 are I-singular maps.
However $(\pi_1, \pi_2) : R^2 \to R^3$ is the identity map on R^2 which is not I-singular.

Remark 3.7. *Like continuous maps the restriction of a I-singular map need not be always I-singular.*

Let $\pi_1 : R^2 \to R$ be the map defined by $\pi_1(x, y) = x$.
Let $y \in R$. Then

$$\pi^{-1}\{y\} = \{y\} \times R.$$

Which is non I-compact.
Hence π_1 is a I-singular map.
However the restriction of π_1 on $R \times \{0\}$ is not I-singular.

Definition 3.8. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. Then A is called an $L_I - Subspace$ of X, if the restriction of each I-singular map $f : X \to Y$ on A is I-singular, where Y is any space.*

Theorem 3.9. *Let (X, τ, I) be an ideal topological space and K be an I-compact set of X, then $X - K$ is an L_I-subspace of X.*

Proof. Let $f : X \to Y$ be an I-singular map, where Y is a *locally* $I - compact$ space.
Denote by g, the restriction of f on $X - K$. To prove that g is an $I - singular$ *map*, take a non-empty open set U of Y.
Consider $g^{-1}(U)$, if $g^{-1}(U) = f^{-1}(U)$, then $g^{-1}(U)$ is not contained in an I-compact set of $X - K$ and hence that of X since the map is I-singular.
If $g^{-1}(U) \neq f^{-1}(U)$, then $f^{-1}(U) \subset g^{-1}(U) \cup K$.
If $g^{-1}(U)$ is contained in an I-compact set F of $X - K$, then $f^{-1}(U) \subset F \cup K$.
Which is a union of two I-compact sets of X, and hence I-compact. Hence g is an $I - singular$ *map*.
Since the singular map f and the the space Y are arbitrary, $X - K$ is an $L_I - subspace$ of X.
□

Theorem 3.10. *Let (X, τ, I) be an ideal topological space and let A be a closed $L_I - subspace$ of X. If B is a subset of X such that $A \subseteq B$, then B is also an $L_I - subspace$ of X.*

Proof. Let $f : X \to Y$ be a I-singular map denote by f_A and f_B respectively, the restrictions of f on A and B.
Since A is an $L_I - subspace$ of X, f_A is a I-singular map.
To prove that f_B is a singular map take an open set U of Y. Since $A \subseteq B$,

$$f_A^{-1}(U) \subseteq f_B^{-1}(U)$$

If $f_B^{-1}(U)$ is contained in an I-compact set F of B, then $f_A^{-1}(U)$ is contained in $F \cap A$, which is closed set of an I-compact space and hence I-compact.
This contradicts the hypothesis that f_A is an I-singular map. Hence f_B is an I-singular map.
Since the I-singular map f and the space Y arbitrary, B is an $L_I - subspace$ of X.
Conclude from the above theorems that the collections of closed $L_I - subspace$ is closed under finite unions and supersets.
□

The following theorem gives an analog example of pasting Lemma for I-singular maps.

Theorem 3.11. *Let X, Y be locally I-compact space and let $f : A \to Y$ and $g : B \to Y$ be two I-singular maps such that $f(x) = g(x)$ for $x \in A \cup B$, then the map $h : X \to Y$ defined by*

$$h(x) = \begin{cases} f(x) & if \quad x \in A \\ g(x) & if \quad x \in B \end{cases}$$

is I-singular.

Proof. Let U be an open set of Y.
Then $h^{-1}(U) = f^{-1}(U) \cup g^{-1}(U)$.
Hence $cl\ h^{-1}(U) = cl\ f^{-1}(U) \cup cl\ g^{-1}(U)$.
Since f, g are I-singular maps, $cl\ f^{-1}(U)$ and $cl\ g^{-1}(U)$ both are non I-compact.
Hence the open set U of Y is arbitrary, h is an I- singular map.

□

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