Geometry of Submanifolds of locally metallic Riemannnian manifolds

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Abstract

The purpose of present paper is to focus on some properties of submanifolds of locally metallic Riemannian manifolds. A Riemannian manifold $(\overline{M}, J, \overline{g})$ is called a locally metallic Riemannian manifold if the $(1,1)-$ tensor field J is parallel on $\overline{M}$ satisfies $J^2 = pJ + qI$ and the metric $\overline{g}$ is $J-$compatible. First, we investigate some new results on submanifolds of Riemannian manifold with metallic structure. Later, we establish certain characterizations on submanifolds of codimension 2 and provide an example on submanifolds of locally metallic Riemannian manifolds.

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1 Introduction

In recent years, one of the most studied manifolds are Riemannian manifolds with metallic structure. Metallic structure on Riemannian manifolds involves metallic means family as generalization of golden mean, which contains the silver mean, the brounze mean, the copper mean and the nickel mean etc was introduced by V. W. de Spinadel [9] in 2002. The positive solution of the equation $x^2 - px - q = 0$ for some positive integers p and q is called a (p, q)-metallic number which has the form

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2}.$$

For particular, $p = 1, q = 1$ and $p = 2, q = 1$, it is well known that we have the golden mean and silver mean respectively.

C. E. Hretcanu and M. Crasmareanu introduced the notion of golden structure [7] . Golden structures are further studied by geometers in [5], [11], [12], [14], [16], [20] and some generalization of golden structure is called metallic structure as particular cases of polynomial structure given by S. I Goldberg and K. Yano [15]. Being inspired by metallic mean, the notion of metallic manifold $\overline{M}$ was defined in [19] by a $(1,1)$-tensor field on $\overline{M}$, which satisfies

$$J^2 = pJ + qI, \tag{1.1}$$

where I is the identity operator on lie algebra $\chi(\overline{M})$ of vector fields on $\overline{M}$ and p, q are fixed positive integers. A Riemannian metric $\overline{g}$ on $\overline{M}$ is $J-$ compatible if

$$\overline{g}(JX, Y) = g(X, JY) \tag{1.2}$$

for every $X, Y \in \chi(\overline{M})$. This condition is equivalent to

$$\overline{g}(JX, JY) = p\overline{g}(X, JY) + q\overline{g}(X, Y).$$

The structure $(\overline{g}, J)$ satisfying (1.1) and (1.2) is called metallic Riemannian structure and $(\overline{M}, \overline{g}, J)$ is called Riemannian metallic manifold. The metallic manifolds are extensively studied by several geometers (see [1], [13], [19], [17]).

The submanifolds are one of the most interesting topics in differential geometry. It is well known that a submanifold of a Riemannian manifold is always a Riemannian one. Metallic structure on ambient Riemannian manifold provides important geometrical results on the submanifolds as it is important tool while investigating the geometry of submanifolds. Some properties of submanifolds in almost contact and almost complex manifolds are studied by geometers (see [2], [3], [4], [8], [10], [22]). Hretcanu and Blaga studied submanifolds of metallic Riemannian manifolds in [20]. Also they studied slant and semi-slant submanifold of metallic Riemannian manifolds in [17], [18].

Motivated by above studies in this paper, we study some properties of submanifolds of a locally metallic Riemannian manifold which are not covered in [20]. The paper is organized as follows:

In section 3, we establish several properties of induced structure (P, g, ξ, u, a) on the submanifolds immersed in locally metallic Riemannian manifolds further, we construct an example of metallic Riemannian structure on Euclidean space and its submanifolds.

2 Preliminaries.

Let us consider that M is an n-dimensional submanifold of codimension r, isometrically immersed in an $(n+r)$-dimensional locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ with $n, r \in N$.

We denote by T_xM the tangent space of M in a point $x \in M$ and by $T_x^{\perp}M$ the normal space of M in x. Let i be the immersion $i : M \to \overline{M}$. The induced Riemannian metric g on M is given by $g(X, Y) = g(iX, iY)$ for every $X, Y \in \chi(M)$.

We consider a local orthonormal basis $N_1, N_2, ..., N_r$ of the normal space $T_x^{\perp}M$. We assume that the indices α, β, γ run over the range $1, 2, ..., r$.

For every $X \in T_xM$ the vector fields $J(iX)$ and $J(N_\alpha)$ can be decomposed in tangential and normal components as follows:

$$J(i_*(X)) = i_*(P(X)) + \sum_{\alpha=1}^{r} N_\alpha, \tag{2.1}$$

$$J(N_\alpha) = i_*(\xi_\alpha) + \sum_{\beta=1}^{r} N_\beta, \tag{2.2}$$

where P is a $(1,1)$ tensor field on M, $\xi \in \xi(M)$, u_α are $1-forms$ on M and $(a_{\alpha\beta})_r$ is an $r \times r$ matrix of smooth real functions on M.
In the rest of paper we shall simply denote by X the vector field i_*X, for $X \in \chi(M)$.

Proposition 2.1. *[20] The structure $\sum = (P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced on the submanifold M by the metallic Riemannian structure $(\overline{g}, J)$ on $\overline{M}$ satisfies the following equalities:*

$$P^2(X) = pP(X) + qX - \sum_\alpha u_\alpha(X)\xi_\alpha, \tag{2.3}$$

$$u_\alpha(P(X)) = pu_\alpha(X) - \sum_\beta a_{\alpha\beta}u_\beta(X), \tag{2.4}$$

$$a_{\alpha\beta} = a_{\beta\alpha}, \tag{2.5}$$

$$u_\beta(\xi_\alpha) = q\delta_{\alpha\beta} + pa_{\alpha\beta} - \sum_\gamma a_{\alpha\gamma}a_{\gamma\beta}, \tag{2.6}$$

$$P(\xi_\alpha) = p\xi_\alpha - \sum_\beta a_{\alpha\beta}\xi_\beta, \tag{2.7}$$

$$u_\alpha(X) = g(X, \xi_\alpha), \tag{2.8}$$

$$g(PX, Y) = g(X, PY), \tag{2.9}$$

$$g(PX, PY) = pg(X, PY) + qg(X, Y) + \sum_\alpha u_\alpha(X)u_\alpha(Y), \tag{2.10}$$

for every $X, Y \in \chi(M)$, where $\delta_{\alpha\beta}$ is the Kronecker delta.

The Gauss and Weingarten formulae are

$$\overline{\nabla}_X Y = \nabla_X Y + \sum_{\alpha=0}^{r} h_\alpha(X,Y)N_\alpha, \tag{2.11}$$

$$\overline{\nabla}_X N_\alpha = -A_\alpha X + \nabla_X^{\perp} N_\alpha. \tag{2.12}$$

If $\{N_1,...,N_r\}$ and $\{N'_1,...,N'_r\}$ are two local orthogonal basis on a normal space $T_x^{\perp}M$, then the decomposition of N'_α in the basis $\{N_1,...,N_r\}$ is

$$N'_\alpha = \sum_{\gamma=1}^{r} k_\alpha^\gamma N_\gamma$$

for any $\alpha \in \{1,...,r\}$, where (k_α^γ) is an $r \times r$ orthogonal matrix and we have $u'_\alpha = \sum_\gamma k_\alpha^\gamma u_\gamma$, $\xi'_\gamma = \sum_\gamma k_\alpha^\gamma \xi_\gamma$ and $a'_{\alpha\beta} = \sum_\gamma k_\alpha^\gamma a_{\gamma\delta} k_\beta^\delta$.
Thus, if $\xi_1, \xi_2, ..., \xi_r$ are linearly independent vector fields, then $\xi'_1, \xi'_2, ..., \xi'_r$ are also linearly independent.
We know that $a_{\alpha\beta}$ is symmetric in α and β, under a suitable transformation, we can find that $a_{\alpha\beta}$ can be reduced to $a'_{\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta}$, where $\lambda_\alpha(\alpha \in \{1,..,r\}$ are eigen values of the matrix $(a_{\alpha\beta})_r$ and in this case we have
$u'_\beta(\xi_\alpha) = \delta_{\alpha\beta}(q + p\lambda_\alpha - \lambda_\alpha\lambda_\beta)$ and from this we obtain $u'_\alpha(\xi_\alpha) = (q + p\lambda_\alpha - \lambda_\alpha^2)$.

Theorem 2.2. *[20] Let M is an n-dimensional submanifold of codimension r in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$. If the structure J is parallel with respect to the Levi-Civita connection $\overline{\nabla}$ defined on $\overline{g}$, then the structure $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced on M by the structure J has the following properties:*

$$(\nabla_X P)(Y) = \sum_\alpha [g(A_\alpha X, Y)\xi_\alpha + u_\alpha(Y)A_\alpha X], \tag{2.13}$$

$$(\nabla_X u_\alpha)(Y) = \sum_\beta [h_\beta(X,Y)a_{\beta\alpha} - u_\beta(Y)l_{\alpha\beta}(X)] - h_\alpha(X, PY), \tag{2.14}$$

$$\nabla_X \xi_\alpha = -P(A_\alpha X) + \sum_\beta a_{\alpha\beta}A_\beta X + \sum_\beta l_{\alpha\beta}(X)\xi_\beta, \tag{2.15}$$

$$X(a_{\alpha\beta}) = -u_\alpha(A_\beta X) - u_\beta(A_\alpha X) + \sum_\gamma [l_{\alpha\gamma}(X)a_{\gamma\beta} + l_{\beta\gamma}(X)a_{\alpha\gamma}], \tag{2.16}$$

for any $X \in \chi(M)$.

Definition 2.3. *The induced structure* $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ *on submanifold* M *in a locally metallic Riemannian manifold* $(\overline{M}, \overline{g}, J)$ *is said to be normal if*

$$N_P(X,Y) - 2\sum_\alpha du_\alpha(X,Y)\xi_\alpha = 0$$

for any $X, Y \in \chi(M)$.

Remark 2.4. *The compatibility condition* $\overline{\nabla}J = 0$, *where* $\overline{\nabla}$ *is Levi-Civita connection with respect to the metric* $\overline{g}$ *implies the integrability of the structure* J *which is equivalent with the vanishing of the Nijenhuis torsion tensor field of* J*:*

$$N_J(X,Y) = [JX, JY] + J^2[X,Y] - J[JX,Y] - J[X,JY].$$

For this assumption, we must have the next general lemma:

Lemma 2.5. *Let* J *be the parallel metallic structure on a Riemannian manifold* $\overline{M}$ *and* D *be a linear connection with the torsion* T*. If* N_J *is Nijenhuis torsion tensor field of J, then*

$$N_J(X,Y) = (D_{JX}J)Y - (D_{JY}J)X - T(JX,JY) - pJT(X,Y) -$$

$$qT(X,Y) + J(D_YJ)X + JT(JX,Y) - J(D_XJ)Y + JT(X,JY).$$

Theorem 2.6. *[20] Let* M *be a submanifold of codimension* r *in a metallic Riemannian manifold* $(\overline{M}, \overline{g}, J)$, *then*

$$N_p(X,Y) - 2\sum_{\alpha=1}^{r} du_\alpha(X,Y)\xi_\alpha = \sum_{\alpha=1}^{r}[g(x,\xi_\alpha)(PA_\alpha - A_\alpha P)Y$$

$$-g(Y,\xi_\alpha)(PA_\alpha - A_\alpha P)] - \sum_{\alpha=1}^{r}\sum_{\beta=1}^{r}[l_{\alpha\beta}(X)g(Y,\xi_\beta) - l_{\alpha\beta}(Y)g(X,\xi_\beta)]\xi_\alpha,$$

for any $X, Y \in \chi(M)$, *where* $l_{\alpha\beta}$ *are the coefficient of the normal connection in the normal bundle* $T^\perp(M)$.

Corollary 2.7. *Let* M *be a submanifold of codimension* r *in a locally metallic Riemannian manifold* $(\overline{M}, \overline{g}, J)$*. If the induced structure* $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ *on* M *is normal and the normal connection* $\nabla^\perp$ *on* M *vanishes identically (i.e.* $l_{\alpha\beta} = 0$*), then we obtain the equality*

$$\sum_\alpha g(X,\xi_\alpha)(PA_\alpha - A_\alpha P)(Y) = \sum_\alpha g(Y,\xi_\alpha)(PA_\alpha - A_\alpha P)(X)$$

for any $X, Y \in \chi(M)$.

3 Submanifolds of codimension 2:

Lemma 3.1. *Let M be a submanifold in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$. Let $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ be the induced structure on M. Then*

$$g((PA_\alpha - A_\alpha P)(X), Y)$$

on M is skew-symmetric for any $X, Y \in \chi(M)$.

Proof.

$$g(PA_\alpha X - A_\alpha PX, Y) = g(PA_\alpha X, Y) - g(A_\alpha PX, Y)$$
$$g(PA_\alpha X - A_\alpha PX, Y) = g(X, A_\alpha PY) - g(PA_\alpha Y, X)$$
$$g(PA_\alpha X - A_\alpha PX, Y) = -g((PA_\alpha - A_\alpha P)(Y), X).$$

So, $g((PA_\alpha - A_\alpha P)(X), Y)$ is skew-symmetric. □

Proposition 3.2. *Let M be a submanifold of codimension r $(r \geq 2)$ in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ and structure J is parallel to Levi-Civita connection $\overline{\nabla}$ defined on $\overline{M}$ with $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ induced structure on M by J. If the normal connection $\nabla^\perp$ vanishes identically on the normal bundle $T^\perp(M)$ (i.e. $l_{\alpha\beta} = 0$,) then the tangent vector fields $\{\xi_1, \xi_2, ..., \xi_r\}$ are linearly independent if and only if the determinant of the matrix $(qI_r + pA - A^2)$ does not vanish in any $x \in M$, (where I_r is the $r \times r$ identity matrix).*

Proof. Let $k_1, ..., k_r$ be the real number with the properties that

$$k_1\xi_1 + k_2\xi_2 + ... + k_r\xi_r = 0 \tag{3.1}$$

in any point $x \in M$.
From the equality (2.6), we obtain

$$g(\xi_\alpha, \xi_\beta) = u_\beta(\xi_\alpha) = q\delta_{\alpha\beta} + pa_{\alpha\beta} - \sum_\gamma a_{\alpha\gamma}a_{\gamma\beta}. \tag{3.2}$$

Multiplying the equality (3.1) by ξ_α { for any $\alpha \in (1, 2..., r)$ } and using the equality (3.2), we obtain

$$\begin{cases} k_1(q + pa_{11} - \sum_\gamma a_{1\gamma}a_{\gamma 1}) + k_2(pa_{12} - \sum_\gamma a_{1\gamma}a_{\gamma 2}) + ... + k_r(pa_{1r} - \sum_\gamma a_{1\gamma}a_{\gamma r}) = 0 \\ k_1(pa_{21} - \sum_\gamma a_{2\gamma}a_{\gamma 2}) + k_2(q + pa_{22} - \sum_\gamma a_{2\gamma}a_{\gamma 2}) + ... + k_r(pa_{2r} - \sum_\gamma a_{r\gamma}a_{\gamma r}) = 0 \\ ---------------------------------- \\ k_1(pa_{r1} - \sum_\gamma a_{r\gamma}a_{\gamma 1}) + k_2(pa_{r2} - \sum_\gamma a_{r\gamma}a_{\gamma 2}) + ... + k_r(q + pa_{rr} - \sum_\gamma a_{r\gamma}a_{\gamma r}) = 0. \end{cases}$$

This linear system of equations has the unique solution $k_1 = k_2 = ... = k_r = 0$ if and only if it does not have a vanishing determinant. Furthermore, the determinant of the linear system of equations is the determinant of the following matrix

$$qI_r + p\begin{pmatrix} a_{11} & a_{12} & a_{13}.....a_{1r} \\ a_{21} & a_{22} & a_{23}.....a_{2r} \\ . & . & ...... \\ . & . & ..... \\ a_{r1} & a_{r2} & a_{r3}.....a_{rr} \end{pmatrix} - \begin{pmatrix} a_{11} & a_{12}.....a_{1r} \\ a_{21} & a_{22}.....a_{2r} \\ . & . \\ . & . \\ a_{r1} & a_{r2}.....a_{rr} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12}.....a_{1r} \\ a_{21} & a_{22}.....a_{2r} \\ . & . \\ . & . \\ a_{r1} & a_{r2}.....a_{rr} \end{pmatrix},$$

that is determinant of the matrix

$$qI_r + pA - A^2.$$

□

Lemma 3.3. *Let M be an n-dimensional submanifold of co-dimension 2 in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, with the normal induced structure $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta}))$ and structure J is parallel to Levi-Civita connection $\overline{\nabla}$. If the normal connection $\nabla^\perp$ vanishes identically (i.e. $l_{\alpha\beta}$) then the following equation is hold good*

$$g(Y,\xi_1)(PA_1 - A_1P)(X) + g(Y,\xi_2)(PA_2 - A_2P)(X) + g((PA_1 - A_1P)(X), Y)\xi_1$$

$$+g((PA_2 - A_2P)X, Y)\xi_2 = 0 \tag{3.3}$$

for any $X, Y \in \chi(M)$.

Proof. By virtue of Proposition 2.7 we obtain

$$g(X,\xi_1)(PA_1 - A_1P)(Y) + g(X,\xi_2)(PA_2 - A_2P)(Y)$$
$$= g(Y,\xi_1)(PA_1 - A_1P)(X) + g(Y,\xi_2)(PA_2 - A_2P)(X)$$

for any $X, Y \in \chi(M)$.
Taking inner product with $Z \in \chi(M)$, we have

$$g(X,\xi_1)g((PA_1 - A_1P)(Y), Z) + g(X,\xi_2)g((PA_2 - A_2P)(Y), Z) =$$

$$g(Y,\xi_1)g((PA_1 - A_1P)(X), Z) + g(Y,\xi_2)g((PA_2 - A_2P)(X), Z) \tag{3.4}$$

for any $X, Y, Z \in \chi(M)$.
Interchanging Y and Z in the last equality, we obtain

$$g(X,\xi_1)g((PA_1 - A_1P)Z, Y) + g(X,\xi_2)g((PA_2 - A_2P)Z, Y)$$

$$= g(Z,\xi_1)g((PA_1 - A_1P)X, Y) + g(Z,\xi_2)g((PA_2 - A_2P)(X), Y). \tag{3.5}$$

Adding equalities (3.4) and (3.5) we obtain

$$g(X,\xi_1)g((PA_1 - A_1P)Z, Y) + g(X,\xi_2)g((PA_2 - A_2P)Z, Y)$$
$$+g(X,\xi_1)g((PA_1 - A_1P)Y, Z) + g(X,\xi_2)g((PA_2 - A_2P)Y, Z)$$
$$= g(Y,\xi_1)g((PA_1 - A_1P)X, Z) + g(Y,\xi_2)g((PA_2 - A_2P)X, Z)$$
$$+g(Z,\xi_1)g((PA_1 - A_1P)X, Y) + g(Z,\xi_2)g((PA_2 - A_2P)X, Y).$$

By virtue of Lemma 3.1 and using property of skew-symmetry, we obtain

$$g([g(Y,\xi_1)((PA_1 - A_1P)(X)) + (g(Y,\xi_2)(PA_2 - A_2P)(X))$$
$$+g((PA_1 - A_1P)X, Y)\xi_1 + g((PA_2 - A_2P)X, Y)\xi_2], Z) = 0$$

for any $Z \in \chi(M)$. Which gives equality (3.3). □

Lemma 3.4. *Let M be an n-dimensional submanifold of codimension 2 in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, with the normal induced structure $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ and structure J is parallel to Levi-Civita connection $\overline{\nabla}$. If the normal connection $\nabla^\perp$ vanishes identically (i.e., $l_{\alpha\beta} = 0$) and $\sigma \neq 0$, then the following equations are hold good*

$$(PA_1 - A_1P)\xi_1 = 0, \tag{3.6}$$

$$(PA_2 - A_2P)\xi_2 = 0, \tag{3.7}$$

$$(PA_1 - A_1P)\xi_2 = 0, \tag{3.8}$$

$$(PA_2 - A_2P)\xi_1 = 0. \tag{3.9}$$

Proof. With $X = Y = \xi_1$ in equality (3.3), we get

$$g(\xi_1, \xi_1)(PA_1 - A_1P)(\xi_1) + g(\xi_1, \xi_2)(PA_2 - A_2P)(\xi_1)$$

$$+g((PA_1 - A_1P)(\xi_1), \xi_1)\xi_1 + g((PA_2 - A_2P)(\xi_1), \xi_1)\xi_2 = 0.$$

Using $g(\xi_1, \xi_1) = ap + \sigma \neq 0$, $g(\xi_1, \xi_2) = 0$ and Lemma 3.1 in above equation, we obtain

$$(ap + \sigma)(PA_1 - A_1P)\xi_1 = 0,$$

as $(ap + \sigma) \neq 0$

$$(PA_1 - A_1P)\xi_1 = 0$$

which is (3.6).
With $X = Y = \xi_2$, in equality (3.3), we obtain

$$g(\xi_1, \xi_2)(PA_1 - A_1P)\xi_2 + g(\xi_2, \xi_2)(PA_2 - A_2P)\xi_2$$

$$+g((PA_1 - A_1P)\xi_2, \xi_2)\xi_1 + g((PA_2 - A_2P)\xi_2, \xi_2)\xi_2 = 0.$$

Using $g(\xi_2, \xi_2) = -ap + \sigma \neq 0$, $g(\xi_1, \xi_2) = 0$, and Lemma 3.1, we have

$$(PA_2 - A_2P)\xi_2 = 0,$$

which is (3.7).
If we put $X = \xi_1$ and $Y = \xi_2$ in equality (3.3), we get

$$g(\xi_2, \xi_1)(PA_1 - A_1P)\xi_1 + g(\xi_2, \xi_2)(PA_2 - A_2P)\xi_1$$

$$+g((PA_1 - A_1P)\xi_1, \xi_2)\xi_1 + g((PA_2 - A_2P)\xi_1, \xi_2)\xi_2 = 0.$$

Using that $g(\xi_2, \xi_2) = b + \sigma \neq 0$, $g(\xi_1, \xi_2) = 0$, we obtain

$$(PA_2 - A_2P)\xi_1 = 0,$$

which is (3.9).
Again, put $X = \xi_2$ and $Y = \xi_1$, we obtain

$$g(\xi_1, \xi_1)(PA_1 - A_1P)\xi_2 + g(\xi_1, \xi_2)(PA_2 - A_2P)\xi_2$$

$$+g((PA_1 - A_1P)\xi_2, \xi_1)\xi_1 + g((PA_2 - A_2P)\xi_2, \xi_1)\xi_2 = 0$$

Using that $g(\xi_1, \xi_1) = a + \sigma \neq 0$, $g(\xi_1, \xi_2) = 0$, we obtain

$$(PA_1 - A_1P)\xi_2 = 0.$$

which is (3.8). □

Proposition 3.5. *Let M be an n-dimensional submanifold of codimension 2 in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, with the normal induced structure $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_r)$ and structure J is parallel to Levi - Civita connection $\overline{\nabla}$. The normal connection $\nabla^\perp$ vanishes identically (i.e., $l_{\alpha\beta} = 0$) and $\sigma \neq 0$, and trace $A = 0$. Then P commutes with the Weingartan operator A_α ($\alpha \in \{1, 2\}$), thus the following relations take place*

$$(i)(PA_1 - A_1P)(X) = 0, \tag{3.10}$$

$$(ii)(PA_2 - A_2P)(X) = 0 \tag{3.11}$$

$\forall X \epsilon \chi(M)$.

Proof. .

$$g((PA_\alpha - A_\alpha P)X, \xi_\beta) = g(PA_\alpha X, \xi_\beta) - g((A_\alpha P)X, \xi_\beta)$$

$$g((PA_\alpha - A_\alpha P)X, \xi_\beta) = -[g(PA_\alpha \xi_\beta, X) - g((A_\alpha P)\xi_\beta, X)$$

$$g((PA_\alpha - A_\alpha P)X, \xi_\beta) = -g((PA_\alpha - A_\alpha P)\xi_\beta, X),$$

where $\alpha\ \beta \in \{1, 2\}$. From the Lemma 3.4, we have

$$(PA_\alpha - A_\alpha P)\xi_\beta = 0$$

for any $\alpha\ \beta \in \{1, 2\}$. Then

$$g((PA_1 - A_1P)X, \xi_\beta) = 0$$

$$(PA_1 - A_1P)X = 0.$$

Similarly,

$$(PA_2 - A_2P)X = 0$$

for any $\alpha\ \beta \in \{1, 2\}$.

□

In the following we assume that M is an n-dimensional submanifold of codimension 2 in locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ with induced structure $(P, g, u_\alpha, \xi_\alpha, (a_{\alpha\beta})_2)$ on M $(\alpha, \beta \in \{1, 2\})$. We suppose that the normal connection vanishes identically, $(i.e l_{\alpha\beta} = 0)$. In these conditions, the relations of Proposition 2.1 have the following forms:

$$P^2X = pP(X) + qX - u_1(X)\xi_1 - u_2(X)\xi_2, \tag{3.12}$$

$$u_1(PX) = pu_1(X) - a_{11}u_1(X) - a_{12}u_2(X), \tag{3.13}$$

$$u_2(PX) = pu_2(X) - a_{21}u_1(X) - a_{22}u_2(X), \tag{3.14}$$

$$u_1(\xi_1) = q + pa_{11} - a_{11}^2 - a_{12}^2, \tag{3.15}$$

$$u_2(\xi_2) = q + pa_{22} - a_{12}^2 - a_{22}^2, \tag{3.16}$$

$$u_1(\xi_2) = u_2(\xi_1) = pa_{21} - a_{21}(a_{11} + a_{22}), \tag{3.17}$$

$$P(\xi_1) = p\xi_1 - a_{11}\xi_1 - a_{12}\xi_2, \tag{3.18}$$

$$P(\xi_2) = p\xi_2 - a_{21}\xi_1 - a_{22}\xi_2, \tag{3.19}$$

$$g(PX, PY) = pg(X, PY) + qg(X, Y) + u_1(X)u_1(Y) + u_2(X)u_2(Y). \tag{3.20}$$

Furthermore, from Theorem 2.2 under the assumption that the normal connection $\nabla^\perp$ vanishes identically $(i.e.\ l_{\alpha\beta} = 0)$, we obtain

$$(\nabla_X P)(Y) = g(A_1X, Y)\xi_1 + g(A_2X, Y)\xi_2 + g(Y, \xi_1)A_1X + g(Y, \xi_2)A_2X, \tag{3.21}$$

$$(\nabla_X u_1)(Y) = -g(A_1X, PY) + a_{11}g(A_1X, Y) + a_{21}g(A_2X, Y), \tag{3.22}$$

$$(\nabla_X u_2)(Y) = -g(A_2X, PY) + a_{12}g(A_1X, Y) + a_{22}g(A_2X, Y), \tag{3.23}$$

$$\nabla_X \xi_1 = -P(A_1X) + a_{11}A_1X + a_{12}A_2X, \tag{3.24}$$

$$\nabla_X \xi_2 = -P(A_2X) + a_{21}A_1X + a_{22}A_2X, \tag{3.25}$$

$$X(a_{12}) = -2u_1(A_1X), \tag{3.26}$$

$$X(a_{22}) = -2u_2(A_2X) \tag{3.27}$$

for any $X, Y \in \chi(M)$. We denote by $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Remark 3.6. *A simplifier assumption for these relations is* $a_{11}+a_{22}=0$. *Thus, trace* $A=0$. *Under this assumption, if we denote* $a_{11}=-a_{22}=a$, $a_{12}=a_{21}=b$ *and* $q-a^2-b^2=\sigma$, *from the relations , we easily see that*

$$u_1(\xi_1)=ap+\sigma \Leftrightarrow g(\xi_1,\xi_1)=ap+\sigma, \tag{3.28}$$

$$u_2(\xi_2)=-ap+\sigma \Leftrightarrow g(\xi_2,\xi_2)=-ap+\sigma \tag{3.29}$$

$$u_1(\xi_2)=u_2(\xi_1)=pb, \tag{3.30}$$

$$u_1(PX)=(p-a)u_1(X)-bu_2(X), \tag{3.31}$$

$$u_2(PX)=(p+a)u_2(X)-bu_1(X), \tag{3.32}$$

$$P(\xi_1)=(p-a)\xi_1-b\xi_2, \tag{3.33}$$

$$P(\xi_2)=(p+a)\xi_2-b\xi_1. \tag{3.34}$$

Proposition 3.7. *Let* M *be a submanifold of codimension 2 in a locally metallic Riemannian manifold* $(\overline{M},\overline{g},J)$ *and structure* J *is parallel to Levi - Civita connection* $\overline{\nabla}$ *defined on* $\overline{M}$ *with the normal induced structure* $(P,g,u_\alpha,\xi_\alpha,(a_{\alpha\beta})_2)$. *If the normal connection* $\nabla^\perp$ *vanishes identically, that is* $l_{\alpha\beta}=0$, *trace* $A=0$ *and* $\sigma\neq 0$, *then the following relations occur:*

$$(\sigma+pa)A_1\xi_1+pbA_1\xi_2=h_1(\xi_1,\xi_1)\xi_1+h_1(\xi_1,\xi_2)\xi_2, \tag{3.35}$$

$$(\sigma-pa)A_1\xi_2+pbA_1\xi_1=h_1(\xi_1,\xi_2)\xi_1+h_1(\xi_2,\xi_2)\xi_2, \tag{3.36}$$

$$(\sigma+pa)A_2\xi_1+pbA_2\xi_2=h_2(\xi_1,\xi_1)\xi_1+h_2(\xi_1,\xi_2)\xi_2, \tag{3.37}$$

$$(\sigma-pa)A_2\xi_2+pbA_2\xi_1=h_2(\xi_1,\xi_2)\xi_1+h_2(\xi_2,\xi_2)\xi_2. \tag{3.38}$$

Proof. . Using (3.10) and applying P it follows that

$$P^2A_1X=PA_1PX$$

for any $X\in\chi(M)$.
Using the equality (3.12) and if we put $X=\xi_1$ and $X=\xi_2$ respectively, we obtain

$$pP(A_1\xi_1)+qA_1\xi_1-u_1(A_1\xi_1)\xi_1-u_2(A_1\xi_1)\xi_2=PA_1P\xi_1.$$

Using equality (3.33), we get

$$(P^2+q-pP)A_1\xi_1+(P-p)aA_1\xi_1+(P-p)bA_1\xi_2=h_1(\xi_1,\xi_1)\xi_1+h_1(\xi_1,\xi_2)\xi_2. \tag{3.39}$$

Now,

$$pP(A_1\xi_2)+qA_1\xi_2-u_1(A_1\xi_2)\xi_1-u_2(A_1\xi_2)\xi_2=PA_1P\xi_2. \tag{3.40}$$

Using (3.34), we obtain

$$(P^2+q-pP)A_1\xi_2+(P-p)aA_1\xi_2+(P-p)bA_1\xi_2=h_1(\xi_1,\xi_2)\xi_1+h_1(\xi_2,\xi_2)\xi_2. \tag{3.41}$$

We replace $X\to PX$ in the equality (3.10), so

$$PA_1PX=A_1P^2X. \tag{3.42}$$

Using equality (3.12) and if we put $X=\xi_1$ and $X=\xi_2$ respectively, we get

$$PA_1P\xi_1=pA_1P\xi_1+qA_1\xi_1-u_1(\xi_1)A_1\xi_1-u_2(\xi_1)A_1\xi_2. \tag{3.43}$$

Using (3.33), we obtain

$$PA_1(p\xi_1-a\xi_1-b\xi_2)=pA_1(p\xi_1-a\xi_1-b\xi_2)+qA_1\xi_1-u_1(\xi_1)A_1\xi_1-u_2(\xi_1)A_1\xi_2, \tag{3.44}$$

$$(pP-P^2-q+\sigma)A_1\xi_1+(2p-P)aA_1\xi_1+(2p-P)bA_1\xi_2=0 \tag{3.45}$$

and

$$PA_1PA\xi_2=A_1P^2\xi_2.$$

Using (3.33), we obtain

$$PA_1((p-a)\xi_2-b\xi_1)=A_1p((p-a)\xi_2-b\xi_1)+qA_1\xi_2-pbA_1\xi_1-(pa+\sigma)A_1\xi_2,$$

$$(pP-p^2-q+\sigma)A_1\xi_2+(2p-P)A_1a\xi_2+(2p-P)bA_1\xi_1=0. \tag{3.46}$$

Adding the relations (3.39) and (3.45), we obtain (3.35).
Adding (3.41) and (3.46), we obtain (3.36).
Applying P in the equality (3.11), it follows that

$$P^2A_2X=PA_2PX$$

for any $X\in\chi(M)$ and using in (3.12) and for $X=\xi_1$ and $X=\xi_2$ respectively we obtain

$$(p^2+q-pP)A_2\xi_1+(P-p)aA_2\xi_1+(P-p)bA_2\xi_2=h_2(\xi_1,\xi_1)\xi_1+h_2(\xi_1,\xi_2)\xi_2, \tag{3.47}$$

$$(p^2+q-pP)A_2\xi_2+(P-p)aA_2\xi_2+(P-p)bA_2\xi_1=h_2(\xi_1,\xi_2)+h_2(\xi_2,\xi_2)\xi_2. \tag{3.48}$$

We replace $X \to PX$ in the equality (3.11), so

$$PA_2PX = A_2P^2X$$

and using equality (3.12),(3.34) and if we put $X = \xi_1$ and $X = \xi_2$ we obtain

$$(pP - p^2 - q + \sigma)A_2\xi_1 + (2p - P)aA_2\xi_1 + (2p - P)bA_2\xi_2 = 0 \tag{3.49}$$

and

$$(pP - p^2 - q + \sigma)A_2\xi_2 + (2p - P)aA_2\xi_2 + (2p - P)bA_2\xi_1 = 0. \tag{3.50}$$

Adding (3.47) and (3.49), we obtain (3.37).
Adding the relation (3.48) and (3.50), we obtain (3.38).

□

Theorem 3.8. *Let M be a submanifold with codimension $(r \geq 2)$ of a locally metallic Riemannian manifold $\overline{M}$ and structure J is parallel to Levi-Civita connection $\overline{\nabla}$ defined on M (i.e $\overline{\nabla}J = 0$). If ξ_α $(\alpha = 1, 2, 3, , , , r)$ are linearly independent, $T_r(P) = constant$ and M is totally umbilical, then M is totally geodesic.*

Proof. Since

$$\nabla_X(a_{\alpha\beta}) = -u_\alpha(A_\beta X) - u_\beta(A_\alpha X) + \sum_\gamma [l_{\alpha\gamma}(X)a_{\gamma\beta} + l_{\beta\gamma}(X)a_{\alpha\gamma}].$$

Putting $\alpha = \beta$, we have

$$\nabla_X(a_{\alpha\alpha}) = -2u_\alpha(A_\alpha X) + \sum_\gamma [l_{\alpha\gamma}(X)a_{\gamma\alpha} + l_{\alpha\gamma}(X)a_{\alpha\gamma}]. \tag{3.51}$$

Since $a_{\alpha\beta}$ is symmetric and $l_{\alpha\beta}$ is skew-symmetric in α, β, then $\sum_{\alpha\gamma} a_{\alpha\gamma}l_{\alpha\gamma}(X) = 0$.
Since, $T_r(P) = constant$, we have $\sum_\alpha a_{\alpha\alpha} = constant$.
Hence,

$$\sum_\alpha u_\alpha(A_{\alpha X}) = 0$$

$$\sum_\alpha g(X, A_\alpha\xi_\alpha) = 0$$

$$\sum_\alpha A_\alpha\xi_\alpha = 0.$$

Since, ξ_α are linearly independent, then

$$A_\alpha = 0.$$

Hence M is totally geodesic.

□

Theorem 3.9. *Let M be a submanifold with codimension $(r \geq 2)$ of a locally metallic Riemannian manifold $\overline{M}$ and J is parallel to Levi-Civita connection $\overline{\nabla}$ (i.e $\overline{\nabla}J = 0$). If ξ_α $(\alpha = 1, 2, .., r)$ are linearly independent, $\sum_j (\nabla_{e_j} P)e_j = 0$ and $T_r(P) = constant$, then M is minimal.*

Proof. Since,

$$(\nabla_X P)(Y) = \sum_\alpha [g(A_\alpha X, Y)\xi_\alpha + u_\alpha(Y)A_\alpha X].$$

Putting $X = Y = e_j$, we obtain

$$\sum_j (\nabla_{e_j} P)(e_j) = \sum_\alpha [A_\alpha \sum_j u_\alpha(e_j)e_j + \sum_j h_\alpha(e_j, e_j)\xi_\alpha].$$

Using (2.8), we obtain

$$\sum_j (\nabla_{e_j} P)(e_j) = \sum_\alpha [A_\alpha \xi_\alpha + \sum_j h_\alpha(e_j, e_j)\xi_\alpha].$$

Since,

$$T_r(P) = constant,$$

then from Theorem (3.8), we have

$$\sum_\alpha h_\alpha(X, \xi_\alpha) = 0.$$

Therefore,

$$\sum_\alpha g(A_\alpha X, \xi_\alpha) = 0.$$

Then

$$\sum_\alpha A_\alpha \xi_\alpha = 0.$$

Thus,

$$\sum_\alpha \sum_j h_\alpha(e_j, e_j 0)\xi_\alpha = 0.$$

Since, ξ_α are linearly independent, then

$$h_\alpha(e_j, e_j) = 0.$$

Hence, M is minimal. □

Lemma 3.10. *Let M be a submanifold of codimension $(r \geq 2)$ of a locally metallic Riemannian manifold $\overline{M}$. If ξ_α $(\alpha = 1, 2, ..., r)$ are linearly independent, then we have*

$$T_r(P) = -T_r(a_{\alpha\beta}), \tag{3.52}$$

Lemma 3.11. *Let M be a submanifoldof codimension $(r \geq 2)$ of a locally metallic Riemannian manifold $\overline{M}$. If ξ_α $(\alpha = 1, 2, ..., r)$ are linearly independent and $\nabla_X P = 0$, then $T_r(a_{\alpha\beta}) = constant$.*

Proof. Let $\{e_1, e_2, ..., e_n\}$ be an orthogonal basis of T_P and extended e_j $(j = 1, 2, ..n)$ to local vector field E_j which are covariantly constant at $p \in M$.
Then at $p \in M$,

$$\nabla_X T_r(P) = \nabla_X \sum_j g(Pe_j, e_j) \tag{3.53}$$

$$\nabla_X T_r(P) = \{\sum_j g(\nabla_X(PE_j, E_j))\}_P$$

$$\nabla_X T_r(P) = \sum_j [g((\nabla_X P)E_j + P\nabla_X E_j, E_j) + g(PE_j, \nabla_X E_j)]$$

$$\nabla_X T_r(P) = \sum_j ((\nabla_X P)E_j, E_j) + \sum_j g(\nabla_X E_j, PE_j) + \sum_j g(\nabla_X E_j, PE_j)$$

$$\nabla_X T_r(P) = 0.$$

Then,

$$T_r(P) = constant.$$

From Lemma 3.10, we have

$$T_r(a_{\alpha\beta}) = constant.$$

□

4 Example of submanifolds of Metallic Riemannian manifold.

Example 4.1 We consider that ambient space is a $(2a + b)-$ dimensional Euclidean space E^{2a+b} $(a, b \in N$. Let $J : E^{2a+b} \to E^{2a+b}$ be an $(1, 1)$ tensor field defined by

$$J(x^1, .., x^a, y^1, .., y^b, z^1, .., z^b) = (\sigma x^1, .., \sigma x^a, \sigma y^1, .., \sigma y^b, .., (p - \sigma)z^1, .., (p - \sigma)z^b)$$

for every point $(x^1, .., x^a, y^1, .., y^a, z^1, .., z^b) \in E^{2a+b}$, where $\sigma = \frac{p+\sqrt{p^2+4q}}{2}$ and $p - \sigma = \frac{p-\sqrt{p^2+4q}}{2}$ are roots of the equation $x^2 = px + q$.
For $(x^1, .., x^a, y^1, .., y^a, z^1, .., z^b) \in E^{2a+b}$,
we have

$$J^2(x^1, .., x^a, y^1, .., y^a, z^1, .., z^b) = (\sigma^2 x^1, \sigma^2 x^2, .., \sigma^2 x^a, \sigma^2 y^1, .., \sigma^2 y^a, (p-\sigma)^2 z^1, .., (p-\sigma)^2 z^b)$$

$$J^2(x^1, .., x^a, y^1, .., y^a, z^1, .., z^b)$$

$$= (qx^1, .., qx^a, qy^1, .., qy^a, qz^1, .., qz^b) + (p\sigma x^1, .., p\sigma x^a, p\sigma y^1, .., p\sigma y^a, p(p-\sigma)z^1, .., p(p-\sigma)z^b)$$

(4.1) $$J^2 = pJ + qI.$$

Therefore, J is a metallic structure defined on $(E^{2a+b}, \langle\rangle)$ and $(E^{2a+b}, \langle\rangle, J)$ is a metallic Riemannian manifold.
In the following issue, we identify iX with X (where $X \in \chi(E^{2a+b})$). It is obvious that $E^{2a+b} = E^a \times E^a \times E^b$ and in each of spaces E^a, E^a and E^b respectively, we can get a hypersphere

$$S^{a-1}(r_1) = \{(x^1, ..., x^a\}, \sum_{i=1}^{a}(x^i)^2 = r_1^2\},$$

$$S^{a-1}(r_2) = \{(y^1, ..., y^a), \sum_{i=1}^{a}(y^i)^2 = r_2^2\},$$

$$S^{b-1}(r_3) = \{(z^1, ..., z^b\}, \sum_{i=1}^{a}(z^j)^2 = r_3^2$$

respectively, where $r_1^2 + r_2^2 + r_3^2 = R^2$.
We construct the product manifold $S^{a-1}(r_1)\times S^{a-1}(r_2)\times S^{b-1}(r_3)$. Every point of $S^{a-1}(r_1)\times S^{a-1}(r_2) \times S^{b-1}(r_3)$ has the coordinate $(x^1, .., x^a, y^1, .., y^a, z^1, .., z^b) = (x^i, y^i, z^j)$ $(i \in \{1, .., a\}, j\epsilon\{1, .., b\})$ such that:

(4.2) $$\sum_{i=1}^{a}(x^i)^2 + \sum_{i=1}^{a}(y^i)^2 + \sum_{j=1}^{b}(z^j)^2 = R^2.$$

Thus, $S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)$ is a submanifold of codimension 3 in E^{2a+b} and $S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)$ is a submanifold of codimension 2 in $S^{2a+b-1}(R)$ and $S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)$ is a hypersurface in $S^{2a+b-1}(R)$. Therefore, we have

$$S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3) \hookrightarrow S^{2a+b-2}(r) \hookrightarrow S^{2a+b-1}(R) \hookrightarrow E^{2a+b}$$

The tangent space in a point $(x^1, .., x^a, y^1, .., y^a, z^1, .., z^b) = (x^i, y^i, z^j)$ at the product of spheres $S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3)$ is

$$T_(x^1, .., x^a, 0, .., 0, 0, .., 0)S^{a-1}(r_1) \bigoplus T_(0, .., 0, y^1, .., y^a, 0, .., 0)S^{a-1}(r_2)$$

$$\bigoplus T_(0, .., 0, 0, .., 0, z^1, .., z^b)S^{b-1}(r_3).$$

A vector $(X^1, .., X^a)$ from $T_{(x^1,..,x^a)}E^a$ is tangent to $S^{a-1}(r_1)$ if and only if we have

$$\sum_{i=1}^{a} x^i X^i = 0$$

and it can be identified by $(X^1,..,X^a,0,..,0,0,..,0)$ from E^{2a+b}.
A vector $(Y^1,..,Y^a)$ from $T_{(y^1,..,y^a)}E^a$ is tangent to $S^{a-1}(r_2)$ if and only if we have

$$\sum_{i=1}^{a} y^i Y^i = 0$$

and it can be identified by $(0,..,Y^1,..,Y^a,0,..,0)$ from E^{2a+b}.
A vector $(Z^1,..,Z^b)$ from $T_{(z^1,..,z^b)}E^b$ is tangent to $S^{b-1}(r_3)$ if and only if we have

$$\sum_{i=1}^{b} z^i Z^i = 0$$

and it can be identified by $(0,..,0,0,..,0,Z^1,..,Z^b)$ from E^{2a+b}. Consequently, for every point $(x^i,y^i,z^j) \in S^{a-1}(r_1)\times S^{a-1}(r_2)\times S^{b-1}(r_3)$, we have
$(X^1,..,X^a,Y^1,..,Y^a,Z^1,..,Z^b) = (X^i,Y^i,Z^j) \in T_(x^1,..,x^2,y^1,..,y^a,z^1,..,z^b)(S^{a-1}(r_1)\times S^{a-1}(r_2)\times S^{b-1}(r_3))$.
If the above relations are satisfied, we remark that (X^i,Y^i,Z^j) is a tangent vector field at S^{2a+b-1} and from this it follows that

$$T_{(x^i,y^i,z^j)}(S^{a-1}(r_1)\times S^{a-1}(r_2)\times S^{b-1}(r_3)) \subset T_{(x^i,y^i,z^j)}S^{2a+b}(r)$$

for every point $(x^i,y^i,z^j) \in S^{a-1}(r_1)\times S^{a-1}(r_2)\times S^{b-1}(r_3)$. We consider a local orthonormal basis (N_1,N_2,N_3) of $T^{\perp}_{(x^i,y^i,z^j)}(S^{a-1}(r_1)\times S^{a-1}(r_2)\times S^{b-1}(r_3))$ in every point $(x^i,y^i,z^j)\ \epsilon\ S^{a-1}(r_1)\times S^{a-1}(r_2)\times S^{b-1}(r_3)$ given by

$$N_1 = \frac{1}{R}(x^i,y^i,z^j),$$

$$N_2 = \frac{1}{R}(x^i,y^i,-z^j),$$

$$N_3 = \frac{1}{r_3}(\frac{r_2}{r_1}x^i,\frac{-r_1}{r_2},0).$$

From decomposition of $J(N_\alpha)$ $(\alpha \in \{1,2,3\})$ in tangential and normal components at $S^{a-1}(r_1)\times S^{a-1}(r_2)\times S^{b-1}(r_3)$, we obtain

$$J(N_\alpha) = \xi_\alpha + a_{\alpha 1}N_1 + a_{\alpha 2}N_2 + a_{\alpha 3}N_3,$$

where $\alpha \in \{1,2,3\}$.
(i) From $a_{\alpha\beta} = \langle J(N_\alpha),N_\beta\rangle$ $(\alpha,\beta \in \{1,2,3\})$, we obtain

$$a_{11} = a_{22} = \frac{1}{R^2}(\sigma r_1^2 + \sigma r_2^2 + (p-\sigma)r_3^2),$$

$$a_{12} = a_{21} = \frac{1}{R^2}(\sigma r_1^2 + \sigma r_2^2 - (p-\sigma) r_3^2),$$
$$a_{13} = a_{23} = 0 = a_{31} = a_{32},$$
$$a_{33} = \frac{\sigma r_2^2 + \sigma r_1^2}{r_3^2}.$$

Thus, the matrix A $= (a_{\alpha\beta})_3$ is given by

$$\begin{pmatrix} \frac{1}{R^2}(\sigma r_1^2 + \sigma r_2^2 + (p-\sigma) r_3^2) & \frac{1}{R^2}(\sigma r_1^2 + \sigma r_2^2 - (p-\sigma) r_3^2) & 0 \\ \frac{1}{R^2}(\sigma r_1^2 + \sigma r_2^2 - (p-\sigma) r_3^2) & \frac{1}{R^2}(\sigma r_1^2 + \sigma r_2^2 + (p-\sigma) r_3^2) & 0 \\ 0 & 0 & \frac{\sigma r_2^2 + \sigma r_1^2}{r_3^2} \end{pmatrix}. \tag{4.3}$$

$$(ii)\xi_1 = \frac{1}{R^3}[(R^2 - 2(r_1^2 + r_2^2))\sigma x^i, (R^2 - 2(r_1^2 + r_2^2))\sigma y^i, (p-\sigma)(R^2 - 2r_3^2) z^j)], \tag{4.4}$$

$$\xi_2 = \frac{1}{R^3}[(R^2 - 2(r_1^2 + r_2^2))\sigma x^i, (R^2 - 2(r_1^2 + r_2^2))\sigma y^i, -(p-\sigma)(R^2 - 2r_3^2) z^j)], \tag{4.5}$$

$$\xi_3 = [\frac{r_2\sigma}{r_3^2 r_1}(r_3 - r_1^2 - r_2^2) x^i, \frac{r_1\sigma}{r_2 r_3^2}(-r_3 + r_1^2 + r_2^2) y^i, 0]. \tag{4.6}$$

(iii) From $u_\alpha(X) = u(X^i, Y^i, Z^j) = \langle (X^i, Y^i, Z^j), \xi_\alpha \rangle$, we obtain

$$u_1 = \frac{1}{R}(\sigma X^i x^i + \sigma Y^i y^i + (p-\sigma) Z^j z^j), \tag{4.7}$$

$$u_2 = \frac{1}{R}(\sigma X^i x^i + \sigma Y^i y^i - (p-\sigma) Z^j z^j), \tag{4.8}$$

$$u_3(X) = \frac{1}{r_3}(\frac{r_2}{r_1}\sigma X^i x^i - r_1 r_2 \sigma Y^i y^i). \tag{4.9}$$

(iv)

$$P(X) = (\sigma X^i - [\frac{2\sigma}{R^2}(X^i x^i + Y^i y^i) - \frac{r_2\sigma}{r_1 r_3^2}(\frac{r_2}{r_1} X^i x^i - \frac{r_1}{r_2} Y^i y^i) x^i],$$
$$\sigma Y^i - [\frac{2\sigma}{R^2}(X^i x^i + Y^i y^i) - \frac{r_1\sigma}{r_2 r_3^2}(\frac{r_2}{r_1} X^i x^i - \frac{r_1}{r_2} Y^i y^i) y^i],$$
$$(p-\sigma) Z^j - [\frac{2(p-\sigma)}{R^2} Z^j z^j] Z^j). \tag{4.10}$$

Thus, we have $J(T_{(x^i,y^j)}(S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3))) \subseteq (T_{(x^i,y^j)}(S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3))$ and we obtain $(P, \xi_\alpha, u_\alpha, (a_{\alpha\beta}))$ induced structure on $(S^{a-1}(r_1) \times S^{a-1}(r_2) \times S^{b-1}(r_3))$ by the metallic Riemannian structure $(J, \langle\rangle)$ on E^{2a+b}, which is effectively determined by the relations , (4.3), (4.4), (4.5), (4.6), (4.7), (4.8) (4.9)and (4.10).

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