Couette flow in a composite porous cylindrical channel of variable permeability

*Sanjeeva Kumar Singh [1] and Vineet Kumar Verma [2]

[1] *Department of Mathematics*
Government Polytechnic, Ayodhya
sanjeevakumars1@gmail.com

[2] *Department of Mathematics and Astronomy*
University of Lucknow, Lucknow
vinlkouniv@gmail.com

Abstract

In the present article we have consider the steady flow of a viscous incompressible fluid in a composite porous cylindrical channel. Inner and outer part of the cylindrical channel are of different permeability. The porous channel consist of two regions. The inner porous cylinder is of variable permeability which is covered by outer porous layer of uniform permeability k_0. We have consider two cases of permeability variation of the inner porous cylinder; (i) quadratic variation, $k = k_0 r^2$ and (ii) linear variation, $k = k_0 r$. Important and useful flow namely; Couette-Poiseuille flow is investigated and exact analytical expressions for the velocity, rate of volume flow, average velocity and shear stress on the impermeable boundary are obtained and exhibited graphically. Effect of permeability parameter and gap parameter on the flow characteristics has been discussed. It is found that these parameters have very strong effect on the flow.

Subject Classification:[2020] 76D03, 76S05
Keywords: Heterogeneous porous medium, Brinkman equation, Modified Bessel function, Couette flow.

1 Introduction

A porous medium usually consists of a large number of interconnected voids (pores), each of which is saturated with the fluid. The flow through porous medium has gained impetus in recent years because of their natural occurrence. Flow in a porous media is an ordered flow in a disordered geometry and therefore, has been an area of intensive investigation. Vafai and Kim (1989) found exact solution for forced convection in a channel filled with porous medium. Al-Hadhrami et al. (2001) have presented fully developed flow in both rectangular and circular cross-sectional domains using Darcy model. They found analytical solutions for different cases and compared them and discussed the effect of the Darcy parameter. Pantokratoras and Fang (2010) investigate the fully developed flow in a fluid-saturated porous medium channel with an electrically conducting fluid under the action of a parallel

Lorentz force. The Lorentz force varies exponentially in the vertical direction due to low fluid electrical conductivity and the special arrangement of the magnetic and electric fields at the lower plate. They found exact analytical solutions for fluid velocity and effect of various parameters on the flow has been investigated. Chauhan and Kumar (2011) studied that heat transfer effects in a 3-D Couttue flow through a composite channel parallel porous plate channel partly filled by a porous medium. The flow in 3-D in the channel because of the application of a transverse sinusoidal injection velocity of a particular form at the lower stationary plate. They solved the governing equations using perturbation series expansion method. They found solutions effects for relevant quantities. Verma and Datta (2012) found analytical solution for fully developed laminar flow of a viscous incompressible fluid in an annular region between two coaxial cylindrical tubes filled with a porous medium of variable permeability when the permeability of the porous medium varies with the radial distance. Wang (2010) investigated a fully developed flow and constant flux heat transfer in super-elliptic ducts filled with a Darcy Brinkman porous medium. He used Ritz method to determine the velocity and temperature fields. Wang and Yu (2013) investigated for forced flow through a channel with bumpy walls which sandwich a porous medium. The problem models micro-fluidics where, due to the small size of the channel width, the surface roughness of the walls is amplified. They found problem depends heavily on the non-dimensional porous medium parameter k which represents the importance of length scale to the square root of permeability and solutions reduce to the clear fluid limit when k is zero and to the Darcy limit when k approaches infinity. Verma and Singh (2014) studied laminar flow of a viscous incompressible fluid in an annular region between two infinitely long coaxial circular cylinders filled by a porous medium of variable permeability. They presented permeability of the porous medium varies with the radial distance and flow within the porous annular region is governed by the Brinkman law. They found analytical solutions for relevant quantities. Verma and Singh (2020) studied steady flow of viscous, incompressible fluid in a composite cylindrical channel. Inner and outer part of the cylinder is of different permeability. They used Brinkman equation as governing equation of motion in the porous cylinder and obtained analytical expressions for the two important cases, Poiseuille and Couette flow. They found analytical solutions for relevant quantities and they discussed about flow characteristics.

In the present paper we have considered steady flow of a viscous, incompressible fluid in a porous composite cylindrical channel. We have been considered two cases (i) permeability of inner part is $k = k_0 r^2$ and of outer part is $k = k_0$ and (ii) permeability of inner part is $k = k_0 r$ and of outer part is $k = k_0$. Analytical solutions are obtained by using Brinkman equation for the Couette flow.

Flow characteristics such as velocity, rate of volume flow, average velocity and shear stress on the boundaries are obtained. The effect various parameters on flow are discussed and presented graphically.

2 Mathematical Formulation:

In the present problem we consider steady flow of viscous, incompressible fluid flow in a porous cylinder of radius a which is embedded in another porous cylinder of radius b $(b > a)$ as shown in Fig.(1). Permeability of region I is quadratic $(k = k_0 r^2)$ and linear $(k = k_0 r)$ and permeability of region II is constant $(k = k_0)$ and both regions have common pressure gradient. The Brinkman (1947) momentum equation for a fully developed flow

$$\mu_e \left(\frac{d^2 u^*}{dr^{*2}} + \frac{1}{r^*} \frac{du^*}{dr^*} \right) - \frac{\mu}{k} u^* = \frac{\partial p^*}{\partial z^*} \tag{2.1}$$

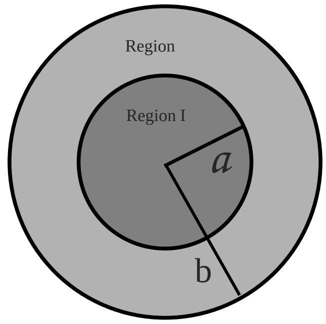

Fig. 1: Cross-sections of the porous channel.

where u^* is the fluid velocity, μ_e is the effective viscosity, μ is the fluid viscosity, k is the permeability of the porous medium and $\partial p^*/\partial z^*$ is the constant applied pressure gradient. We follow Brinkman (1947) and Chikh et al. (1995) and assume that $\mu_e = \mu$ (for high porosity cases). Therefore, Eq.(2.1) becomes

$$\frac{d^2u^*}{dr^{*2}} + \frac{1}{r^*}\frac{du^*}{dr^*} - \frac{u^*}{k} = \frac{1}{\mu}\frac{\partial p^*}{\partial z^*} \tag{2.2}$$

Now we introduce dimensionless variables as follows

$$r = \frac{r^*}{a} \quad \text{and} \quad u = \frac{\mu u^*}{a^2(-\partial p^*/\partial z^*)}$$

the characteristic velocity being determined by $\frac{a^2}{\mu}(-\partial p^*/\partial z^*)$.

Using the above dimensionless variables in Eq.(2.2) and after dropping the star index for our convenience, we get governing Brinkman equation of motion as

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{a^2}{k}u = -1 \tag{2.3}$$

Thus for inner porous cylinder of permeability $k_0 r^2$, the governing equation of motion is

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} - \alpha^2 u = -r^2; \qquad (0 \le r \le 1) \tag{2.4}$$

and again, for inner porous cylinder of permeability $k_0 r$, the governing equation of motion is

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} - \alpha^2 u = -r; \qquad (0 \le r \le 1) \tag{2.5}$$

where $a^2/k_{0.}= \alpha^2$ is called permeability variation parameter for region I and u is the velocity in region I.

Similarly, for outer porous cylinder is of permeability k_0, the governing equation of motion is

$$r\frac{d^2v}{dr^2}+\frac{dv}{dr}-\alpha^2 rv=-r; \qquad (1\leq r\leq q=b/a) \tag{2.6}$$

where $a^2/k_0=\alpha^2$ is called permeability variation parameter for region II and v is the velocity in region II. Eqs.(2.4), (2.5) and (2.6) are modified Bessel's equations of order zero.

3 Solution and Results:

Now we consider two different cases (i) when permeability of region I is k_0r^2 and permeability of region II is k_0 (ii) when permeability of region I is k_0r and permeability of region II is k_0.

Case I

When permeability of region I is k_0r^2 and permeability of region II is k_0.

For Couette-Poiseuille flow the outer cylinder is moving with the constant velocity $U=\frac{a^2}{u}\left(\frac{\partial p^*}{\partial z^*}\right)$ with fix pressure gradient and the inner cylinder has been kept fixed. The no slip conditions at boundaries are

$$\begin{aligned} v(r) &= 1 && \text{at } r=q\\ u(r) &= v(r) && \text{at } r=1\\ u^{'}(r) &= v^{'}(r) && \text{at } r=1\\ u^{'}(r) &= 0 && \text{at } r=0 \end{aligned} \tag{3.1}$$

where, $q=\left(\frac{b}{a}\right)$. The general solutions of Eqs.(2.4) and (2.6) are given by

$$u(r)=A_1r^\alpha+\frac{A_2}{r^\alpha}-\frac{r^2}{(4-\alpha^2)}; \qquad \text{for } \alpha\neq 2,\quad 0\leq r\leq 1 \tag{3.2}$$

$$\begin{aligned} u(r) &= A_1\cosh(2\log r)+A_2\sinh(2\log r)\\ &\quad+\frac{1}{16}[-r^4\sinh(2\log r)+r^4\cosh(2\log r)\\ &\quad-4\log r\sinh(2\log r)-4\log r\cosh(2\log r)]; \qquad \text{for } \alpha=2 \end{aligned}$$

and

$$v(r)=B_1I_0(\alpha r)+B_2K_0(\alpha r)+\frac{1}{\alpha^2}; \quad 1\leq r\leq q \tag{3.3}$$

where, I_o and K_o are the modified Bessel functions of zeroth order of first and second kind respectively. Here A_1, A_2, B_1 and B_2 are constants of integration determined by using above boundary conditions (3.1), we get constants as

$$
\begin{aligned}
A_1 &= \frac{1}{\Delta}\Big[\frac{I_1(\alpha)\{\left(\alpha^4-5\alpha^2+4\right)K_0(\alpha)-4K_0(q\alpha)\}}{(\alpha-2)}-\frac{2\{\alpha K_0(\alpha)+2K_1(\alpha)\}I_0(q\alpha)}{(\alpha-2)}\\
&\quad+\frac{I_0(\alpha)\{\left(\alpha^4-5\alpha^2+4\right)K_1(\alpha)+2\alpha K_0(q\alpha)\}}{(\alpha-2)}\Big]\\
A_2 &= 0\\
B_1 &= \frac{1}{\Delta}\left[\left(\alpha^3+2\alpha^2-\alpha-2\right)K_0(\alpha)-\left(-\alpha^3-2\alpha^2+\alpha+2\right)K_1(\alpha)+2K_0(q\alpha)\right]\\
&\text{and}\\
B_2 &= \frac{1}{\Delta}\left[\left(-\alpha^3-2\alpha^2+\alpha+2\right)I_0(\alpha)+\left(\alpha^3+2\alpha^2-\alpha-2\right)I_1(\alpha)-2I_0(q\alpha)\right]
\end{aligned}
\tag{3.4}
$$

where, $\Delta=\alpha^2(\alpha+2)[\{I_1(\alpha)-I_0(\alpha)\}K_0(q\alpha)+\{K_0(\alpha)+K_1(\alpha)\}I_0(q\alpha)]$

The dimensionless velocities of the fluid at any point within the region I and II (inner and outer) when permeability of the medium is k_0r^2 and k_0 respectively given by Eqs.(3.2) and (3.3) on insertion of the preceding values of A_1, A_2, B_1 and B_2. The graphical presentation of velocity profiles for different α is given in Fig.(2).

In the limiting case, when $\alpha\to 0$ (i.e., when permeability of the medium is infinite) in Eqs.(3.2) and (3.3), the classical velocity, u_0 (inner cylinder) and v_0 (outer cylinder) for clear fluid flow is obtained as

$$\lim_{\alpha\to 0} u=\lim_{\alpha\to 0} v=\frac{\left(q^2-r^2+4\right)}{4} \tag{3.5}$$

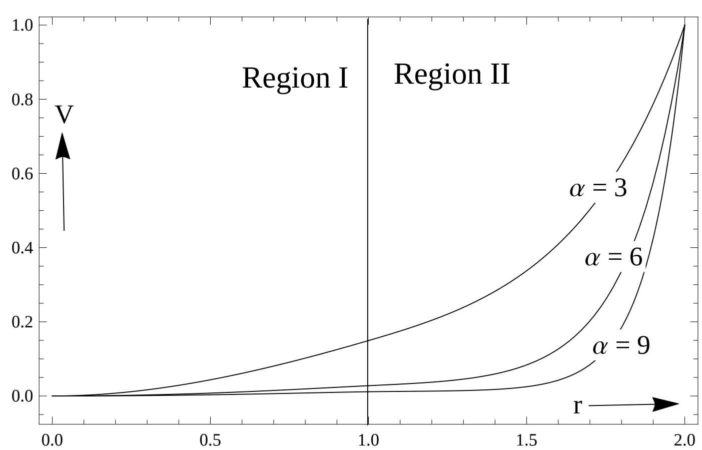

Fig. 2: Variation of velocity with radial distance r for different α for fixed $q=2$.

3.0.1 Rate of volume flow:

The dimensionless rate of volume flow through cross-section of the inner cylinder is given by

$$Q_1 = 2\pi \int_0^1 u(r) r dr \tag{3.6}$$

Substituting $u(r)$ from Eq.(3.2) and after integration [Ref. Abramowitz and Stegun (1972)], we obtain

$$Q_1 = \frac{\pi\left[4A_1(\alpha-2)+1\right]}{2(\alpha^2-4)} \tag{3.7}$$

Similarly, the dimensionless rate of volume flow through cross-section of the outer cylinder is given by

$$Q_2 = 2\pi \int_1^q v(r) r \ dr \tag{3.8}$$

Substituting $v(r)$ from Eq.(3.3) and after integration [Ref. Abramowitz and Stegun (1972)], we obtain

$$Q_2 = \frac{\pi}{\alpha^2}\left[2\alpha\{-B_1 I_1(\alpha) + B_1 q I_1(q\alpha) + B_2(K_1(\alpha) - qK_1(q\alpha))\} + q^2 - 1\right] \tag{3.9}$$

where I_1 and K_1 are the modified Bessel functions of first kind of order one and A_1, A_2, B_1 and B_2. are given by Eq.(3.4). In the evaluation of above integrals the following identity [Ref. Abramowitz and Stegun (1970)] has been used

$$\left(\frac{1}{z}\frac{d}{dz}\right)^m \{z^\nu \pounds_\nu(z)\} = z^{\nu-m}\pounds_{\nu-m}(z) \tag{3.10}$$

with $m = 1$ and $\nu = 1$. $\pounds_\nu$ denotes I_ν and $e^{\nu\pi i}K_\nu$.

The dimensionless rate of volume flow is given by

$$Q = Q_1 + Q_2$$

or,

$$Q = 2\pi\left(\int_0^1 u(r) r dr + \int_1^q v(r) r dr\right) \tag{3.11}$$

Hence, total dimensionless rate of volume flow is

$$\begin{aligned} Q &= \frac{\pi}{2\alpha^2(\alpha^2-4)}[\{4(\alpha-2)A_1+1\} + 2(\alpha^2-4)\{-2\alpha B_1 I_1(\alpha) + 2\alpha B_1 q I_1(q\alpha) \\ &\quad +2\alpha B_2(K_1(\alpha) - qK_1(q\alpha)) + q^2 - 1\}] \end{aligned} \tag{3.12}$$

The dimensionless volume flow rate Q_0 for clear fluid flow (when permeability is infinite) can be obtained by taking limit $\alpha \to 0$ in Eq.(3.12). We get

$$Q_0 = \lim_{\alpha\to 0} Q = \frac{\pi q^2}{8}\left(q^2 + 8\right) \tag{3.13}$$

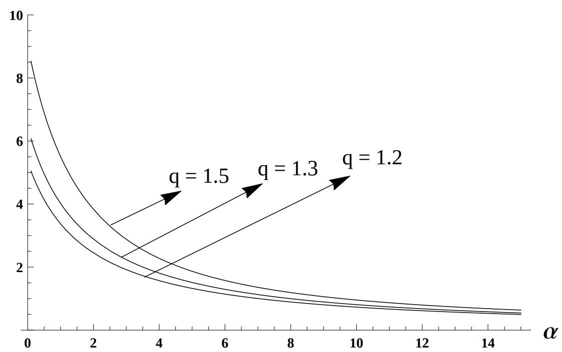

Fig. 3: Variation of volume flow rate with α for different values of q.

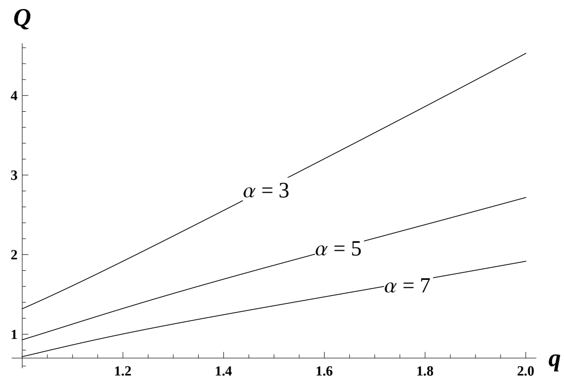

Fig. 4: Variation of volume flow rate with q for different values of α.

3.0.2 Average velocity

The dimensionless average velocity of the flow is defined as

$$u_{avg} = \frac{Q}{\pi q^2} \tag{3.14}$$

Substituting Q from the Eq.(3.12) into preceding equation, the average velocity of the flow with permeability of the porous medium is obtained as

$$u_{avg} = \frac{1}{2q^2\alpha^2(\alpha^2-4)}[\{4(\alpha-2)A_1+1\}+2(\alpha^2-4)\{-2\alpha B_1 I_1(\alpha)+2\alpha B_1 q I_1(q\alpha) +2\alpha B_2(K_1(\alpha)-qK_1(q\alpha))+q^2-1\}] \tag{3.15}$$

For clear fluid flow average velocity of the flow is obtained by taking limit $\alpha \to 0$ in Eq.(3.15). Hence

$$\lim_{\alpha\to 0} u_{avg} = \frac{\left(q^2+8\right)}{8} \tag{3.16}$$

3.0.3 Shearing stress on the surface of pipe

The dimensionless shearing stress at any point of inner cylinder is given by,

$$\tau_{rz}(r) = -\frac{du}{dr} \tag{3.17}$$

Substituting u from Eq.(3.2) and and differentiating the modified Bessel functions $I_o(\alpha r)$ and $K_o(\alpha r)$ with use of the identity $\frac{d}{dr}I_o(r) = I_1(r)$ and $\frac{d}{dr}K_o(r) = -K_1(r)$ [Ref. Abramowitz and Stegun (1970)], we obtain

$$\tau_{rz}(r) = \frac{2r}{4-\alpha^2} - \alpha A_1 r^{\alpha-1} \tag{3.18}$$

Similarly, the dimensionless shearing stress at any point of outer cylinder is given by,

$$\tau_{rz}(r) = -\frac{dv}{dr} \tag{3.19}$$

Substituting v from Eq.(3.3) in above equation and after differentiation we get,

$$\tau_{rz}(r) = -\alpha[B_1 I_1(\alpha r) - B_2 K_1(\alpha r)] \tag{3.20}$$

where I_1 and K_1 are modified Bessel function of order one. Shear stress on the surface of inner and outer cylinder is obtained by putting $r = 1$ and $r = q$ in Eqs.(3.18) and (3.20), respectively and using the appropriate sign. Thus, providing

$$\tau_{rz}(1) = \frac{\alpha^3 A_1 - 4\alpha A_1 + 2}{(4 - \alpha^2)} \tag{3.21}$$

and

$$\tau_{rz}(q) = -\alpha[B_1 I_1(\alpha q) - B_2 K_1(\alpha q)] \tag{3.22}$$

where A_1, B_1 and B_2 are given by Eq.(3.4).
Dimensionless shearing stress on the surface of outer cylinder for clear fluid flow (i.e. when $\alpha = 0$) is obtained by taking limit $\alpha \to 0$ in Eq.(3.22). We get

$$\lim_{\alpha \to 0} \tau_{rz}(q) = \frac{q}{2} \tag{3.23}$$

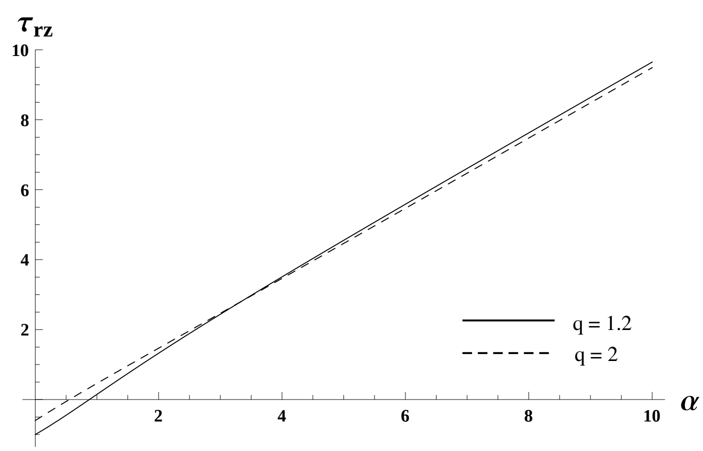

Fig. 5: Variation of shear stress with permeability parameter α for different values of q.

Case II

When permeability of region I is $k_0 r$ and permeability of region II is k_0.

For Couette-Poiseuille flow the outer cylinder is moving with the constant velocity $U = \frac{a^2}{u}\left(\frac{\partial p^*}{\partial z^*}\right)$ with fix pressure gradient and the inner cylinder has been kept fixed. The no slip conditions at boundaries are

$$\begin{aligned} v(r) &= 1 \quad \text{at } r = q \\ u(r) &= v(r) \quad \text{at } r = 1 \\ u'(r) &= v'(r) \quad \text{at } r = 1 \\ u'(r) &= 0 \quad \text{at } r = 0 \end{aligned} \tag{3.24}$$

where, $q = \left(\frac{b}{a}\right)$. The general solutions of equation (2.5) and (2.6) are given by

$$u(r) = C_1 I_0(2\alpha\sqrt{r}) + C_2 K_0(2\alpha\sqrt{r}) + \frac{1}{\alpha^4}(1 + r\alpha^2); \quad 0 \le r \le 1 \tag{3.25}$$

and,

$$v(r) = D_1 I_0(\alpha r) + D_2 K_0(\alpha r) + \frac{1}{\alpha^2}; \quad 1 \le r \le q \tag{3.26}$$

where, I_o and K_o are the modified Bessel functions of zeroth order of first and second kind respectively. Chikh et al. (1995) find similar solution for the forced convection in an annular duct partially filled with a porous medium.

Here C_1, C_2, D_1 and D_2 are constants of integration determined by using above boundary conditions (3.24), we get constants C_1, C_2, D_1 and D_2 as

$$\begin{aligned} C_1 &= \frac{1}{\lambda}[I_1(\alpha)\{\alpha^2(\alpha^2-1)K_0(\alpha) - K_0(q\alpha)\} + \alpha I_0(\alpha)\{\alpha(\alpha^2-1)K_1(\alpha) \\ &\quad + K_0(q\alpha)\} - \{\alpha K_0(\alpha) + K_1(\alpha)\}I_0(q\alpha)] \\ C_2 &= 0 \\ D_1 &= \frac{1}{\lambda}[I_1(2\alpha)\{\alpha^2(\alpha^2-1)K_0(\alpha) - K_0(q\alpha)\} + \alpha I_0(2\alpha)\{\alpha(\alpha^2-1)K_1(\alpha) + K_0(q\alpha)\}] \\ &\quad \text{and} \\ D_2 &= \frac{1}{\lambda}[\alpha I_0(2\alpha)\{\alpha(\alpha^2-1)I_1(\alpha) - I_0(q\alpha)\} + \alpha^2 I_1(2\alpha)\{(1-\alpha^2)I_0(\alpha) + I_0(q\alpha)\}] \end{aligned} \tag{3.27}$$

where,

$$\begin{aligned} \lambda &= \alpha^4[\{I_0(2\alpha)I_1(\alpha) - I_0(\alpha)I_1(2\alpha)\}K_0(q\alpha) + \{I_1(2\alpha)K_0(\alpha) \\ &\quad + I_0(2\alpha)K_1(\alpha)\}I_0(q\alpha)] \end{aligned} \tag{3.28}$$

The dimensionless velocities of the fluid at any point within the region I and II (inner and outer) when permeability of the medium is $k_0 r$ and k_0 respectively given by Eqs.(3.25) and (3.26) on insertion of the preceding values of C_1, C_2, D_1 and D_2.The graphical presentation of velocity profiles for different α is given in Fig.(6).

In the limiting case, when $\alpha \to 0$ (i.e., when permeability of the medium is infinite) in Eqs.(3.25) and (3.26), the classical velocity, u_0 (inner cylinder) and v_0 (outer cylinder) for clear fluid flow is obtained as

(3.29) $$\lim_{\alpha\to 0} u = \lim_{\alpha\to 0} v = \frac{\left(q^2 - r^2 + 4\right)}{4}$$

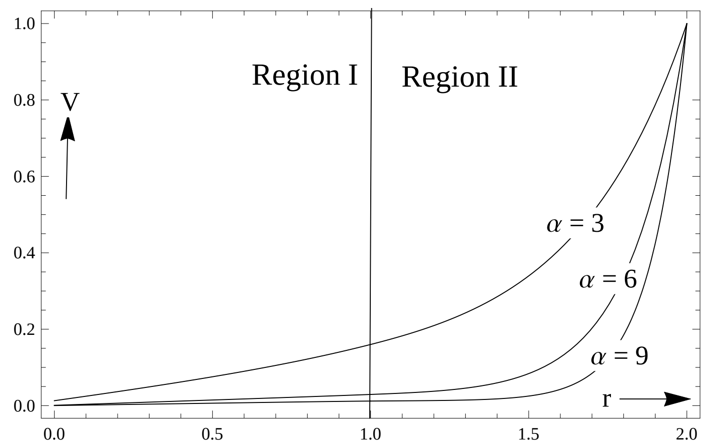

Fig. 6: Variation of velocity with radial distance r for different α for fixed $q = 2$.

3.0.4 Rate of volume flow

The dimensionless rate of volume flow through cross-section of the inner cylinder is given by

(3.30) $$Q_1 = 2\pi \int_0^1 u(r) r dr$$

Substituting $u(r)$ from Eq.(3.25) and after integration [Ref. Abramowitz and Stegun (1972)], we obtain

(3.31) $$Q_1 = \frac{\pi}{3\alpha^4}\left[(3 + 2\alpha^2) + 6\alpha^2 C_1\{\alpha I_1(2\alpha) - I_2(2\alpha)\}\right]$$

Similarly, the dimensionless rate of volume flow through cross-section of the outer cylinder is given by

(3.32) $$Q_2 = 2\pi \int_1^q v(r) r \; dr$$

Substituting $v(r)$ from Eq.(3.26) and after integration [Ref. Abramowitz and Stegun (1972)], we obtain

(3.33) $$Q_2 = \frac{\pi}{\alpha^2}\left[2\alpha\{-D_1 I_1(\alpha) + qD_1 I_1(q\alpha) + D_2(K_1(\alpha) - qK_1(q\alpha))\} + q^2 - 1\right]$$

where I_1 and K_1 are the modified Bessel functions of first kind of order one and C_1, C_2, D_1 and D_2. are given by Eq.(3.27). In the evaluation of above integrals the following identity [Ref. Abramowitz and Stegun (1970)] has been used

$$\left(\frac{1}{z}\frac{d}{dz}\right)^m \{z^\nu \pounds_\nu(z)\} = z^{\nu-m}\pounds_{\nu-m}(z) \tag{3.34}$$

with $m = 1$ and $\nu = 1$. $\pounds_\nu$ denotes I_ν and $e^{\nu\pi i}K_\nu$.

The dimensionless rate of volume flow is given by

$$Q = Q_1 + Q_2$$

or,

$$Q = 2\pi\left(\int_0^1 u(r)rdr + \int_1^q v(r)rdr\right) \tag{3.35}$$

Hence, total dimensionless rate of volume flow is

$$\begin{aligned} Q \;=\; & \frac{\pi}{3\alpha^4}[2\alpha^2 + 6\alpha^2 C_1\{\alpha I_1(2\alpha) - I_2(2\alpha)\} + 3\alpha^2\{2\alpha\{-D_1 I_1(\alpha) \\ & +qD_1 I_1(q\alpha) + D_2(K_1(\alpha) - qK_1(q\alpha))\} + q^2 - 1\} + 3] \end{aligned} \tag{3.36}$$

The dimensionless volume flow rate Q_0 for clear fluid flow (when permeability is infinite) can be obtained by taking limit $\alpha \to 0$ in Eq.(3.36). We get

$$Q_0 = \lim_{\alpha\to 0} Q = \frac{\pi q^2}{8}\left(q^2 + 8\right) \tag{3.37}$$

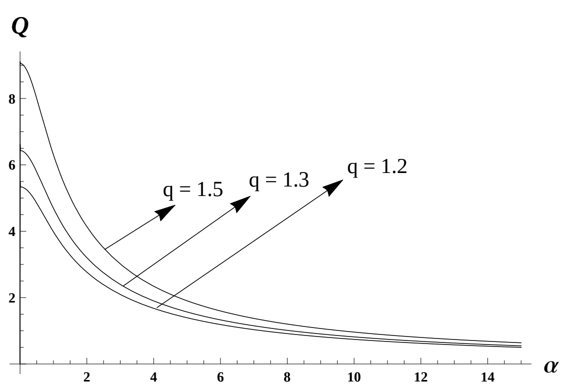

Fig. 7: Variation of volume flow rate with q for different values of α.

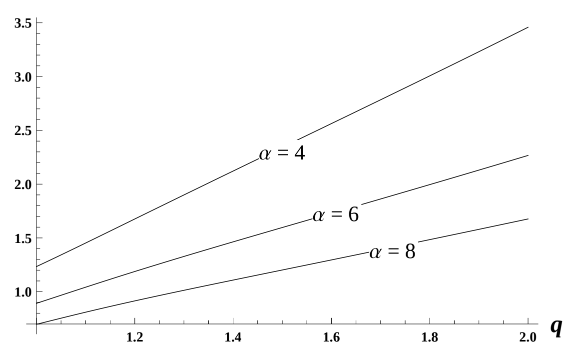

Fig. 8: Variation of volume flow rate with α for different values of q.

3.0.5 Average velocity

The dimensionless average velocity of the flow is defined as

$$u_{avg} = \frac{Q}{\pi q^2} \tag{3.38}$$

Substituting Q from the Eq.(3.36) into preceding equation, the average velocity of the flow with permeability of the porous medium is obtained as

$$\begin{aligned} u_{avg} &= \frac{1}{3q^2\alpha^4}[2\alpha^2 + 6\alpha^2 C_1\{\alpha I_1(2\alpha) - I_2(2\alpha)\} + 3\alpha^2\{2\alpha\{-D_1 I_1(\alpha) \\ &\quad + D_1 q I_1(q\alpha) + D_2(K_1(\alpha) - qK_1(q\alpha))\} + q^2 - 1\} + 3] \end{aligned} \tag{3.39}$$

For clear fluid flow average velocity of the flow is obtained by taking limit $\alpha \to 0$ in Eq.(3.39). Hence

$$\lim_{\alpha \to 0} u_{avg} = \frac{(q^2+8)}{8} \tag{3.40}$$

3.0.6 Shearing stress on the surface of pipe

The dimensionless shearing stress at any point of inner cylinder is given by,

$$\tau_{rz}(r) = -\frac{du}{dr} \tag{3.41}$$

Substituting u from Eq.(3.25) and and differentiating the modified Bessel functions $I_o(\alpha r)$ and $K_o(\alpha r)$ with use of the identity $\frac{d}{dr}I_o(r) = I_1(r)$ and $\frac{d}{dr}K_o(r) = -K_1(r)$ [Ref. Abramowitz and Stegun (1970)], we obtain

$$\tau_{rz}(r) = -\left[\frac{1}{\alpha^2} + \frac{\alpha C_1 I_1\left(2\sqrt{r}\alpha\right)}{\sqrt{r}}\right] \tag{3.42}$$

Similarly, the dimensionless shearing stress at any point of outer cylinder is given by,

$$\tau_{rz}(r) = -\frac{dv}{dr} \tag{3.43}$$

Substituting v from equation (3.26) in above equation and after differentiation we get,

$$\tau_{rz}(r) = -\left[\alpha D_1 I_1(r\alpha) - \alpha D_2 K_1(r\alpha)\right] \tag{3.44}$$

where I_1 and K_1 are modified Bessel function of order one. Shear stress on the surface of inner and outer cylinder is obtained by putting $r = 1$ and $r = q$ in Eqs.(3.41) and (3.44), respectively and using the appropriate sign. Thus, providing

$$\tau_{rz}(1) = -\left[\frac{1}{\alpha^2} + \alpha C_1 I_1(2\alpha)\right] \tag{3.45}$$

and,

$$\tau_{rz}(q) = -\alpha[D_1 I_1(\alpha q) - D_2 K_1(\alpha q)] \tag{3.46}$$

where C_1, D_1 and D_2 are given by Eq.(3.27). Dimensionless shearing stress on the surface of outer cylinder for clear fluid flow (i.e. when $\alpha = 0$) is obtained by taking limit $\alpha \to 0$ in Eq.(3.46). We get

$$\lim_{\alpha\to 0} \tau_{rz}(q) = \frac{q}{2} \tag{3.47}$$

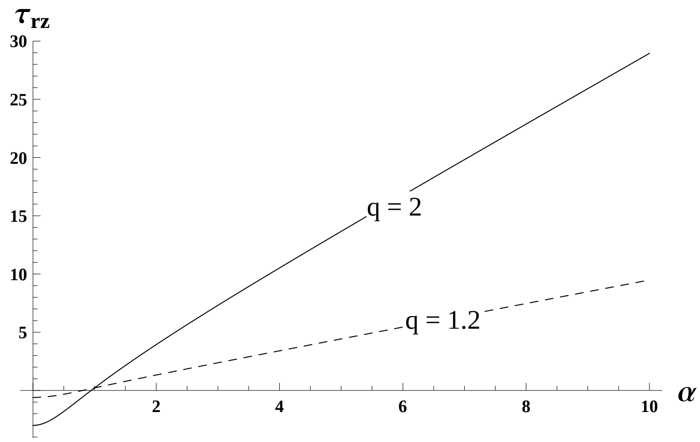

Fig. 9: Variation of shear stress with permeability parameter α for different values of q.

4 Discussion:

Case I : When permeability of inner region is $\mathbf{k_0 r^2}$ and of outer region is $\mathbf{k_0}$.

Fig.2 shows the velocity profiles of Couette-Poiseuille flow in inner (Region I) and outer (Region II) are decreasing continuously for different values of $\alpha = 3, 6, 9$ and they also incorporate the role of the gap parameter q.

Fig.2 also reveals that velocity is decreasing for inner and outer cylinder continuously when permeability is decreasing.

Fig.3 represents a variation of volume flow rate Q with gap parameter q for for different values of $\alpha = 3, 5, 7$. It also shows that Q decreases as α increases (permeability decreases).

Fig.4 represents a variation of volume flow rate Q with permeability parameter α for different $q = 1.2, 1.3, 1.5$. It also shows that Q decreases as α increases (permeability decreases).

Fig.5 represents variation of stress τ_{rz} with permeability parameter α for different $q = 1.2, 2$. It also shows that τ_{rz} increases as α increases (permeability decreases).

Case II : When permeability of inner region is $\mathbf{k_0 r}$ and of outer region is $\mathbf{k_0}$.

Fig.6 shows the velocity profiles of Couette-Poiseuille flow in inner (Region I) and outer (Region II) are decreasing continuously for different values of $\alpha = 3, 6, 9$ and they also incorporate the role of the gap parameter q.

Fig.6 also reveals that velocity is decreasing for inner and outer cylinder continuously when permeability is decreasing.

Fig.7 represents a variation of volume flow rate Q with permeability parameter α for different $q = 1.2, 1.3, 1.5$. It also shows that Q decreases as α increases (permeability decreases).

Fig.8 represents a variation of volume flow rate Q with gap parameter q for different $\alpha = 4, 6, 8$. It also shows that Q decreases as α increases (permeability decreases).

Fig.9 represents variation of stress τ_{rz} with permeability parameter α for different $q = 1.2, 2$. It also shows that τ_{rz} increases as α increases (permeability decreases).

5 Conclusion:

It is observed that in case I and II both permeability variation parameter α has strong effect on the flow. Because when α increases velocity decreases in both cases. Similarly, in both cases variation of volume flow also decreases when permeability variation parameter α increases (for different values of gap parameter q). And, variation of stress also increases with the increase of permeability variation parameter α (in both cases).

References

[1] Abramowitz and Stegun (1972): A hand book of mathematical functions, Wiley-Interscience, NewYork.

[2] Al-Hadhrami, A. K., Elliot, L., Ingham, D. B. and Wen, X. (2001): Analytical solutions of fluid flows through composite channels, Journal of Porous Media, vol. 4, issue 2.

[3] Chauhan D. S. and Kumar V. (2011), Heat transfer effects in a Couttue flow through a composite channel partly filled by a porous medium with a transverse sinusoidal injection velocity and heat source, Thermal Science, vol. 15, issue 2, pp. S175-S186.

[4] Kaviany, M. (1985): Laminar flow through a porous channel bounded by isothermal parallel plates, Int. J. Heat Mass Transfer, vol. 28, issue 4, pp. 851–858.

[5] Nield, D. A. and Bejan A. (2006): Convection in Porous Media, 3rd ed., Springer, New York.

[6] Pop I. and Cheng P. (1992): Flow past a circular cylinder embedded in a porous medium based on the brinkman model, International Journal of Engineering Science, vol 30, issue 2, pp. 257-262.

[7] Pantokratoras, A. and Fang, T. (2010): Flow of a weakly conducting fluid in a channel filled with a porous medium, Trans. Porous Media, vol. 83, pp. 667-676.

[8] Singh, S. K. and Verma, V. K (2020): Exact solution of flow in a composite porous channel, Archieve of mechanical engineering, vol. 67, issue 1, pp. 96-110.

[9] Vafai, K. and Kim, S. J. (1989): Forced convection in a channel filled with porous medium: An exact solution, ASME J. Heat Transfer, vol. 111, pp. 1103-1106.

[10] Verma V.K. and Datta, S. (2012): Flow in an annular channel filled with a porous medium of a variable permeability, Journal of Porous Media, vol. 15, issue 10, pp. 891–899.

[11] Verma, V.K. (2014): Analytical Solution of Magnetohydrodynamic Flow with varying viscosity in an annular channel, Advances in Theoretical and Applied Mathematics, vol. 9, pp. 105-122.

[12] Verma, V.K. and Singh, S.K. (2014): Flow between coaxial rotating cylinders filled by porous medium of varaible permeability, Special Topics & Reviews in Porous Media, An International Journal, vol. 5, issue 4, pp. 355-359.

[13] Wang, C.Y. (2010): Flow Through Super-elliptic Ducts Filled with a Darcy- Brinkman Medium, Transport Porous Medium, vol. 81, pp. 207-217.

[14] Wang, C.Y. and Yu. L. H. (2013): Darcy-Brinkman Flow Through a Bumpy Channel, Transport Porous Medium, vol. 97, pp. 281-294.