Strongly Summable Fibonacci Difference Geometric Sequences defined by Orlicz Functions

Saubhagyalaxmi Singh [1,*], Salila Dutta [2], Sagarika Dash[3] and Ram Prakash Sharma [4,]**

[1]*Department of Mathematics*
Centurion University of Technology and Management, Bhubaneswar, Odisha, India
saubhagyalaxmi.singh@cutm.ac.in

[2,3] *Department of Mathematics*
Utkal University, Bhubaneswar, Odisha, India
saliladutta516@gmail.com; sagarika05021993@gmail.com

[3]*Department of Mechanical Engineering*
NIT Arunachal Pradesh, Arunachal Pradesh-791112, India
Corresponding Author Email: rpsharma@nitap.ac.in

Abstract

The aim of this paper is to introduce some new geometric sequence spaces by using de la Vallée-Poussin mean, Fibonacci difference sequences and Orlicz functions . Also here certain inclusion relations, geometric properties and some topological properties of these spaces are studied .

Subject Classification:[2020] Primary 46A45 ; Secondary 46B45, 46A35
Keywords: Orlicz function, Fibonacci sequence space, Summability, Geometric sequence space.

1 Introduction

By ω, we denote the space of all real or complex valued sequences and any subspace of ω is called a sequence space. Let l_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with real or complex terms respectively.
In 1968 Cesro sequence space was brought to lime light by Dutch Mathematical Society which was of the form

$$[C,1] = \left\{ \xi = (\xi_k) \ \in \ \omega : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |\xi_k - \ell| = 0 \ \ , for \ \ some \ \ \ell \right\},$$

$$(C,1) = \left\{ \xi = (\xi_k) \ \in \ \omega : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (\xi_k - \ell) = 0 \ \ , for \ \ some \ \ \ell \right\}.$$

Then Shiue [18, 19] studied the Cesro sequence spaces and Cesro function spaces in 1970. Since then Cesro sequence spaces attracted the attention of researchers in various directions.

The strongly $(V,\lambda)-$ summable sequence spaces were introduced by Leindler [15] with

$$[V,\lambda]=\left\{\xi=(\xi_k)\ \in\ \omega:\lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k\,\in\,I_n}|\xi_k-\ell|=0\ ,\ for\ \ some\ \ \ell\right\}$$

and

$$(V,\lambda)=\left\{\xi=(\xi_k)\ \in\ \omega:\lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{k\,\in\,I_n}(\xi_k-\ell)=0\ ,\ for\ \ some\ \ \ell\right\},$$

where the non-decreasing sequence (λ_n) of positive reals which tend to infinity with $\lambda_1=1$ and $\lambda_{n+1}\le\lambda_n+1$ and $I_n=[n-\lambda_n+1,n], n=1,2,3,......$
Then de la Vallee-Pousin means is defined by $t_n(\xi)=\frac{1}{\lambda_n}\sum_{k\,\in\,I_n}\xi_k$.

It is observed that when $\lambda_n=n$ the $(V,\lambda)-$ summable sequence spaces reduces to Cesro sequences spaces.

The notion of difference sequence spaces was introduced by Kızmaz [13] as:

$$X(\Delta)=\{\xi=(\xi_k):\Delta\xi\in X,\ \text{for}\ X=\{l_\infty,c,c_0\}\}$$

where $\Delta\xi=(\xi_k-\xi_{k+1})$. These sequence spaces are Banach spaces with the norm

$$\|\xi\|_\Delta=|\xi_1|+||\Delta\xi||_\infty \tag{1.1}$$

The difference sequence spaces attracted several researchers among them are Atlay and Başar [1],Et and Çolak [9], Et and Bektas [8] and Dutta and Baliarsingh [3, 4, 5, 6].

Definition 1.1. *An Orlicz function is a function* $M:[0,\infty)\to[0,\infty)$, *if it is*
- *continuous,*
- *non-decreasing,*
- *convex and*
- $M(0)=0, M(x)>0,$ *for* $x>0$ *with* $M(x)\to\infty$ *as* $x\to\infty$.

Orlicz sequence space was introduced by Lindenstrauss and Tzafriri [15] as

$$\ell_M=\left\{\xi=(\xi_k):\sum_{k=1}^{\infty}M\Big(\frac{|\xi_k|}{\rho}\Big)<\infty\quad for\ some\ \ \rho>0\right\}$$

which is a Banach Space with the norm :
$$\|\xi\|=\inf\left\{\rho>0:\sum_{k=1}^{\infty}M\Big(\frac{|\xi_k|}{\rho}\Big)\le 1\right\}$$
ℓ_M coincides with ℓ_p for $1\le p\le\infty$ and $M(\xi)=\xi^p$

Parashar and Choudhary [17] introduced the sequence spaces

$$W(M,p)_0 = \left\{ \xi \in \omega : \frac{1}{n}\sum_{k=1}^{n}\left(M\Big(\frac{|\xi_k|}{\rho}\Big)\right)^{p_k} \to 0 \;\; as\; n \to \infty,\; for some\; \rho\; and\; \ell > 0 \right\}$$

$$W(M,p) = \left\{ \xi \in \omega : \frac{1}{n}\sum_{k=1}^{n}\left(M\Big(\frac{|\xi_k - \ell|}{\rho}\Big)\right)^{p_k} \to 0 \;\; as\; n \to \infty,\; for some\; \rho\; and\; \ell > 0 \right\}$$

$$W(M,p)_\infty = \left\{ \xi \in \omega : sup_n \frac{1}{n}\sum_{k=1}^{n}\left(M\Big(\frac{|\xi_k|}{\rho}\Big)\right)^{p_k} \to 0 \;\; as\; n \to \infty,\; for some\; \rho\; and\; \ell > 0 \right\}$$

which are complete paranormed spaces.

2 Fibonacci difference sequence spaces

The sequence of Fibonacci numbers have wide range of applications in arts,science and architecture and it satisfies Golden ratio [14].

The Fibonacci infinite matrix [11] $\widehat{F} = (\widehat{f_{nk}})$ is defined as

$$(\widehat{f_{nk}}) = \begin{cases} -\frac{f_{n+1}}{f_n} & (k = n-1), \\ \frac{f_n}{f_{n+1}} & (k = n), \qquad (n.k \in \mathbb{N}). \\ 0 & (0 \leq k < n-1 \text{ or } k > n), \end{cases}$$

Robert Katz and Michael Grossman invented the geometric calculus, the first non-Newtonian calculus system, in 1967 and built up an endless family of non-Newtonian calculi by 1970, each of which differed significantly from Newton and Leibniz's classical calculus.

Non-Newtonian or Geometric calculus is also known as multiplicative calculus [10], and it is based on multiplication rather than addition for differentiation and integration.

Geometric sequence spaces were introduced by Türkmen and Başar [25] with geometric zero one, geometric identity e, geometric addition ($\oplus$), geometric subtraction ($\ominus$) , geometric multiplication ($\odot$) and geometric limit $_G\lim_{n\to\infty}$ defined in [2, 20, 21, 22, 23, 24].

They also defined the sets of geometric integers, geometric real numbers and geometric complex numbers $\mathbb{R}(G)$ and $\mathbb{C}(G)$, respectively as follows:

$$\mathbb{R}(G) = \{e^x : x \in \mathbb{R}\} = \mathbb{R}^+/0,$$
$$\mathbb{C}(G) = \{e^x : x \in \mathbb{C}\} = \mathbb{C}/0.$$

The extended real number line $\mathbb{R}(G) = [0, \infty]$.
The geometric complex number $\mathbb{C}(G)$ is defined by Türkmen and Başar [4] as $\mathbb{C}(G) = \{\ e^z : z \in \mathbb{C}\ \} = \{\mathbb{C}\}/0$. Then $(\mathbb{C}(G), \oplus, \odot)$ is a field with geometric zero 1 and geometric identity e.

The geometric sequence spaces introduced by Türkmen and Başar [25] are

$$\omega(G) = \{\xi = (\xi_k) \in \mathbb{C}(G) \text{ for all } k \in \mathbb{N}\},$$
$$l_\infty(G) = \left\{\xi = (\xi_k) \in \omega(G) : \sup_{k\in N} |\xi_k|^G < \infty\right\},$$
$$c(G) = \left\{\xi = (\xi_k) \in \omega(G) : G\lim_{k\to\infty} |\xi_k \ominus l|^G = 1\right\},$$
$$c_0(G) = \left\{\xi = (\xi_k) \in \omega(G) : G\lim_{k\to\infty} \xi_k = 1\right\},$$
$$l_p(G) = \left\{\xi = (\xi_k) \in \omega(G) : {}_G\sum_{k=0}^{\infty} |\xi_k|_G^{p^G} < \infty\right\}.$$

Now we introduce the new Fibonacci difference geometric sequence spaces as follows:

$$[V,\lambda,M,\widehat{F},p]_0^G = \left\{\xi = (\xi_k) \in \omega(G) : G\lim_{n\to\infty} \frac{1}{\lambda_n} G \sum_{k\,\in\, I_n} \left[M\left(\frac{|\frac{f_k}{f_{k+1}}\xi_k \ominus \frac{f_{k+1}}{f_k}\xi_{k-1}|_G}{\rho}\right)\right]^{p_k^G} = 1,\ \rho > 1\right\}$$
$$[V,\lambda,M,\widehat{F},p]^G = \left\{\xi = (\xi_k) \in \omega(G) : G\lim_{n\to\infty} \frac{1}{\lambda_n} G \sum_{k\,\in\, I_n} \left[M\left(\frac{|(\frac{f_k}{f_{k+1}}\xi_k \ominus \frac{f_{k+1}}{f_k}\xi_{k-1}) \ominus \ell|}{\rho}\right)\right]^{p_k^G} = 1,\ \rho > 1\right\}$$
$$[V,\lambda,M,\widehat{F},p]_\infty^G = \left\{\xi = (\xi_k) \in \omega(G) : G\lim_{n\to\infty^G} \frac{1}{\lambda_n} G \sum_{k\,\in\, I_n} \left[M\left(\frac{|\frac{f_k}{f_{k+1}}\xi_k \ominus \frac{f_{k+1}}{f_k}\xi_{k-1}|}{\rho}\right)\right]^{p_k^G} < \infty,\ \rho > 1\right\}$$

3 Main Results

Theorem 3.1. *$[V,\lambda,M,\widehat{F},p]_0^G$, $[V,\lambda,M,\widehat{F},p]^G$ and $[V,\lambda,M,\widehat{F},p]_\infty^G$ are linear spaces over $\mathbb{C}(G)$ for any sequence $p^G = (p_k)^G$ of strictly positive real numbers.*

Proof. It is a routine verification and hence omitted.

□

Theorem 3.2. *The space $[V,\lambda,M,\widehat{F},p]_0^G$ is a paranormed space with a paranorm*

$$g(\xi) = inf\left\{\rho^{\frac{p_n}{H}} G : \left(\frac{1}{\lambda_n} G \sum_{k\,\in\, I_n} \left[M\left(\frac{|\frac{f_k}{f_{k+1}}\xi_k \ominus \frac{f_{k+1}}{f_k}\xi_{k-1}|}{\rho}\right)\right]^{p_k^G}\right)^{\frac{1}{H}} \leq 1, n = 1,2,3,...\right\}$$

where $H = max(1, sup\, p_k^G)$.

Proof. For $\xi, \eta \in [V,\lambda,M,\widehat{F},p]_0^G$, now $g(\xi) = g(\ominus\xi)$. For any two scalars $\alpha, \beta \in \mathbb{C}(G)$, and from linearity we have if $\alpha = \beta = 1$ then $g(\xi \oplus \eta) \leq g(\xi) \oplus g(\eta)$.
Since $\frac{1}{\lambda_n} M(1) = 1$, we get $inf\left\{\rho^{\frac{p_n}{H}}\right\} = 1$ $for\ \xi = 1$.

Conversely, suppose $g(\xi)=1$, then

$$g(\xi)=inf\left\{\rho^{\frac{p_n}{H}}G:\left(\frac{1}{\lambda_n}G\sum_{k\,\in\,I_n}\left[M\left(\frac{|\frac{f_k}{f_{k+1}}\xi_k\ominus\frac{f_{k+1}}{f_k}\xi_{k-1}|}{\rho}\right)\right]^{p_k^G}\right)^{\frac{1}{H}}\leq 1\right\}=1.$$

Now for a given $\varepsilon>1$, there is some $\rho_\varepsilon(1<\rho_\varepsilon<\varepsilon)$ such that

$$\left(\frac{1}{\lambda_n}G\sum_{k\,\in\,I_n}\left[M\left(\frac{|\frac{f_k}{f_{k+1}}\xi_k\ominus\frac{f_{k+1}}{f_k}\xi_{k-1}|}{\rho_\varepsilon}\right)\right]^{p_k^G}\right)^{\frac{1}{H}}\leq 1.$$

Then

$$\left(\frac{1}{\lambda_n}G\sum_{k\,\in\,I_n}\left[M\left(\frac{|\frac{f_k}{f_{k+1}}\xi_k\ominus\frac{f_{k+1}}{f_k}\xi_{k-1}|}{\varepsilon}\right)\right]^{p_k^G}\right)^{\frac{1}{H}}$$

$$\leq\left(\frac{1}{\lambda_n}G\sum_{k\,\in\,I_n}\left[M\left(\frac{|\frac{f_k}{f_{k+1}}\xi_k\ominus\frac{f_{k+1}}{f_k}\xi_{k-1}|}{\rho_\varepsilon}\right)\right]^{p_k^G}\right)^{\frac{1}{H}}$$

$$\leq 1\quad for\ each\ n.$$

Suppose $y_k\neq 1$ for some $k\ \in\ I_n$ where $y_k=\frac{f_k}{f_{k+1}}\xi_k\ominus\frac{f_{k+1}}{f_k}\xi_{k-1}$,
and $\varepsilon\to 1$, then

$$\left(\frac{1}{\lambda_n}G\sum_{k\,\in\,I_n}\left[M\left(\frac{|y_k|}{\varepsilon}\right)\right]^{p_k^G}\right)^{\frac{1}{H}}\neq 1,$$

which is a contradiction.
This implies $\eta_k=1$ for each k.
For continuity of scalar multiplication for any complex number μ , consider

$$g(\mu\odot\xi)=inf\left\{\rho^{\frac{p_n}{H}}:\left(\frac{1}{\lambda_n}G\sum_{k\,\in\,I_n}\left[M\left(\frac{|\mu(\frac{f_k}{f_{k+1}}\xi_k\ominus\frac{f_{k+1}}{f_k}\xi_{k-1})|}{\rho}\right)\right]^{p_k^G}\right)^{\frac{1}{H}}\leq 1,\ \ n=1,2,3...\right\}$$

$$=inf\left\{(|\mu|\odot s)^{\frac{p_n}{H}}:\left(\frac{1}{\lambda_n}G\sum_{k\,\in\,I_n}\left[M\left(\frac{|\frac{f_k}{f_{k+1}}\xi_k\ominus\frac{f_{k+1}}{f_k}\xi_{k-1}|}{s}\right)\right]^{p_k^G}\right)^{\frac{1}{H}}\leq 1,\ n=1,2,3,...\right\}$$

where $s=\frac{\rho}{|\mu|}$.

Now $g(\mu\odot\xi)\leq(max(1,|\mu|^{sup\ p_n^G}))^{\frac{1}{H}}$

$$\times\ \ inf\left\{s^{\frac{p_n}{H}}:\left(\frac{1}{\lambda_n}G\sum_{k\,\in\,I_n}\left[M\left(\frac{|\frac{f_k}{f_{k+1}}\xi_k\ominus\frac{f_{k+1}}{f_k}\xi_{k-1}|}{s}\right)\right]^{p_k^G}\right)^{\frac{1}{H}}\leq 1,\ n=1,2,3,...\right\}$$

as $|\mu|^{p_n^G} \leq max(1, |\mu|^{sup\, p_n^G})$.
Here $g(\mu \odot \xi) \to 1$ as $g(\xi) \to 1$ in $[V, \lambda, M, \widehat{F}, p]_0^G$.
Suppose $\mu_m \to 1$ as $m \to \infty$ and y_k be a sequence fixed in $[V, \lambda, M, \widehat{F}, p]_0^G$, then

$$\frac{1}{\lambda_n} G \sum_{k \in I_n} \left[M\left(\frac{|\frac{f_k}{f_{k+1}}\xi_k \ominus \frac{f_{k+1}}{f_k}\xi_{k-1}|}{\rho} \right) \right]^{p_k^G} < \left(\frac{\varepsilon}{2}\right)^H$$

that is for some $\rho > 1$ and all $n > \mathbb{N}$.

$$\frac{1}{\lambda_n} G \sum_{k \in I_n} \left[M\left(\frac{|\frac{f_k}{f_{k+1}}\xi_k \ominus \frac{f_{k+1}}{f_k}\xi_{k-1}|}{\rho} \right) \right]^{p_k^G} < \frac{\varepsilon}{2}$$

for some $\rho > 1$ and all $n > \mathbb{N}$.
Let $1 < |\mu| < e$, $n > \mathbb{N}$, we get

$$\frac{1}{\lambda_n} G \sum_{k \in I_n} \left[M\left(\frac{|\mu\left(\frac{f_k}{f_{k+1}}\xi_k \ominus \frac{f_{k+1}}{f_k}\xi_{k-1}\right)|}{\rho} \right) \right]^{p_k^G}$$

$$< \frac{1}{\lambda_n} G \sum_{k \in I_n} \left[|\mu| M\left(\frac{|\frac{f_k}{f_{k+1}}\xi_k \ominus \frac{f_{k+1}}{f_k}\xi_{k-1}|}{\rho} \right) \right]^{p_k^G} < \left(\frac{\varepsilon}{2}\right)^H$$

as M is convex.
As M appears to be continuous everywhere in $[1, \infty)$, for $n \leq \mathbb{N}$ and any scalar t we have

$$f(t \odot \xi) = \frac{1}{\lambda_n} G \sum_{k \in I_n} \left[M\left(\frac{|t\left(\frac{f_k}{f_{k+1}}\xi_k \ominus \frac{f_{k+1}}{f_k}\xi_{k-1}\right)|}{\rho} \right) \right]^{p_k^G}$$

is continuous at 1. So $e > \delta > 1$ such that $|f(t)| < \left(\frac{\varepsilon}{2}\right)^H$ for $1 < t < \delta$.

Now

$$g\{(\mu_m \odot \xi^m) = \left(\frac{1}{\lambda_n} G \sum_{k \in I_n} \left[M\left(\frac{|\mu_m\left(\frac{f_k}{f_{k+1}}\xi_k \ominus \frac{f_{k+1}}{f_k}\xi_{k-1}\right)|}{\rho} \right) \right]^{p_k^G} \right)^{\frac{1}{H}} < \frac{\varepsilon}{2}.$$

Which implies

$$\left(\frac{1}{\lambda_n} G \sum_{k \in I_n} \left[M\left(\frac{|\mu_m\left(\frac{f_k}{f_{k+1}}\xi_k \ominus \frac{f_{k+1}}{f_k}\xi_{k-1}\right)|}{\rho} \right) \right]^{p_k^G} \right)^{\frac{1}{H}} < \varepsilon,$$

as $\varepsilon \to 1$ so $g(\mu \odot \xi) \to 1$.
Hence $g\left\{(\mu_m \odot \xi^m) \ominus (\mu \odot \xi)\right\} \leq \left\{(\mu_m \ominus \mu) \odot g(\xi^m)\right\} \oplus \left\{|\mu| \odot g(\xi^m \ominus \xi)\right\}$. Hence the theorem. □

4 $\lambda-$ Geometric Statistical Convergence

We say a sequence $\xi = (\xi_k)$ is $S_\lambda^G(\widehat{F})-$ statistically convergent to ℓ if

$$G\lim_{n\to\infty}\frac{1}{\lambda_n}\Big|\Big\{k\leq n:\Big(|\frac{f_k}{f_{k+1}}\xi_k\ominus\frac{f_{k+1}}{f_k}\xi_{k-1}\ominus\ell|\Big)\geq\varepsilon\Big\}\Big|^G=1.$$

for every $\varepsilon > 1$, where the cardinality of the set is indicated by the bars.
Write for simplicity $y_k = \frac{f_k}{f_{k+1}}\xi_k \ominus \frac{f_{k+1}}{f_k}\xi_{k-1}$.

Theorem 4.1. $[V, \lambda, M, \widehat{F}]^G \subset S_\lambda^G(\widehat{F})$

Proof. Let $\xi \in [V, \lambda, M, \widehat{F}]^G$ and $\varepsilon > 1$ be given.Then for $y_k = \frac{f_k}{f_{k+1}}\xi_k \ominus \frac{f_{k+1}}{f_k}\xi_{k-1}$ consider

$$\frac{1}{\lambda_n}G\sum_{k\in I_n}\Big[M\Big(\frac{|y_k\ominus\ell|}{\rho}\Big)\Big]\geq\frac{1}{\lambda_n}G\sum_{\substack{k\in I_n\\|y_k\ominus\ell|\geq\varepsilon}}\Big[M\Big(\frac{|y_k\ominus\ell|}{\rho}\Big)\Big]$$

$$>\frac{1}{\lambda_n}\odot M\odot(\frac{\varepsilon}{\rho})|\{k\in I_n:|y_k\ominus\ell|\geq\varepsilon\}|.$$

Hence $\xi \in S_\lambda^G(\widehat{F})$. □

Definition 4.2. *If $M(2u) \leq KM(u)$, $u \geq 0$, for any positive constant $K > 2$ then we say M satisfy Δ_2-condition.*

Theorem 4.3. $[V, \lambda, \widehat{F}, p]^G \subseteq [V, \lambda, M, \widehat{F}, p]^G$., *if M satisfies Δ_2-condition.*

Proof. Suppose $\xi \in [V, \lambda, \widehat{F}, p]^G$. Then

$$A_n\equiv\frac{1}{\lambda_n}G\sum_{k\in I_n}\Big|\Big(\frac{f_k}{f_{k+1}}\xi_k\ominus\frac{f_{k+1}}{f_k}\xi_{k-1}\Big)\ominus\ell\Big|\to 1$$

as $n \to \infty$ for some ℓ.
Let $\varepsilon > 1$ and choose δ with $1 < \delta < e$ such that $M(t) < \varepsilon$ for $1 \leq t \leq \delta$.

Let $y_k = \Big|\Big(\frac{f_k}{f_{k+1}}\xi_k \ominus \frac{f_{k+1}}{f_k}\xi_{k-1}\Big) \ominus \ell\Big|_G$ and consider

$$\frac{1}{\lambda_n}G\sum_{k\in I_n}M(|y_k|)=\frac{1}{\lambda_n}G\sum_{\substack{k\in I_n\\|y_k|\leq\delta}}M(|y_k|)\ \oplus\ \frac{1}{\lambda_n}G\sum_{\substack{k\in I_n\\|y_k|>\delta}}M(|y_k|).$$

Since M is continuous, $\frac{1}{\lambda_n}G\sum_{\substack{k\in I_n\\|y_k|\leq\delta}}M(|y_k|) < \lambda_n \odot \varepsilon$ and for $y_k > \delta$ we use

$y_k < \frac{y_k}{\delta} < 1 \oplus \frac{y_k}{\delta}$.

Now

$$M(y_k) < M(1 \oplus \delta^{-1} y_k) < \frac{1}{2} M(2) \oplus \frac{1}{2} M(2\delta^{-1} t y_k)_G.$$

as M is non-decreasing and convex. By definition (4.2) for $K > 2$, $M(2\delta^{-1}y_k)_G \leq (\frac{1}{2}K\delta^{-1}y_k M(2))_G$,
so

$$M(y_k) < (\frac{1}{2}K\delta^{-1}y_k M(2))_G \oplus (\frac{1}{2}K\delta^{-1}y_k M(2))_G = (\frac{1}{4}K\delta^{-1}y_k M(2))_G.$$

Hence $\frac{1}{\lambda_n} G \sum_{\substack{k \in I_n \\ |y_k| > \delta}} M(y_k) \leq \frac{1}{4} K\delta^{-1} y_k M(2) \lambda_n A_n$

which together with $\frac{1}{\lambda_n} G \sum_{\substack{k \in I_n \\ |y_k| > \delta}} M(|y_k|) \leq \varepsilon \lambda_n$

yields $[V, \lambda, \widehat{F}, p]^G \subseteq [V, M, \widehat{F}, p]^G$.

□

Similarly the result holds for

$[V, \lambda, \widehat{F}, p]_0^G \subset [V, \lambda, M, \widehat{F}, p]_0^G$ and $[V, \lambda, \widehat{F}, p]_\infty^G \subset [V, \lambda, M, \widehat{F}, p]_\infty^G$.

Theorem 4.4. *For* $1 \leq p_k^G \leq q_k^G$ *and* $(\frac{q_k}{p_k})^G$ *bounded,* $[V, \lambda, M, \widehat{F}, q]^G \subset [V, \lambda, M, \widehat{F}, p]^G$.

Proof. Let $\xi = (\xi_k) \in [V, \lambda, M, \widehat{F}, q]^G$, consider

$$t_k = \left[M \left(\frac{\left| \frac{f_k}{f_{k+1}} \xi_k \ominus \frac{f_{k+1}}{f_k} \xi_{k-1} \right|}{\rho} \right) \right]^{q_k^G} \quad and$$

$\mu_k = \frac{p_k}{q_k}$, for all $k \in \mathbb{N}$.
Then $1 < \mu_k \leq e$.
Let $1 < \mu < \mu_k$, define the sequences (u_k) and (v_k) as follows:
$u_k = t_k$ and $v_k = 1$ for $t_k \geq e$,

$u_k = 1$ and $v_k = t_k$ for $t_k < e$.

Then

$$t_k = u_k \oplus v_k, \quad t_k^{\mu_k} = u_k^{\mu_k} \oplus v_k^{\mu_k}.$$

This implies $u_k^{\mu_k} \leq u_k \leq t_k$ and $v_k^{\mu_k} \leq v_k^{\mu}$. Hence

$$\frac{1}{\lambda_n} G \sum_{k \in I_n} t_k^{\mu_k} = \frac{1}{\lambda_n} G \sum_{k \in I_n} (u_k^{\mu_k} \oplus v_k^{\mu_k})$$

$$\leq \frac{1}{\lambda_n} G \sum_{k \in I_n} t_k \oplus \frac{1}{\lambda_n} G \sum_{k \in I_n} v_k^{\mu}.$$

for each k,

$$\frac{1}{\lambda_n} G \sum_{k \in I_n} v_k^{\mu} = G \sum_{k \in I_n} \Big(\frac{1}{\lambda_n} v_k\Big)^{\mu} \Big(\frac{1}{\lambda_n}\Big)^{1-\mu}$$

$$\leq \left(G \sum_{k \in I_n} \Big[\Big(\frac{1}{\lambda_n} v_k\Big)^{\mu} \Big]^{\frac{1}{\mu}} \right)^{\mu} \odot \left(G \sum_{k \in I_n} \Big[\Big(\frac{1}{\lambda_n}\Big)^{1-\mu} \Big]^{\frac{1}{1-\mu}} \right)^{1-\mu} = \left(\frac{1}{\lambda_n} G \sum_{k \in I_n} v_k \right)^{\mu},$$

and

$$\frac{1}{\lambda_n} G \sum_{k \in I_n} t_k^{\mu_k} \leq \frac{1}{\lambda_n} G \sum_{k \in I_n} t_k \oplus \left(\frac{1}{\lambda_n} G \sum_{k \in I_n} v_k \right)^{\mu}.$$

Hence $\xi = (\xi_k) \in [V, \lambda, M, \widehat{F}, p]^G$. □

5 Conclusion

In this paper, certain results on some Fibonacci difference geometric sequences using Orlicz function have been extended . The results presented in this paper not only generalize the earlier works done by several authors [7, 11, 12] but also give a new prospective regarding the development of geometric difference sequences. As a future work we will study certain matrix transformations and certain inclusion relations of these spaces, and the present results will be extended to the binomial sequence spaces of fractional order .Also these results can be extended including lacunary sequences including a Riesz and Norlund summability methods. The new results will provide new tools to deal with the convergence problems of sequences occurring in many branches of science and engineering.

6 Acknowledgment

We are very much grateful to the referees for their valuable suggestions and comments, which have improved the presentation of this article.

References

[1] Atlay , B. and Başar,F. (2007): The matrix domain and the fine spectrum of the difference operator Δ on the sequence space ℓ_p $(0 < p < 1)$, *Commun.Math.Anal.*, **2**(2), 1-11.

[2] Boruah, K. and Hazarika, B. (2017): Application of geometric calculus in numerical analysis and difference sequence spaces, *J. Math. Anal. Appl.*, **449**,1265-1285.

[3] Dutta, S. and Baliarsingh, P. (2012): On the fine spectra of the generalized rth difference operator Δ_ν^r on the sequence space ℓ_1, *Appl. Math. Comput.*, **219**, 1776-1784.

[4] Dutta, S. and Baliarsingh, P. (2013): On the Spectrum of 2nd order generalized difference operator Δ^2 over the sequence space c_0, *Bol. Soc. Paran. Mat.*, **31** (2), 235-244.

[5] Dutta, S., Baliarsingh, P., (2014): On some Toeplitz matrices and their inversion, *J. Egypt. Math. Soc.*, **22**,420-423.

[6] Dutta, S., Baliarsingh, P., (2014): A note on paranormed difference sequence spaces of fractional order and their matrix transformations, *J. Egypt. Math. Soc.*, **22**,249-253.

[7] Esi, A., (2009): Strongly generalized difference $(V, \lambda, \Delta^m, p)$ -summable sequence spaces defined by a sequence of moduli, *Nihonkai Math. J.*, **20**, 99-108.

[8] Et, M. and Bektas, C.A. (2004): Generalized strongly $(V, \lambda)-$summable sequences defined by Orlicz functions , *Math.Slovaca* , **54**(4), 411-422.

[9] Et, M. and Çolak, R. (1995): On generalized difference sequence spaces, *Soochow J. Math.*, **21**, 377-386.

[10] Grossmann, M. and Katz, R. (1972): Non-newtonian Calculus, *Lee press Pigeon cove Massachusetts.*

[11] Gulsen, K. and Murat, C. (2017): Some generalized fibonacci difference spaces defined by a sequence of modulus function, *FACTA UNIVERSITATIS (NIS), Ser. Math. Inform.* , **32** (1), 95116. DOI:10.22190/FUMI1701095K

[12] Kara, E.E., Başarir,M., (2012): An application of fibonacci numbers into infinite toeplitz matrices, *Casp. J. Math. Sci.* ,**1**(1), 43-47.

[13] Kızmaz,H. (1981): On certain sequence spaces, *Can.Math.Bull.*,**24**(2), 169-176.

[14] Koshy,T. (2001): Fibonacci and Lucas numbers with applications, *Wiley, New York.*

[15] Leindler,L. (1965): Über de la Vallee Pousinsche Summierbarkeit allgemeiner Orthogonalreihen, *Acta Math.Hung.*, **16**, 375-378.

[16] Maddox, I.J. (1988): Elements of Functional Analysis, *2nd ed. Cambridge, Cambridge University Press.*

[17] Parashar,S.D. and Choudhary, B. (1994): Sequence spaces defined by Orlicz functions, *Indian J. Pure Appl. Math.*, **25**(14) ,419-428.

[18] Shiue, J.S. (1970): On the Cesro sequence spaces, *Tamkang J. Math..* **1**, 19-25.

[19] Shiue, J.S. (1970): A note on Cesro function spaces, *Tamkang J. Math.*, **1**, 91-95.

[20] Singh, S. and Dutta, S. (2019): On certain generalized mth order geometric difference sequence spaces, *FJMS*, **116** (1), 83-100.

[21] Singh, S. and Dutta, S. (2019): Geometric zweier convergent lacunary sequence spaces,*iJET*, **10**(2b) , 126-129.

[22] Singh, S. and Dutta, S. (2020):Classes of new geometric difference sequence spaces,*Test Engineering and Management*, **83** (January - February 2020), 14358-14364.

[23] Singh, S. and Dutta, S. (2020): On some fractional order Binomial sequence spaces with infinite matrices,`arXiv:2012.07364 [math.FA]`.

[24] Singh, S. and Dutta, S.(2021): Strongly summable Fibonacci difference geometric sequences defined by Orlicz functions ,`arXiv:2101.03811 [math.FA]`.

[25] Türkmen, C. and Başar, F. (2012): Some basic results on the sets of sequences with geometric calculus, *Commun. Fac. Fci. Univ. Ank. series A1.*, **2**, 17-34.