D-Gap Function and Error Bound in Fuzzy Settings

Abdul Faiz Ansari[1], **Mohd Asif**[2] **and Zubair Khan**[3]

[1] *Department of Mathematics and Astronomy*
University of Lucknow, Lucknow, 226007, India
imfaizofficial@gmail.com

[2] *Department of Mathematics & Statistics*
Integral University, Lucknow, 226026, India
asifzakir007@gmail.com

[3] *Department of Applied Mathematics*
Aligarh Muslim University, Aligarh, 202002, India
zubkhan403@gmail.com

Abstract

In this paper, a D-gap function in fuzzy map settings is constructed by using a generalized version of the regularized generalized gap function (RGF). We devoted and focused to study the theory of one of the main tools in optimization, that is, gap function and also the error bound for random generalized variational inequality (RGVIP) is calculated in fuzzy map settings.

Subject Classification:[2010]Primary 49J40; 49J52; Secondary 90C30;
Keywords: Gap function, error bound, fuzzy map.

1 Introduction

Zadeh [11], in 1965, gave a very beautiful concept known to be fuzzy theory. There are many applications of fuzzy theory in pure as well as applied mathematics. The fuzzy logic concept is very useful in control engineering, artificial intelligence, decision theory, optimization problems, and others. Initiated by Chang and Zha [12], many authors have studied the concept of variational inequality problem (VIP) in fuzzy maps settings. In the theory of optimization, a gap function is a very useful tool which can change VIPs into identical equivalent optimization problems. In the past few years, so many authors studied and constructed different varieties of gap functions for different types of variational inequality problems [1, 2, 3, 4, 5, 6, 7, 8, 9, 14, 15, 16, 17, 18, 19, 20, 21, 22].
Fukushima [1] constructed a regularized gap function (RGF) for classical VIPs in 1992, which is further extended and studied by Wu et al [10] in 1993, in a slightly different way. In fuzzy maps environment, the tool, the gap function is first studied with error bound for RGVIP by Khan et al [13].
We wrote RGVIP, in this paper throughout, for random generalized VIP.
In this paper, Let M be a real Hilbert space such that (s.t.) the notation $\|.\|$ denoted the norm on M and $\langle .,. \rangle$ denoted the inner product on M. Let T be a σ-algebra of subsets

of any set δ s.t., (δ, T) be a measurable space. Also consider 2^M, $B(M)$ and $CB(M)$ as the collection of non-empty subsets of M, the collection of Borel σ-fields in M and the collection of all non-empty closed and bounded subsets of M, respectively. Let the collection of all fuzzy sets over M be denoted by F. A map $S : M \longrightarrow F(M)$ is known as fuzzy map on M. Also $S(q)$ or S_q is fuzzy set if map S is a fuzzy map on M and $S_q(r)$ is a characteristic function or a membership function of y in S_q. Suppose a set $V \in F(M)$ and $\alpha \in [0,1]$ s.t., the set $(V)_\alpha = \{q \in M : V(q) \geq \alpha\}$ is known as α-set of M.

2 Preliminaries:

This section provides definitions, formulations and lemmas for our main results.

Definition 2.1. *A fuzzy map $S : \delta \longrightarrow F(M)$ is known to be measurable if $\forall\ \alpha \in (0,1], (S(.))_\alpha : \delta \longrightarrow F(M)$ is a measurable set-valued or multi-valued map.*

Definition 2.2. *A fuzzy map $S : \delta \times M \longrightarrow F(M)$ is known to be random fuzzy set-valued or multi-valued map if $\forall\ q \in M$, fuzzy map $S(., q) : \delta \longrightarrow F(M)$ is measurable.*

From above definitions, we can easily conclude that, random fuzzy maps contain set-valued or multi-valued maps, fuzzy maps and random set-valued or multi-valued maps as particular cases.

Suppose a random fuzzy map $S^* : \delta \times M \longrightarrow F(M)$ satisfying the property,
(C_1): $\exists$ a map $e : M \longrightarrow [0,1]$ s.t., $(S^*_{j,q})_{e(q)} \in CB(M), \forall\ (j,q) \in \delta \times M$.
From the definition of random fuzzy map S^*, define random set-valued or multi-valued map S, $S : \delta \times M \longrightarrow CB(M), (j,q) \in (S^*_{j,q})_{e(q)}, \forall\ (j,q) \in \delta \times M$. S is known to be random set-valued or multi-valued map induced by S^*.
Now for given map $e : M \longrightarrow [0,1]$, a random fuzzy map $S^* : \delta \times M \longrightarrow F(M)$ satisfying the property (C_1), and random operator $m : \delta \times M \longrightarrow M$ with Img$(m)\cap$domain$(\partial\psi)\neq\phi$, consider RGVIP,
to find measurable maps $q, v : \delta \longrightarrow M$ s.t., $\forall\ j \in \delta, r(j) \in M$,

$$S^*_{j,q}(v(j)) \geq e(q(j)), \langle v(j), r(j) - m(j, q(j))\rangle + \psi(r(j)) - \psi(m(j, q(j))) \geq 0, \tag{2.1}$$

where $\psi : M \longrightarrow \mathbb{R}\cup\{+\infty\}$ be a proper, convex and lower semi-continuos function with its effective domain is being closed and $\partial\psi$ is sub-differential of function ψ. The sub-differential of ψ at point $q \in M$ that is $\partial\psi$, is given as, $\partial\psi(q) = \{v \in M : \psi(r) \geq \psi(q) + \langle v, r - q\rangle, \forall\ r \in M\}$ and point $v \in \partial\psi$ is known as sub-gradient of ψ at q.

Definition 2.3. *A random set-valued or multi-valued map $S : \delta \times M \longrightarrow M$ is known as strongly m-monotone if $\exists$ a measurable map $a : \delta \longrightarrow (0, +\infty)$ s.t., $\forall\ j \in \delta, q_i(j) \in M$, and $\forall\ v_i(j) \in S(j, q_i(j))$, $(i = 1, 2.)$,*

$$\langle v_1(j) - v_2(j), m(j, q_1(j)) - m(j, q_2(j))\rangle \geq a(j)\|q_1(j) - q_2(j)\|^2.$$

Definition 2.4. *A random operator $m : \delta \times M \longrightarrow M$ is known as Lipschitz continuos if $\exists\ k : \delta \longrightarrow (0, +\infty)$, a measurable function s.t., $\forall\ q_1(j), q_2(j) \in M, \forall\ j \in \delta$*

$$\|m(j, q_1(j)) - m(j, q_2(j))\| \leq k(j)\|q_1(j) - q_2(j)\|.$$

Definition 2.5. *Any function $g : M \longrightarrow \mathbb{R}$ is called gap function for a VIP if, $g(u) \geq 0, \forall\, u \in M$ and $g(w) = 0$, if and only if, $w \in M$ is the solution of that VIP.*

Khan et al [13] gave the the following generalized RGF $g_\epsilon : \delta \times M \longrightarrow M$ associated with RGVIP (2.1). For a measurable map $\epsilon : \delta \longrightarrow (0, +\infty)$,

$$g_{\epsilon(j)}(q(j)) = max_{r(j)\in M}\Psi_{\epsilon(j)}(q(j), r(j)),$$

$$g_{\epsilon(j)}(q(j)) = max_{r(j)\in M}\{\langle v(j), m(j, q(j)) - r(j)\rangle + \psi(m(j, q(j))) - \psi(r(j)) - \epsilon(j)G(m(j, q(j), r(j))\}. \tag{2.2}$$

This can also be written as,

$$g_{\epsilon(j)}(q(j)) = \{\langle v(j), m(j, q(j)) - \pi_{\epsilon(j)}(q(j))\rangle + \psi(m(j, q(j))) - \psi(\pi_{\epsilon(j)}(q(j))) - \epsilon(j)G(m(j, q(j), \pi_{\epsilon(j)}(q(j))).$$

Where the unique minimiser of $-\Psi_{\epsilon(j)}(q(j), .)$ on M is denoted by $\pi_{\epsilon(j)}(q(j))\ \forall\ j \in \delta$ and the function $G : M^2 = M \times M \longrightarrow \mathbb{R}$ satisfying (P1-5) properties as:
(P1-2) G is continuously differentiable and non-negative on M^2,
(P3) $G(q(j), .)$ is strongly convex uniformly in first slot that is in $q(j)\ \forall\ j \in \delta$ which means $\forall\ j \in \delta, q(j) \in M, \exists$ a measurable function $\theta : \delta \longrightarrow (0, +\infty)$ s.t.,

$$G(q(j), r_1(j)) - G(q(j), r_2(j)) - \theta(j)\|r_1(j) - r_2(j)\|^2 \geq \langle\nabla_2 G(q(j), r_2(j)), r_1(j) - r_2(j)\rangle$$

where $\nabla_2 G(q(j), r(j))$ denotes the partial derivative with respect to $r(j)$;
(P4) $G(q(j), r(j)) = 0$ if and only if $q(j) - r(j) = 0,\ \forall\ j \in \delta$,
(P5) $\nabla_2 G(q(j), .)$ is uniformly Lipschitz continuous, that is, $\forall\ j \in \delta, q(j) \in M, \exists$ a measurable function $\mu : \delta \longrightarrow (0, +\infty)$ s.t.,

$$\|\nabla_2 G(q(j), r_1(j) - \nabla_2 G(q(j), r_2(j)\| \leq \mu(j)\|r_1(j) - r_2(j)\|, r_1(j), r_2(j) \in M.$$

Lemma 2.6. *[13] Let us consider the function G satisfies the properties (P1-4) then $\nabla_2 G(q(j), r(j)) = 0$ if and only if $q(j) - r(j) = 0, \forall\ j \in \delta$.*

Lemma 2.7. *[13] Let us consider the function G satisfies the property (P3) then $\forall\ j \in \delta, r_1(j), r_2(j) \in M$,*

$$\langle\nabla_2 G(q(j), r_1(j)) - \nabla_2 G(q(j), r_2(j)), r_1(j) - r_2(j)\rangle - 2\theta(j)\|r_1(j) - r_2(j)\|^2 \geq 0.$$

Lemma 2.8. *[13] Let us consider the function G satisfies the properties (P1-5) then, $G(q(j), r(j)) \leq (\mu(j) - \theta(j))\|q(j) - r(j)\|^2, \forall j \in \delta, q(j), r(j) \in M$.*

Lemma 2.9. *[13] Let us consider the function G satisfies the properties (P1-4) then a measurable map $q : \delta \longrightarrow M$ solves RGVIP (2.1) if and only if, $m(j, q(j)) = \pi_{\epsilon(j)}(q(j))$, $\forall\ j \in \delta$.*

3 Main Result (D-Gap Function and Error Bound):

Let the two measurable functions $\epsilon_1, \epsilon_2 : \delta \longrightarrow (0, +\infty)$ s.t., $\epsilon_1(j) > \epsilon_2(j) > 0, \forall\ j \in \delta$, then the D-gap function [5, 6] is a difference of two generalized RGF, that is, $g_{\epsilon_1(j)}(q(j)) - g_{\epsilon_2(j)}(q(j))$. Therefore the D-gap function for RGVIP (2.1) with the help of (2.2) is given by,

$$(3.1)\quad \begin{aligned} D_{\epsilon_1(j),\epsilon_2(j)}(q(j)) = max_{r(j)\in M}\{\langle v(j), m(j,q(j)) - r(j)\rangle + \psi(m(j,q(j)) - \psi(r(j)) \\ + \epsilon_2(j)G(m(j,q(j)), r(j)) - \epsilon_1(j)G(m(j,q(j),r(j)))\}. \end{aligned}$$

Which can also be written as,

$$\begin{aligned} D_{\epsilon_1(j),\epsilon_2(j)}(q(j)) = \langle v(j), \pi_{\epsilon_2(j)}(q(j)) - \pi_{\epsilon_1(j)}(q(j))\rangle + \psi(\pi_{\epsilon_2(j)}(q(j)) - \psi(\pi_{\epsilon_1(j)}(q(j)) \\ + \epsilon_2(j)G(m(j,q(j)), \pi_{\epsilon_2(j)}(q(j))) - \epsilon_1(j)G(m(j,q(j),\pi_{\epsilon_1(j)}(q(j))). \end{aligned}$$

Theorem 3.1. *Let the function G satisfies the properties (P1-4) then $\forall\ q(j) \in M, j \in \delta$,*

1. $D_{\epsilon_1(j),\epsilon_2(j)}(q(j)) \geq 2\epsilon_1(j)(\theta(j) - \mu(j))\|m(j,q(j)) - \pi_{\epsilon_1(j)}(q(j))\|^2$,
2. $D_{\epsilon_1(j),\epsilon_2(j)}(q(j)) \leq \epsilon_2(j)(\mu(j) - \theta(j))\|m(j,q(j)) - \pi_{\epsilon_1(j)}(q(j))\|^2$,
3. $D_{\epsilon_1(j),\epsilon_2(j)}(q(j)) \leq 2\epsilon_2(j)\mu(j)\|m(j,q(j)) - \pi_{\epsilon_2(j)}(q(j))\|^2$
$+ \epsilon_2(j)\mu(j)\|m(j,q(j)) - \pi_{\epsilon_1(j)}(q(j))\|\|m(j,q(j)) - \pi_{\epsilon_2(j)}(q(j))\|$.

Proof. Let $\forall\ q(j) \in M, j \in \delta$, we know that the unique minimiser of $-\Psi_{\epsilon_1(j)}(q(j), .)$ on M is denoted by $\pi_{\epsilon_1(j)}(q(j))$ and also $-\Psi_{\epsilon_1(j)}(q(j), .)$ is convex, then
$\forall\ r(j) \in M, j \in \delta$, we have

$$0 \in \Psi_{\epsilon_1(j)}(q(j), r(j)) = v(j) + \partial\psi(\pi_{\epsilon_1(j)}(q(j))) + \epsilon_1(j)\nabla_2 G(m(j,q(j),\pi_{\epsilon_1(j)}(q(j))),$$

equivalently,

$$-v(j) - \epsilon_1(j)\nabla_2 G(m(j,q(j),\pi_{\epsilon_1(j)}(q(j))) \in \partial\psi(\pi_{\epsilon_1(j)}(q(j))).$$

By definition of sub-gradient, we have,

$$\psi(r(j)) \geq \psi(\pi_{\epsilon_1(j)}(q(j))) - \langle v(j) + \epsilon_1(j)\nabla_2 G(m(j,q(j),\pi_{\epsilon_1(j)}(q(j))), r(j) - \pi_{\epsilon_1(j)}(q(j))\rangle,$$

that is,

$$\begin{aligned} \langle v(j), r(j) - \pi_{\epsilon_1(j)}(q(j))\rangle + \psi(r(j)) - \psi(\pi_{\epsilon_1(j)}(q(j))) \\ \geq \epsilon_1(j)\langle -\nabla_2 G(m(j,q(j)), \pi_{\epsilon_1(j)}(q(j))), r(j) - \pi_{\epsilon_1(j)}(q(j))\rangle, \end{aligned}$$

now put $r(j) = \pi_{\epsilon_2(j)}(q(j))$, so the above inequality can be written in term of $D_{\epsilon_1(j),\epsilon_2(j)}(q(j))$ as,

$$\begin{aligned} D_{\epsilon_1(j),\epsilon_2(j)}(q(j)) \geq \epsilon_1(j)\langle -\nabla_2 G(m(j,q(j)), \pi_{\epsilon_1(j)}q(j)), \pi_{\epsilon_2(j)}(q(j)) - \pi_{\epsilon_1(j)}(q(j))\rangle \\ + \epsilon_2(j)G(m(j,q(j)), \pi_{\epsilon_2(j)}(q(j))) - \epsilon_1(j)G(m(j,q(j)), \pi_{\epsilon_1(j)}(q(j))), \end{aligned}$$

by using (P3) of G, we have,

$$D_{\epsilon_1(j),\epsilon_2(j)}(q(j)) \geq (\epsilon_1(j)+\epsilon_2(j))G(m(j,q(j)),\pi_{\epsilon_2(j)}(q(j))) - 2\epsilon_1(j)G(m(j,q(j),\pi_{\epsilon_2(j)}(q(j))),$$

using lemma 2.8, so that first result follows.
On other hand, in similar way, we have

$$-v(j) - \epsilon_2(j)\nabla_2 G(m(j,q(j),\pi_{\epsilon_2(j)}(q(j))) \in \partial\psi(\pi_{\epsilon_2(j)}(q(j))).$$

by definition of sub-gradient,

$$\psi(r(j)) \geq \psi(\pi_{\epsilon_2(j)}(q(j))) - \langle v(j) + \epsilon_2(j)\nabla_2 G(m(j,q(j)),\pi_{\epsilon_2(j)}(q(j))), r(j) - \pi_{\epsilon_2(j)}(q(j))\rangle,$$

now put $r(j) = \pi_{\epsilon_1(j)}(q(j))$, the above inequality can be written as,

$$\langle v(j), \pi_{\epsilon_2(j)}(q(j)) - \pi_{\epsilon_1(j)}(q(j))\rangle + \psi(\pi_{\epsilon_2(j)}(q(j))) - \psi(\pi_{\epsilon_1(j)}(q(j))) \leq \epsilon_2(j)\langle \nabla_2 G(m(j,q(j)),\psi(\pi_{\epsilon_2(j)}(q(j)))), \psi(\pi_{\epsilon_1(j)}(q(j))) - \psi(\pi_{\epsilon_2(j)}(q(j)))\rangle,$$

so the above inequality can be written in term of $D_{\epsilon_1(j),\epsilon_2(j)}(q(j))$ as

$$\begin{aligned}(3.2)\quad D_{\epsilon_1(j),\epsilon_2(j)}(q(j)) &\leq \epsilon_2(j)\langle \nabla_2 G(m(j,q(j)),\psi(\pi_{\epsilon_2(j)}(q(j)))), \psi(\pi_{\epsilon_1(j)}(q(j))) - \psi(\pi_{\epsilon_2(j)}(q(j)))\rangle \\ &\quad + \epsilon_2(j)G(m(j,q(j)),\pi_{\epsilon_2(j)}(q(j))) - \epsilon_1(j)G(m(j,q(j)),\pi_{\epsilon_1(j)}(q(j))).\end{aligned}$$

Using (P3)

$$\begin{aligned}D_{\epsilon_1(j),\epsilon_2(j)}(q(j)) &\leq \epsilon_2(j)G(m(j,q(j)),\pi_{\epsilon_1(j)}(q(j))) - \epsilon_2(j)G(m(j,q(j)),\pi_{\epsilon_2(j)}(q(j))) \\ &\quad - \theta(j)\epsilon_2(j)||\pi_{\epsilon_1(j)}(q(j)) - \pi_{\epsilon_2(j)}(q(j))||^2 + \epsilon_2(j)G(m(j,q(j)),\pi_{\epsilon_2(j)}(q(j))) \\ &\quad - \epsilon_1(j)G(m(j,q(j)),\pi_{\epsilon_1(j)}(q(j))),\end{aligned}$$

the second result follows by using lemma 2.8.
Now again using inequality 3.2, we have,

$$\begin{aligned}D_{\epsilon_1(j),\epsilon_2(j)}(q(j)) &\leq \epsilon_2(j)\|\nabla_2 G(m(j,q(j)),\psi(\pi_{\epsilon_2(j)}(q(j))))\|\|\psi(\pi_{\epsilon_1(j)}(q(j))) - \psi(\pi_{\epsilon_2(j)}(q(j)))\| \\ &\quad + \epsilon_2(j)G(m(j,q(j)),\pi_{\epsilon_2(j)}(q(j))).\end{aligned}$$

By (P5) of G, we have $\|\nabla_2 G(q(j),q(j)) - \nabla_2 G(q(j),r(j))\| \leq \mu(j)\|q(j) - r(j)\|$ or $\|\nabla_2 G(q(j),r(j))\| \leq \mu(j)\|q(j) - r(j)\|$ (by lemma 2.6), also by lemma 2.8 and triangle inequality, so that the third result follows. □

Also we can easily verify that $D_{\epsilon_1(j),\epsilon_2(j)}(q(j)) = 0$ if and only if $q(j)$ is the solution of RGVIP (2.1) by using above theorem 3.1 and lemma 2.9.
Now we will evaluate error bound.

Theorem 3.2. *Let $q_0(j) \in M$ be the solution of RGVIP (2.1), $\forall\, j \in \delta$, also $m : \delta \times M \longrightarrow M$ be Lipschitz continous function and a random set-valued or multi-valued map S be strongly m-monotone with measurable function $k : \delta \longrightarrow (0, +\infty)$ and $a : \delta \longrightarrow (0, +\infty)$ respectively. If G satisfies the properties (P1-5) then,*

$$\|q(j) - q_0(j)\| \leq \sqrt{\frac{D_{\epsilon_1(j),\epsilon_2(j)}(q(j))}{a(j) + (\epsilon_1(j) - \epsilon_2(j))(\theta(j) - \mu(j))k^2(j)}}.$$

Proof. The D-gap function (3.1) can be written as,

$$D_{\epsilon_1(j),\epsilon_2(j)}(q(j)) \geq \langle v(j), m(j, q(j)) - m(j, q_0(j)) + \psi(m(j, q(j))) - \psi(m(j, q_0(j))) \\ + \epsilon_2(j)G(m(j, q(j)), m(j, q_0(j))) - \epsilon_1(j)G(m(j, q(j)), m(j, q_0(j))),$$

which is written as,

$$D_{\epsilon_1(j),\epsilon_2(j)}(q(j)) \geq \langle v_0(j), m(j, q(j)) - m(j, q_0(j)) + \psi(m(j, q(j))) - \psi(m(j, q_0(j)))\rangle \\ + \langle v(j) - v_0(j), m(j, q(j)) - m(j, q_0(j))\rangle - (\epsilon_1(j) - \epsilon_2(j)G(m(j, q(j)), m(j, q_0(j))).$$

Since the solution of RGVIP (2.1) is $q_0(j) \in M$ then

$$D_{\epsilon_1(j),\epsilon_2(j)}(q(j)) \geq \langle v(j) - v_0(j), m(j, q(j)) - m(j, q_0(j))\rangle \\ - (\epsilon_1(j) - \epsilon_2(j))G(m(j, q(j)), m(j, q_0(j))).$$

By Lipschitz continuity of m, lemma 2.8 and using strongly m-monotonicity, then

$$D_{\epsilon_1(j),\epsilon_2(j)}(q(j)) \geq a(j)\|q(j) - q_0(j)\|^2 - (\epsilon_1(j) - \epsilon_2(j))(\theta(j) - \mu(j))k^2(j)\|q(j) - q_0(j)\|^2,$$

so that result follows,

$$\|q(j) - q_0(j)\| \leq \sqrt{\frac{D_{\epsilon_1(j),\epsilon_2(j)}(q(j))}{a(j) + (\epsilon_1(j) - \epsilon_2(j))(\theta(j) - \mu(j))k^2(j)}}.$$

□

4 Conclusion:

Our focus in this paper is to study the D-gap function for RGVIP in fuzzy maps setting and then we also derived a result for error bound by the D-gap function because it is one of the very important and useful applications of D-gap function. If we put $\epsilon_2 = 0$ in the D-gap function (3.1) then we get the generalized version of RGF (2.2) for RGVIP (2.1) which is studied by Khan et al [13].

Acknowledgment

The first author was supported by UGC's MANF-JRF (201819- MANF-2018-19-UTT-100556), India, and the second author was supported by UGC's NFOBC-SRF (201819-NFO-2018-19-OBC-UTT-71260), India.

References

[1] Fukushima M. (1992): Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems, Math. Program., 53, 99-110 .

[2] Aussel D., Gupta R. and Mehra A. (2013): Gap function and error bound for inverse quasi-variational inequality problems, J. Math. Anal. Appl., 407, 270–280.

[3] Haung L. R. and Ng K. F. (2005): Equivalent optimization formulations and error bound for variational inequality problem, J. Optim Theory Appl., 125, 299 – 314.

[4] Khan S. A., Iqbal J. and Shehu Y. (2015): Mixed quasi-variational inequalities involving error bounds, J. Inequal. Appl., 2015, 417.

[5] Solodov M. V. and Tseng P. (2000): Some methods based on the D-gap function for solving monotone variational inequalities, Comput. Optim. Appl., 17, 255-277.

[6] Solodov M. V. (2003): Merit functions and error bounds for generalized variational inequalities, 287, 405, 2003., J. Math. Anal. Appl., 287, 405 – 414.

[7] Tang G. J. and Haung N. J. (2013): Gap functions and global error bounds for set-valued miqed variational inequalities, Taiwan. J. Math., 17, 1267 – 1286.

[8] Yamashita N.and Fukushima M. (1997): Equivalent unconstraint minimization and global error bounds for variational inequality problems, SIAM J. Control Optim., 35, 273 – 284.

[9] Yamashita N., Taji K. and Fukushima M. (1997): Unconstrained optimization reformulations of variational inequality problems , J. Optim. Theory Appl., 92, 439 – 456.

[10] Wu J. H., Florian M. and Marcotte P. (1993): A general descent framework for the monotone variational inequality problem, Math. Program., 61, 281 – 300.

[11] Zadeh L. A. (1965): Fuzzy sets 8, 338, 1965., Inf. Control, 8, 338 – 353.

[12] Chang S. S. and Zhu Y. G. (1989): On variational inequalities for fuzzy mappings, Fuzzy Sets Syst., 32, 359 – 367.

[13] Khan S. A., Iqbal J. and Shehu Y. (2016): Gap functions and error bound for random generalized variational inequality problems, J. Inqual. Appl., 2016, 47-63.

[14] Khan Z. and Ansari A. F. (2020): Gap function with error bound for random generalized variational-like inequlaity problems, J. Sci. Arts, 53, 917-930 .

[15] Aussel D. and Dutta J. (2011): On gap function for multivalued Stampacchia variational inequalities, J. Optim. Theory Appl., 149, 513–527.

[16] Gupta R. and Mehra A. (2012): Gap functions and error bounds for quasi variational inequalities, J. Glob. Optim., 53, 737–748.

[17] Khan S. A. and Chen J. W. (2015): Gap function and global error bounds for generalized mixed quasi variational inequalities, Appl. Math. Comput., 260, 71–81.

[18] Noor M. A. (2006): Merit functions for general variational inequalities, J. Math. Anal. Appl., 316 , 736–752.

[19] Qu B.,Wang C. Y. and Zhang J. Z. (2003): Convergence and error bound of a method for solving variational inequality problems via the generalized D-gap function, J.Optim. Theory Appl., 119, 535–552.

[20] Asif M., Khan Z., babu F. and Ansari A.F. (2022): Generalized regularized gap functions and error bounds for generalized vector variational-like inequalities, Appl. Set-Valued Anal. Optim., 4(1), 83-93.

[21] Dar A. H. and Ahmad M. K. (2018): Gap Functions and Error Bounds for Random Extended Generalized Variational Inequality Problem, Bull. Math. Anal. Appl., 10(3), 10-24.

[22] Dar A. H., Sarfaraz M. and Ahmad M. K. (2016): Gap Functions and Error Bounds for Generalized Set-Valued Mixed Variational Inequality, Indian Society of Industrial and Applied Mathematics, 7(2), 117-126.