Composite entire functions on the basis of comparative growth analysis of $(p,q)^{th}$ relative Gol'dberg type

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Abstract

The aim of this paper is to discuss about the growth properties and some results based on $(p,q)^{th}$ relative Gol'dberg type of composite entire functions.

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1 Introduction

Let $\mathbb{C}^n$ and $\mathbb{R}^n$ respectively denote the complex and real n-space. Also let us indicate the point $(z_1, z_2, ..., z_n), (m_1, m_2, ..., m_n)$ of $\mathbb{C}^n$ by their corresponding unsuffixed symbols z, m respectively. The modulus of z, denoted by $|z|$, is defined as $|z| = \left(|z_1|^2 + ... + |z_n|^2\right)^{\frac{1}{2}}$. If the coordinates of the vector m are non-negative integers, then z^m will denote $z_1^{m_1}...z_n^{m_n}$ and $||m|| = m_1 + ... + m_n$.

If $D \subset \mathbb{C}^n$ ($\mathbb{C}^n$ denote the n-dimensional complex space) be an arbitrary bounded complex n-circular domain with center at the origin of coordinates then for any entire function $f(z)$ of n-complex variables and $R > 0$, $M_{f,D}(R)$ defined by $M_{f,D} = \sup_{z \epsilon D_R} |f(z)|$,where a point $z\epsilon D_R$ if and only if $\frac{z}{R}\epsilon D$. If $f(z)$ is non-constant, then $M_{f,D}(R)$ is strictly increasing and its inverse $M_{f,D}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists such that $\lim_{R\to\infty} M_{f,D}^{-1}(R) = \infty$

Definition 1.1. *([1],[3]) Let $f(z)$ be an entire function of n-variables with respect to any bounded complete n-circular domain D with center at the origin in $\mathbb{C}^n$. Then Gol'dberg order of an entire function $f(z)$ for any bounded complete n-circular domain D is defined as*

$$\rho_{f,D} = \limsup_{R\to\infty} \frac{log^{[2]} M_{f,D}(R)}{log R}.$$

The Gol'dberg lower order of an entire function $f(z)$ for any bounded complete n-circular domain D is defined as

$$\lambda_{f,D} = \liminf_{R\to\infty} \frac{log^{[2]} M_{f,D}(R)}{log R}.$$

where $log^{[k]} R = log\left(log^{[k-1]} R\right)$ *for* $k = 1, 2, 3, ...; log^{[0]} R = R.$

For any bounded complete n-circular domain D, an entire function of n-complex variables for which Goldberg order and Goldberg lower order are the same is said to be of regular growth. Functions which are not of regular growth are said to be of irregular growth. To compare the relative growth of entire functions of n-complex variables having same non zero finite Goldberg order, we introduce the definition of Gol'dberg type and Gol'dberg lower type in following manner:

Definition 1.2. *([5],[6]) Let $f(z)$ be an entire function of n-variables with respect to any bounded complete n-circular domain D with center at the origin in $\mathbb{C}^n$. Then Gol'dberg type of an entire function $f(z)$ for any bounded complete n-circular domain D is defined as*

$$\Delta_{f,D} = \limsup_{R\to\infty} \frac{log M_{f,D}(R)}{(R)^{\rho_{f,D}}}.$$

The Gol'dberg lower type of an entire function $f(z)$ for any bounded complete n-circular domain D is defined as

$$\overline{\Delta}_{f,D} = \liminf_{R\to\infty} \frac{log M_{f,D}(R)}{(R)^{\rho_{f,D}}}.$$

Definition 1.3. *([3],[6]) Let $f(z)$ be an entire function of n-variables and D be a bounded complete n-circular domain with center at the origin in $\mathbb{C}^n$. Then $(p,q)^{th}$ Gol'dberg type of an entire function $f(z)$ for any bounded complete n-circular domain D is defined as*

$$\Delta_{f,D}(p,q) = \limsup_{R\to\infty} \frac{log^{[p-1]} M_{f,D}(R)}{\left[log^{[q-1]} R\right]^{\rho_{f,D}(p,q)}}, \qquad where\ p \geq q \geq 1.$$

The $(p,q)^{th}$ Gol'dberg lower type of an entire function $f(z)$ for any bounded complete n-circular domain D is defined as

$$\overline{\Delta}_{f,D}(p,q) = \liminf_{R\to\infty} \frac{log^{[p-1]} M_{f,D}(R)}{\left[log^{[q-1]} R\right]^{\rho_{f,D}(p,q)}}, \qquad where\ p \geq q \geq 1.$$

We introduce the definition of $(p,q)^{th}$- relative Goldberg type and $(p,q)^{th}$-relative Goldberg lower type in order to compare the relative growth of two entire functions of n-complex variables having same non zero finite $(p,q)^{th}$- relative Goldberg order with respect to another entire function of n-complex variables.

Definition 1.4. *[9] Let $f(z)$ and $g(z)$ be entire function of n-complex variables with index pair (m,q) and (m,p) respectively, where p, q and m are positive integers such that $m \geq q \geq 1$ and $m \geq p \geq 1$ and D be any bounded complete n-circular domain with center at origin in $\mathbb{C}^n$. Then the $(p,q)^{th}$ relative Gol'dberg type of $f(z)$ with respect to $g(z)$ is defined as*

$$\Delta_{g,D}^{(p,q)}(f) = \limsup_{R\to\infty} \frac{log^{[p-1]} M_{g,D}^{-1} M_{f,D}(R)}{\left[log^{[q-1]} R\right]^{\rho_g^{(p,q)}(f)}}$$

Similarly, the $(p,q)^{th}$ relative Gol'dberg lower type of an entire function $f(z)$with respect to another entire function $g(z)$ is defined as

$$\overline{\Delta}_{g,D}^{(p,q)}(f) = \liminf_{R\to\infty} \frac{log^{[p-1]} M_{g,D}^{-1} M_{f,D}(R)}{\left[log^{[q-1]} R\right]^{\rho_g^{(p,q)}(f)}}, \qquad where\ 0 < \rho_g^{(p,q)}(f) < \infty.$$

During the past decades, several authors (see [2],[3],[7],[8],[9]) made close investigations on the properties of relative type of entire functions of several complex variable using different growth indicator such as Goldberg type, $(p,q)^{th}$ Goldberg type. In this paper we wish to study some relative growth properties of entire functions of n-complex variables using Goldberg type, $(p,q)^{th}$ relative Goldberg type.

2 Main Results

Theorem 2.1. *Let $f(z)$ and $g(z)$ be an entire function of n-complex variables and D be bounded complete n-circular domain with center at origin in $\mathbb{C}^n$.*
Also let $0 < \overline{\Delta}_D^{(m,q)}(fog) < \Delta_D^{(m,q)}(fog) < \infty$ and $0 < \overline{\Delta}_D^{(m,p)}(g) < \Delta_D^{(m,p)}(g) < \infty$
Then

$$\left[\frac{\overline{\Delta}_D^{(m,q)}(fog)}{\Delta_D^{(m,p)}(g)}\right]^{\frac{1}{\rho g(m,p)}} \leq \liminf_{R\to\infty} \frac{log^{[p-1]} M_{g,D}^{-1} M_{fog,D}(R)}{\left[log^{[q-1]} R\right]^{\rho_g^{(p,q)}(fog)}}$$
$$\leq \min\left\{\left[\frac{\overline{\Delta}_D^{(m,q)}(fog)}{\overline{\Delta}_D^{(m,p)}(g)}\right]^{\frac{1}{\rho g(m,p)}}, \left[\frac{\Delta_D^{(m,q)}(fog)}{\Delta_D^{(m,p)}(g)}\right]^{\frac{1}{\rho g(m,p)}}\right\}$$
$$\leq \max\left\{\left[\frac{\overline{\Delta}_D^{(m,q)}(fog)}{\overline{\Delta}_D^{(m,p)}(g)}\right]^{\frac{1}{\rho g(m,p)}}, \left[\frac{\Delta_D^{(m,q)}(fog)}{\Delta_D^{(m,p)}(g)}\right]^{\frac{1}{\rho g(m,p)}}\right\} \leq \limsup_{R\to\infty} \frac{log^{[p-1]} M_{g,D}^{-1} M_{fog,D}(R)}{\left[log^{[q-1]} R\right]^{\rho_g^{(p,q)}(fog)}}$$
$$\leq \left[\frac{\Delta_D^{(m,q)}(fog)}{\overline{\Delta}_D^{(m,p)}(g)}\right]^{\frac{1}{\rho g(m,p)}}$$

Proof. From the definition of $(p,q)^{th}$ Gol'dberg type and $(p,q)^{th}$ Gol'dberg lower type of entire composite functions fog for $\epsilon > 0$ and value of R tending to infinity.

$$M_{fog,D}(R) \leq exp^{[m-1]}\left[\left(\Delta_{fog,D}^{(m,q)}(R) + \epsilon\right)\left[log^{[q-1]} R\right]^{\rho_{fog}^{(m,q)}}\right] \tag{2.1}$$

and

$$M_{fog,D}(R) \geq exp^{[m-1]}\left[\left(\Delta_{fog,D}^{(m,q)}(R) - \epsilon\right)\left[log^{[q-1]} R\right]^{\rho_{fog}^{(m,q)}}\right] \tag{2.2}$$

Again for all sufficiently large value of R

$$M_{fog,D}(R) \geq exp^{[m-1]}\left[\left(\overline{\Delta}_{fog,D}^{(m,q)}(R) - \epsilon\right)\left[log^{[q-1]} R\right]^{\rho_{fog}^{(m,q)}}\right] \tag{2.3}$$

and

$$M_{fog,D}(R) \leq exp^{[m-1]}\left[\left(\overline{\Delta}_{fog,D}^{(m,q)}(R) + \epsilon\right)\left[log^{[q-1]} R\right]^{\rho_{fog}^{(m,q)}}\right] \tag{2.4}$$

Now, from the definition of $(p,q)^{th}$ Gol'dberg type and $(p,q)^{th}$ Gol'dberg lower type of entire function $g(z)$ for $\epsilon > 0$ and for all sufficiently large value of R

$$M_{g,D}(R) \leq exp^{[m-1]}\left[\left(\Delta_{g,D}^{(m,p)}(R)+\epsilon\right)\left[log^{[p-1]}R\right]^{\rho_g^{(m,p)}}\right]$$

$$R \leq M_{g,D}^{-1}\left[exp^{[m-1]}\left[\left(\Delta_{g,D}^{(m,p)}(R)+\epsilon\right)\left[log^{[p-1]}R\right]^{\rho_g^{(m,p)}}\right]\right]$$

i.e.

$$M_{g,D}^{-1}(R) \geq exp^{[p-1]}\left[\frac{log^{[m-1]}R}{\left(\Delta_{g,D}^{(m,p)}(R)+\epsilon\right)}\right]^{\frac{1}{\rho_g^{(m,p)}}} \tag{2.5}$$

and

$$M_{g,D}^{-1}(R) \leq exp^{[p-1]}\left[\frac{log^{[m-1]}R}{\left(\Delta_{g,D}^{(m,p)}(R)-\epsilon\right)}\right]^{\frac{1}{\rho_g^{(m,p)}}} \tag{2.6}$$

Also for a sequence of value of R, tending to infinity

$$M_{g,D}^{-1}(R) \leq exp^{[p-1]}\left[\frac{log^{[m-1]}R}{\left(\overline{\Delta}_{g,D}^{(m,p)}(R)-\epsilon\right)}\right]^{\frac{1}{\rho_g^{(m,p)}}} \tag{2.7}$$

and

$$M_{g,D}^{-1}(R) \geq exp^{[p-1]}\left[\frac{log^{[m-1]}R}{\left(\overline{\Delta}_{g,D}^{(m,p)}(R)+\epsilon\right)}\right]^{\frac{1}{\rho_g^{(m,p)}}} \tag{2.8}$$

Now form (2.3) and (2.5) it follows for $R \to \infty$ that

$$M_{g,D}^{-1}M_{fog,D}(R) \geq M_{g,D}^{-1}\left[exp^{[m-1]}\left[\left(\overline{\Delta}_{fog,D}^{(m,q)}(R)-\epsilon\right)\left[log^{[q-1]}R\right]^{\rho_{fog}^{(m,q)}}\right]\right]$$

i.e.

$$\geq exp^{[p-1]}\left[\frac{log^{[m-1]}exp^{[m-1]}\left(\overline{\Delta}_{fog,D}^{(m,q)}(R)-\epsilon\right)\left[log^{[q-1]}R\right]^{\rho_{fog}^{(m,q)}}}{\left(\Delta_{g,D}^{(m,p)}(R)+\epsilon\right)}\right]^{\frac{1}{\rho_g^{(m,p)}}}$$

$$log^{[p-1]}M_{g,D}^{-1}M_{fog,D}(R) \geq \left[\frac{\overline{\Delta}_{fog,D}^{(m,q)}(R)-\epsilon}{\Delta_{g,D}^{(m,p)}(R)+\epsilon}\right]^{\frac{1}{\rho_g^{(m,p)}}} \left[log^{[q-1]}R\right]^{\frac{\rho_{fog}^{(m,q)}}{\rho_g^{(m,p)}}}$$

$$log^{[p-1]}M_{g,D}^{-1}M_{fog,D}(R) \geq \left[\frac{\overline{\Delta}_{fog,D}^{(m,q)}(R)-\epsilon}{\Delta_{g,D}^{(m,p)}(R)+\epsilon}\right]^{\frac{1}{\rho_g^{(m,p)}}} \left[log^{[q-1]}R\right]^{\rho_g^{(p,q)}(fog)}$$

As $\epsilon(>0)$ is arbitrary we obtained that

$$\liminf_{R\to\infty} \frac{log^{[p-1]}M_{g,D}^{-1}M_{fog,D}(R)}{\left[log^{[q-1]}R\right]^{\rho_g^{(p,q)}(fog)}} \geq \left[\frac{\overline{\Delta}_{fog,D}^{(m,q)}(R)}{\Delta_{g,D}^{(m,p)}(R)}\right]^{\frac{1}{\rho_g^{(m,p)}}} \tag{2.9}$$

Again from (2.4) and (2.7) it follows for $R \to \infty$ that

$$M_{g,D}^{-1}M_{fog,D}(R) \leq M_{g,D}^{-1}\left[exp^{[m-1]}\left[\left(\overline{\Delta}_{fog,D}^{(m,q)}(R)+\epsilon\right)\left[log^{[q-1]}R\right]^{\rho_{fog}^{(m,q)}}\right]\right]$$

i.e.

$$\leq exp^{[p-1]}\left[\frac{log^{[m-1]}exp^{[m-1]}\left(\overline{\Delta}_{fog,D}^{(m,q)}(R)+\epsilon\right)\left[log^{[q-1]}R\right]^{\rho_{fog}^{(m,q)}}}{\left(\overline{\Delta}_{g,D}^{(m,p)}(R)-\epsilon\right)}\right]^{\frac{1}{\rho_g^{(m,p)}}}$$

$$log^{[p-1]}M_{g,D}^{-1}M_{fog,D}(R) \leq \left[\frac{\overline{\Delta}_{fog,D}^{(m,q)}(R)+\epsilon}{\overline{\Delta}_{g,D}^{(m,p)}(R)-\epsilon}\right]^{\frac{1}{\rho_g^{(m,p)}}} \left[log^{[q-1]}R\right]^{\frac{\rho_{fog}^{(m,q)}}{\rho_g^{(m,p)}}}$$

$$log^{[p-1]}M_{g,D}^{-1}M_{fog,D}(R) \leq \left[\frac{\overline{\Delta}_{fog,D}^{(m,q)}(R)+\epsilon}{\overline{\Delta}_{g,D}^{(m,p)}(R)-\epsilon}\right]^{\frac{1}{\rho_g^{(m,p)}}} \left[log^{[q-1]}R\right]^{\rho_g^{(p,q)}(fog)}$$

As $\epsilon(>0)$ is arbitrary we obtained that

$$\liminf_{R\to\infty} \frac{log^{[p-1]}M_{g,D}^{-1}M_{fog,D}(R)}{\left[log^{[q-1]}R\right]^{\rho_g^{(p,q)}(fog)}} \leq \left[\frac{\overline{\Delta}_{fog,D}^{(m,q)}(R)}{\overline{\Delta}_{g,D}^{(m,p)}(R)}\right]^{\frac{1}{\rho_g^{(m,p)}}} \tag{2.10}$$

Similarly from(2.1) and (2.6) it follows for a sequence of values of R tending to infinity that

$$M_{g,D}^{-1}M_{fog,D}(R) \leq M_{g,D}^{-1}\left[exp^{[m-1]}\left[\left(\Delta_{fog,D}^{(m,q)}(R)+\epsilon\right)\left[log^{[q-1]}R\right]^{\rho_{fog}^{(m,q)}}\right]\right]$$

i.e.

$$\leq exp^{[p-1]}\left[\frac{log^{[m-1]}exp^{[m-1]}\left[\left(\Delta_{fog,D}^{(m,q)}(R)+\epsilon\right)\left[log^{[q-1]}R\right]^{\rho_{fog}^{(m,q)}}\right]}{\Delta_{g,D}^{(m,p)}(R)-\epsilon}\right]^{\frac{1}{\rho_g^{(m,p)}}}$$

$$log^{[p-1]}M_{g,D}^{-1}M_{fog,D}(R)\leq\left[\frac{\Delta_{fog,D}^{(m,q)}(R)+\epsilon}{\Delta_{g,D}^{(m,p)}(R)-\epsilon}\right]^{\frac{1}{\rho_g^{(m,p)}}}\left[log^{[q-1]}R\right]^{\frac{\rho_{fog}^{(m,q)}}{\rho_g^{(m,p)}}}$$

$$log^{[p-1]}M_{g,D}^{-1}M_{fog,D}(R)\leq\left[\frac{\Delta_{fog,D}^{(m,q)}(R)+\epsilon}{\Delta_{g,D}^{(m,p)}(R)-\epsilon}\right]^{\frac{1}{\rho_g^{(m,p)}}}\left[log^{[q-1]}R\right]^{\rho_g^{(p,q)}(fog)}$$

As $\epsilon(>0)$, we obtained that

$$\liminf_{R\to\infty}\frac{log^{[p-1]}M_{g,D}^{-1}M_{fog,D}(R)}{\left[log^{[q-1]}R\right]^{\rho_g^{(p,q)}(fog)}}\leq\left[\frac{\Delta_{fog,D}^{(m,q)}(R)}{\Delta_{g,D}^{(m,p)}(R)}\right]^{\frac{1}{\rho_g^{(m,p)}}} \tag{2.11}$$

Now combining (2.9),(2.10) and (2.11) we get that

$$\left[\frac{\overline{\Delta}_{fog,D}^{(m,q)}(R)}{\Delta_{g,D}^{(m,p)}(R)}\right]^{\frac{1}{\rho_g^{(m,p)}}}\leq\liminf_{R\to\infty}\frac{log^{[p-1]}M_{g,D}^{-1}M_{fog,D}(R)}{\left[log^{[q-1]}R\right]^{\rho_g^{(p,q)}(fog)}}$$

$$\leq\min\left\{\left[\frac{\overline{\Delta}_{fog,D}^{(m,q)}(R)}{\overline{\Delta}_{g,D}^{(m,p)}(R)}\right]^{\frac{1}{\rho_g^{(m,p)}}},\left[\frac{\Delta_{fog,D}^{(m,q)}(R)}{\Delta_{g,D}^{(m,p)}(R)}\right]^{\frac{1}{\rho_g^{(m,p)}}}\right\} \tag{2.12}$$

from (2.3) and (2.8) it follows for $R\to\infty$ that

$$M_{g,D}^{-1}M_{fog,D}(R)\geq M_{g,D}^{-1}\left[exp^{[m-1]}\left[\left(\overline{\Delta}_{fog,D}^{(m,q)}(R)-\epsilon\right)\left[log^{[q-1]}R\right]^{\rho_{fog}^{(m,q)}}\right]\right]$$

i.e.

$$\geq exp^{[p-1]}\left[\frac{log^{[m-1]}exp^{[m-1]}\left[\left(\overline{\Delta}_{fog,D}^{(m,q)}(R)-\epsilon\right)\left[log^{[q-1]}R\right]^{\rho_{fog}^{(m,q)}}\right]}{\overline{\Delta}_{g,D}^{(m,p)}(R)+\epsilon}\right]^{\frac{1}{\rho_g^{(m,p)}}}$$

$$log^{[p-1]}M_{g,D}^{-1}M_{fog,D}(R)\geq\left[\frac{\overline{\Delta}_{fog,D}^{(m,q)}(R)-\epsilon}{\overline{\Delta}_{g,D}^{(m,p)}(R)+\epsilon}\right]^{\frac{1}{\rho_g^{(m,p)}}}\left[log^{[q-1]}R\right]^{\frac{\rho_{fog}^{(m,q)}}{\rho_g^{(m,p)}}}$$

$$log^{[p-1]}M_{g,D}^{-1}M_{fog,D}(R) \geq \left[\frac{\overline{\Delta}_{fog,D}^{(m,q)}(R)-\epsilon}{\overline{\Delta}_{g,D}^{(m,p)}(R)+\epsilon}\right]^{\frac{1}{\rho_g^{(m,p)}}} \left[log^{[q-1]}R\right]^{\rho_g^{(p,q)}(fog)}$$

As $\epsilon(>0)$, we obtained that

$$\limsup_{R\to\infty}\frac{log^{[p-1]}M_{g,D}^{-1}M_{fog,D}(R)}{\left[log^{[q-1]}R\right]^{\rho_g^{(p,q)}(fog)}} \geq \left[\frac{\overline{\Delta}_{fog,D}^{(m,q)}(R)}{\overline{\Delta}_{g,D}^{(m,p)}(R)}\right]^{\frac{1}{\rho_g^{(m,p)}}} \tag{2.13}$$

From (2.1) and (2.7) it follows for $R\to\infty$ that

$$M_{g,D}^{-1}M_{fog,D}(R) \leq M_{g,D}^{-1}\left[exp^{[m-1]}\left[\left(\Delta_{fog,D}^{(m,q)}(R)+\epsilon\right)\left[log^{[q-1]}R\right]^{\rho_{fog}^{(m,q)}}\right]\right]$$

i.e.

$$\leq exp^{[p-1]}\left[\frac{log^{[m-1]}exp^{[m-1]}\left[\left(\Delta_{fog,D}^{(m,q)}(R)+\epsilon\right)\left[log^{[q-1]}R\right]^{\rho_{fog}^{(m,q)}}\right]}{\overline{\Delta}_{g,D}^{(m,p)}(R)-\epsilon}\right]^{\frac{1}{\rho_g^{(m,p)}}}$$

$$log^{[p-1]}M_{g,D}^{-1}M_{fog,D}(R) \leq \left[\frac{\Delta_{fog,D}^{(m,q)}(R)+\epsilon}{\overline{\Delta}_{g,D}^{(m,p)}(R)-\epsilon}\right]^{\frac{1}{\rho_g^{(m,p)}}} \left[log^{[q-1]}R\right]^{\frac{\rho_{fog}^{(m,q)}}{\rho_g^{(m,p)}}}$$

$$log^{[p-1]}M_{g,D}^{-1}M_{fog,D}(R) \leq \left[\frac{\Delta_{fog,D}^{(m,q)}(R)+\epsilon}{\overline{\Delta}_{g,D}^{(m,p)}(R)-\epsilon}\right]^{\frac{1}{\rho_g^{(m,p)}}} \left[log^{[q-1]}R\right]^{\rho_g^{(p,q)}(fog)}$$

As $\epsilon(>0)$, we obtained that

$$\limsup_{R\to\infty}\frac{log^{[p-1]}M_{g,D}^{-1}M_{fog,D}(R)}{\left[log^{[q-1]}R\right]^{\rho_g^{(p,q)}(fog)}} \leq \left[\frac{\Delta_{fog,D}^{(m,q)}(R)}{\overline{\Delta}_{g,D}^{(m,p)}(R)}\right]^{\frac{1}{\rho_g^{(m,p)}}} \tag{2.14}$$

Similarly from (2.2) and (2.5) it follows for $R\to\infty$ that

$$M_{g,D}^{-1}M_{fog,D}(R) \geq M_{g,D}^{-1}\left[exp^{[m-1]}\left[\left(\Delta_{fog,D}^{(m,q)}(R)-\epsilon\right)\left[log^{[q-1]}R\right]^{\rho_{fog}^{(m,q)}}\right]\right]$$

i.e.

$$\geq exp^{[p-1]}\left[\frac{log^{[m-1]}exp^{[m-1]}\left[\left(\Delta_{fog,D}^{(m,q)}(R)-\epsilon\right)\left[log^{[q-1]}R\right]^{\rho_{fog}^{(m,q)}}\right]}{\Delta_{g,D}^{(m,p)}(R)+\epsilon}\right]^{\frac{1}{\rho_g^{(m,p)}}}$$

$$log^{[p-1]}M_{g,D}^{-1}M_{fog,D}(R) \geq \left[\frac{\Delta_{fog,D}^{(m,q)}(R)-\epsilon}{\Delta_{g,D}^{(m,p)}(R)+\epsilon}\right]^{\frac{1}{\rho_g^{(m,p)}}} \left[log^{[q-1]}R\right]^{\frac{\rho_{fog}^{(m,q)}}{\rho_g^{(m,p)}}}$$

$$log^{[p-1]}M_{g,D}^{-1}M_{fog,D}(R) \leq \left[\frac{\Delta_{fog,D}^{(m,q)}(R)-\epsilon}{\Delta_{g,D}^{(m,p)}(R)+\epsilon}\right]^{\frac{1}{\rho_g^{(m,p)}}} \left[log^{[q-1]}R\right]^{\rho_g^{(p,q)}(fog)}$$

As $\epsilon(>0)$, we obtained that

$$\limsup_{R\to\infty} \frac{log^{[p-1]}M_{g,D}^{-1}M_{fog,D}(R)}{\left[log^{[q-1]}R\right]^{\rho_g^{(p,q)}(fog)}} \leq \left[\frac{\Delta_{fog,D}^{(m,q)}(R)}{\Delta_{g,D}^{(m,p)}(R)}\right]^{\frac{1}{\rho_g^{(m,p)}}} \tag{2.15}$$

Now combining (2.13),(2.14) and (2.15) we obtained that

$$\max\left\{\left[\frac{\overline{\Delta}_{fog,D}^{(m,q)}(R)}{\overline{\Delta}_{g,D}^{(m,p)}(R)}\right]^{\frac{1}{\rho_g^{(m,p)}}}, \left[\frac{\Delta_{fog,D}^{(m,q)}(R)}{\Delta_{g,D}^{(m,p)}(R)}\right]^{\frac{1}{\rho_g^{(m,p)}}}\right\} \leq \limsup_{R\to\infty} \frac{log^{[p-1]}M_{g,D}^{-1}M_{fog,D}(R)}{\left[log^{[q-1]}R\right]^{\rho_g^{(p,q)}(fog)}}$$

$$\leq \left[\frac{\Delta_{fog,D}^{(m,q)}(R)}{\overline{\Delta}_{g,D}^{(m,p)}(R)}\right]^{\frac{1}{\rho_g^{(m,p)}}} \tag{2.16}$$

Hence theorem follows from (2.12) and (2.16). □

Theorem 2.2. *Let $f(z)$ and $g(z)$ be any two entire functions of ncomplex variables with index pair (m,q) and (m,p), respectively, where $m \geq q \geq 1$ and $m \geq p \geq 1$ and D be bounded complete n-circular domain with center at origin in $\mathbb{C}^n$.*

Then $$\Delta_{g,D}^{(p,q)}(fog) \leq \left[\frac{\Delta_{fog,D}^{(m,q)}(R)}{\overline{\Delta}_{g,D}^{(m,p)}(R)}\right]^{\frac{1}{\rho_g^{(m,p)}}}$$

Proof. The conclusion of the above theorem can be carried out from (2.1) and (2.7) after applying the same technique of Theorem 2.1 and therefore its proof is omitted. □

Theorem 2.3. *Let $f(z)$ and $g(z)$ be any two entire functions of ncomplex variables with index pair (m,q) and (m,p), respectively, where $m \geq q \geq 1$ and $m \geq p \geq 1$ and D be bounded complete n-circular domain with center at origin in $\mathbb{C}^n$.*

Then $$\overline{\Delta}_{g,D}^{(p,q)}(fog) \leq \min\left\{\left[\frac{\overline{\Delta}_{D}^{(m,q)}(fog)}{\overline{\Delta}_{D}^{(m,p)}(g)}\right]^{\frac{1}{\rho g(m,p)}}, \left[\frac{\Delta_{D}^{(m,q)}(fog)}{\Delta_{D}^{(m,p)}(g)}\right]^{\frac{1}{\rho g(m,p)}}\right\}$$

Proof. The conclusion of the above theorem can be carried out from (2.1),(2.6),(2.11) and (2.4),(2.7),(2.10) after applying the same technique of Theorem 2.1 and therefore its proof is omitted. □

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