Some Results on Coupled Fixed Point on Complex Partial $b-$Metric Space

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Abstract

The purpose of this paper is to establish a coupled fixed point results on complex partial $b-$metric space under contractive condition with some examples are presented to illustrate the facts.

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1 Introduction

In 1922 [1] the Banach contraction mapping theorem is popularly known as Banach contraction mapping principle, is a rewarding in result fixed point theory. Bakhtin,[2] and Czerwik [3] introduced b-metric spaces. Azam at al. [4] introduced the concept of complex valued metric spaces. So, many researches [5, 6, 7, 8][10, 11, 12] studied the extension of fixed point results in metric spaces. Hassen Aydi [9] introduced coupled fixed point theorems in partially ordered metric using contractive condition. we refer [13, 14, 15, 16]. In this paper, we further generalize and extend the results of some coupled fixed point results on complex partial $b-$metric spaces under contractive conditions.

2 Preliminaries

Let $\mathbb{C}$ be the set of complex numbers and $\lambda_1, \lambda_2 \in \mathbb{C}$. Define a partial order $\preceq$ on $\mathbb{C}$ as follows:

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$$\lambda_1 \preceq \lambda_2 \text{ if and only if } Re(\lambda_1) \leq Re(\lambda_2) \text{ and } Im(\lambda_1) \leq Im(\lambda_2).$$

Consequently, one can infer that $\lambda_1 \preceq \lambda_2$ if one of the following conditions is satisfied:

(i) $Re(\lambda_1) = Re(\lambda_2)$ and $Im(\lambda) < Im(\lambda_2)$,

(ii) $Re(\lambda_1) < Re(\lambda_2)$ and $Im(\lambda) = Im(\lambda_2)$,

(iii) $Re(\lambda_1) < Re(\lambda_2)$ and $Im(\lambda) < Im(\lambda_2)$,

(iv) $Re(\lambda_1) = Re(\lambda_2)$ and $Im(\lambda) = Im(\lambda_2)$.

In particular, we will write $\lambda_1 \precnsim \lambda_2$ if $\lambda_1 \neq \lambda_2$ and one of $(i), (ii)$ and (iii) is satisfied and we will write $\lambda_1 \prec \lambda_2$ if only (iii) is satisfied. Notice that

(a) If $0 \preceq \lambda_1 \precnsim \lambda_2$, then $|\lambda_1| < |\lambda_2|$,

(b) If $\lambda_1 \preceq \lambda_2$ and $\lambda_2 \prec \lambda_3$ then $\lambda_1 \prec \lambda_3$,

(c) If $a, b \in \mathbb{R}$ and $a \leq b$ then $a\lambda_1 \preceq b\lambda_1$ for all $\lambda_1 \in \mathbb{C}$.

Definition 2.1: [8] A complex partial $b-$metric on a non-void set X is a function $\zeta_{cb} : X \times X \to \mathbb{C}^+$ such that for all $\lambda, \mu, \kappa \in X$:

(i) $0 \preceq \zeta_{cb}(\lambda, \mu) \preceq \zeta_{cb}(\lambda, \mu)(smallself - distances)$

(ii) $\zeta_{cb}(\lambda, \mu) = \zeta_{cb}(\mu, \lambda)(symmetry)$

(iii) $\zeta_{cb}(\lambda, \lambda) = \zeta_{cb}(\lambda, \mu) = \zeta_{cb}(\mu, \mu) \Leftrightarrow \lambda = \mu(equality)$

(iv) $\exists$ a real number $s \geq 1$ such that
$\zeta_{cb}(\lambda, \mu) \preceq s[\zeta_{cb}(\lambda, \kappa) + \zeta_{cb}(\kappa, \mu)] - \zeta_{cb}(\kappa, \kappa)(triangularity)$.

A complex partial $b-$metric space is a pair (X, ζ_{cb}) such that X is a non-void set and ζ_{cb} is complex partial $b-$metric on X. The number s is called the coefficient of (X, ζ_{cb}).

Remark 2.2:In a complex partial $b-$metric space (X, ζ_{cb}) if $\lambda, \mu \in X$ and $\zeta_{cb}(\lambda, \mu) = 0$, then $\lambda = \mu$, but the converse may not be true.

Remark 2.3:It is clear that every complex partial metric space is a complex partial $b-$metric space with coefficient $s = 1$ and every complex valued $b-$metric is a complex partial $b-$metric space with the same coefficient and zero self-distance. However, the converse of this fact need not hold.

Now, we define Cauchy sequence and convergent sequence in complex partial $b-$metric spaces.

Definition 2.4: Let (X, ζ_{cb}) be a complex partial $b-$metric space with coefficient s. Let $\{\lambda_n\}$ be any sequence in X and $\lambda \in X$. Then

(i) The sequence $\{\lambda_n\}$ is said to be convergent with respect to τ_{cb} and converges to λ, if $\lim_{n\to\infty} \zeta_{cb}(\lambda_n, \lambda) = \zeta_{cb}(\lambda, \lambda)$.

(ii) The sequence $\{\lambda_n\}$ is said to be Cauchy sequence in (X, ζ_{cb}) if $\lim_{n,m\to\infty} \zeta_{cb}(\lambda_n, \lambda_m)$ exists and is finite.

(iii) (X, ζ_{cb}) is said to be a complete complex partial $b-$metric space if for every Cauchy sequence $\{\lambda_n\}$ in X there exists $\lambda \in X$ such that $\lim_{n,m\to\infty} \zeta_{cb}(\lambda_n, \lambda_m) = \lim_{n\to\infty} \zeta_{cb}(\lambda_n, \lambda) = \zeta_{cb}(\lambda, \lambda)$.

(iv) A mappings $\xi : X \to X$ is said to be continuous at $\lambda_0 \in X$ if for every $\epsilon > 0$, there exists $\omega > 0$ such that $\xi(B_{\zeta_{cb}}(\lambda_0, \omega)) \subset B_{\zeta_{cb}}(\xi(\lambda_0, \epsilon))$.

Let X be a complex partial $b-$metric space and $B \subseteq X$. A point $\lambda \in X$ is called an interior of set B, if there exists $0 < r \in \mathbb{C}$ such that $B_{\zeta_{cb}}(\lambda, r) = \{\mu \in X : \zeta_{cb}(\lambda, \mu) < \zeta_{cb}(\lambda, \lambda) + r\} \subseteq B$. A subset B is called open, if each point of B is an interior point of B. A point $\lambda \in X$ is said to be a limit point of B, for every $0 < r \in \mathbb{C}$, $B_{\zeta_{cb}}(\lambda, r) \cap (B - \{\lambda\}) \neq \phi$. A subset $B \subseteq X$ is called closed, B contains all its limit points.

Lemma 2.5:[8] Let (X, ζ_{cb}) be a complex partial $b-$metric space. A sequence $\{\lambda_n\}$ is Cauchy sequence in the CPBMS (X, ζ_{cb}) then $\{\lambda_n\}$ is Cauchy in a metric space (X, ζ^t_{cb}).

Definition 2.6:Let (X, ζ_{cb}) be a complex partial $b-$metric space(CPBMS). Then an element $(\lambda, \kappa) \in X \times X$ is said to be a coupled fixed point of the mapping $\xi : X \times X \to X$ if $\xi(\lambda, \kappa) = \lambda$ and $\xi(\kappa, \lambda) = \kappa$.

3 Main Results

Theorem 3.1. *Let (X, ζ_{cb}) be a complete complex partial $b-$metric space. Suppose that the mapping $\xi : X \times X \to X$ satisfies the following contractive condition for all $\lambda, \mu, \kappa, \nu \in X$*

$$\zeta_{cb}(\xi(\lambda, \mu), \xi(\kappa, \nu)) \preceq \alpha\zeta_{cb}(\xi(\lambda, \mu), \kappa) + \beta\zeta_{cb}(\xi(\kappa, \nu), \lambda),$$

where α, β are nonnegative constants with $\alpha + 2\beta < 1$. Then, ξ has a unique coupled fixed point.

Proof. Choose $\lambda_0, \mu_0 \in X$ and set $\lambda_1 = \xi(\lambda_0, \mu_0)$ and $\mu_1 = \xi(\mu_0, \lambda_0)$. Continuing this process, set $\lambda_{n+1} = \xi(\lambda_n, \mu_n)$ and $\mu_{n+1} = \xi(\mu_n, \lambda_n)$.
Then,

$$\begin{aligned}
\zeta_{cb}(\lambda_n, \lambda_{n+1}) &= \zeta_{cb}(\xi(\lambda_{n-1}, \mu_{n-1}), \xi(\lambda_n, \mu_n)) \\
&\preceq \alpha\zeta_{cb}(\xi(\lambda_{n-1}, \mu_{n-1}), \lambda_n) + \beta\zeta_{cb}(\xi(\lambda_n, \mu_n), \lambda_{n-1}) \\
&= \alpha\zeta_{cb}(\lambda_n, \lambda_n) + \beta\zeta_{cb}(\lambda_{n+1}, \lambda_{n-1}) \\
&\preceq \alpha\zeta_{cb}(\lambda_n, \lambda_{n+1}) + \beta\zeta_{cb}(\lambda_{n+1}, \lambda_{n-1}) \\
&\preceq \alpha\zeta_{cb}(\lambda_n, \lambda_{n+1}) + \beta(\zeta_{cb}(\lambda_{n+1}, \lambda_n) + \zeta_{cb}(\lambda_n, \lambda_{n-1}) - \zeta_{cb}(\lambda_n, \lambda_n)) \\
&\preceq \alpha\zeta_{cb}(\lambda_n, \lambda_{n+1}) + \beta(\zeta_{cb}(\lambda_{n+1}, \lambda_n) + \zeta_{cb}(\lambda_n, \lambda_{n-1})) \\
&\preceq \frac{\beta}{1 - (\alpha + \beta)}\zeta_{cb}(\lambda_n, \lambda_{n-1})
\end{aligned}$$

which implies that

$$|\zeta_{cb}(\lambda_n, \lambda_{n+1})| \leq \frac{\beta}{1-(\alpha+\beta)} |\zeta_{cb}(\lambda_n, \lambda_{n-1})| \tag{3.1}$$

Similarly, one can prove that

$$|\zeta_{cb}(\mu_n, \mu_{n+1})| \leq \frac{\beta}{1-(\alpha+\beta)} |\zeta_{cb}(\mu_n, \mu_{n-1})| \tag{3.2}$$

From (3.1) and (3.2), we get

$$\begin{aligned} |\zeta_{cb}(\lambda_n, \lambda_{n+1})| + |\zeta_{cb}(\mu_n, \mu_{n+1})| &\leq \frac{\beta}{1-(\alpha+\beta)} (|\zeta_{cb}(\lambda_n, \lambda_{n-1})| + |\zeta_{cb}(\mu_n, \mu_{n-1})|) \\ &= \rho(|\zeta_{cb}(\lambda_n, \lambda_{n-1})| + |\zeta_{cb}(\mu_n, \mu_{n-1})|) \end{aligned}$$

where $\rho = \frac{\beta}{1-(\alpha+\beta)} < 1$.
Also,

$$|\zeta_{cb}(\lambda_{n+1}, \lambda_{n+2})| \leq \frac{\beta}{1-(\alpha+\beta)} |\zeta_{cb}(\lambda_n, \lambda_{n-1})| \tag{3.3}$$

$$|\zeta_{cb}(\mu_{n+1}, \mu_{n+2})| \leq \frac{\beta}{1-(\alpha+\beta)} |\zeta_{cb}(\mu_n, \mu_{n-1})| \tag{3.4}$$

From (3.3) and (3.4), we get

$$\begin{aligned} |\zeta_{cb}(\lambda_{n+1}, \lambda_{n+2})| + |\zeta_{cb}(\mu_{n+1}, \mu_{n+2})| &\leq \frac{\beta}{1-(\alpha+\beta)} (|\zeta_{cb}(\lambda_n, \lambda_{n-1})| + |\zeta_{cb}(\mu_n, \mu_{n-1})|) \\ &= \rho(|\zeta_{cb}(\lambda_n, \lambda_{n-1})| + |\zeta_{cb}(\mu_n, \mu_{n-1})|) \end{aligned}$$

Repeating this way, we get

$$\begin{aligned} |\zeta_{cb}(\lambda_n, \lambda_{n+1})| + |\zeta_{cb}(\mu_n, \mu_{n+1})| &\leq \rho(|\zeta_{cb}(\lambda_n, \lambda_{n-1})| + |\zeta_{cb}(\mu_n, \mu_{n-1})|) \\ &\leq \rho^2(|\zeta_{cb}(\mu_{n-2}, \mu_{n-1})| + |\zeta_{cb}(\lambda_{n-2}, \lambda_{n-1})|) \\ &\leq \cdots \leq \rho^n(|\zeta_{cb}(\mu_0, \mu_1)| + |\zeta_{cb}(\lambda_0, \lambda_1)|) \end{aligned}$$

Now, if $|\zeta_{cb}(\lambda_n, \lambda_{n+1})| + |\zeta_{cb}(\mu_n, \mu_{n+1})| = \delta_n$, then

$$\delta_n \leq \rho\delta_{n-1} \leq \rho^2\delta_{n-2} \leq \cdots \leq \rho^n\delta_0 \tag{3.5}$$

If $\delta_0 = 0$ then $|\zeta_{cb}(\lambda_0, \lambda_1)| + |\zeta_{cb}(\mu_0, \mu_1)| = 0$. Hence $\lambda_0 = \lambda_1 = \xi(\lambda_0, \mu_0)$ and $\mu_0 = \mu_1 = \xi(\mu_0, \lambda_0)$, which implies that (λ_0, μ_0) is a coupled fixed point of ξ.
Let $\delta_0 > 0$. For each $n \geq m$,we have

$$\begin{aligned} \zeta_{cb}(\lambda_n, \lambda_m) \preceq\ & s\left[\zeta_{cb}(\lambda_n, \lambda_{n-1}) + \zeta_{cb}(\lambda_{n-1}, \lambda_{n-2})\right] - \zeta_{cb}(\lambda_{n-1}, \lambda_{n-1}) \\ &+ s^2\left[\zeta_{cb}(\lambda_{n-2}, \lambda_{n-3}) + \zeta_{cb}(\lambda_{n-3}, \lambda_{n-4})\right] - \zeta_{cb}(\lambda_{n-3}, \lambda_{n-3}) \\ &+ \cdots + s^m\left[\zeta_{cb}(\lambda_{m+2}, \lambda_{m+1}) + \zeta_{cb}(\lambda_{m+1}, \lambda_m)\right] - \zeta_{cb}(\lambda_{m+1}, \lambda_{m+1}) \\ &\preceq s\zeta_{cb}(\lambda_n, \lambda_{n-1}) + s^2\zeta_{cb}(\lambda_{n-1}, \lambda_{n-2}) + \cdots + s^m\zeta_{cb}(\lambda_{m+1}, \lambda_m) \end{aligned}$$

which implies that

$$|\zeta_{cb}(\lambda_n,\lambda_m)| \le s|\zeta_{cb}(\lambda_n,\lambda_{n-1})| + s^2|\zeta_{cb}(\lambda_{n-1},\lambda_{n-2})| + \cdots + s^m|\zeta_{cb}(\lambda_{m+1},\lambda_m)|,$$

Similarly, one can prove that

$$|\zeta_{cb}(\mu_n,\mu_m)| \le s|\zeta_{cb}(\mu_n,\mu_{n-1})| + s^2|\zeta_{cb}(\mu_{n-1},\mu_{n-2})| + \cdots + s^m|\zeta_{cb}(\mu_{m+1},\mu_m)|,$$

Thus,

$$\begin{aligned}
|\zeta_{cb}(\lambda_n,\lambda_m)| + |\zeta_{cb}(\mu_n,\mu_m)| &\le s\delta_{n-1} + s^2\delta_{n-2} + s^3\delta_{n-3} + \cdots + s^m\delta_m \\
&\le (s\rho^{n-1} + s^2\rho^{n-2} + s^3\rho^{n-3} + \cdots + s^m\rho^m)\delta_0 \\
&\le \frac{s\rho^m}{1-s\rho}\delta_0 \to 0 \quad n\to\infty.
\end{aligned}$$

which implies that $\{\lambda_n\}$ and $\{\mu_n\}$ are Cauchy sequence in (X,ζ_{cb}). Since the partial b-metric space (X,ζ_{cb}) is complete, there exists $\lambda,\mu \in X$ such that $\{\lambda_n\} \to \lambda$ and $\{\mu_n\} \to \mu$ as $n\to\infty$ and $\zeta_{cb}(\lambda,\lambda) = \lim_{n\to\infty}\zeta_{cb}(\lambda,\lambda_n) = \lim_{n,m\to\infty}\zeta_{cb}(\lambda_n,\lambda_m) = 0$, $\zeta_{cb}(\mu,\mu) = \lim_{n\to\infty}\zeta_{cb}(\mu,\mu_n) = \lim_{n,m\to\infty}\zeta_{cb}(\mu_n,\mu_m) = 0$.

Now we have to show that $\lambda = \xi(\lambda,\mu)$. We suppose on the contrary that $\lambda \ne \xi(\lambda,\mu)$ and $\mu \ne \xi(\mu,\lambda)$ so that $0 \prec \zeta_{cb}(\lambda,\xi(\lambda,\mu)) = \alpha_1$ and $0 \prec \zeta_{cb}(\mu,\xi(\mu,\lambda)) = \alpha_2$, then

$$\begin{aligned}
\alpha_1 = \zeta_{cb}(\lambda,\xi(\lambda,\mu)) &\preceq \zeta_{cb}(\lambda,\lambda_{n+1}) + \zeta_{cb}(\lambda_{n+1},\xi(\lambda,\mu)) \\
&= \zeta_{cb}(\lambda,\mu_{n+1}) + \zeta_{cb}(\xi(\lambda_n,\mu_n),\xi(\lambda,\mu)) \\
&\preceq \zeta_{cb}(\lambda,\lambda_{n+1}) + \alpha\zeta_{cb}(\xi(\lambda_n,\mu_n),\lambda) + \beta\zeta_{cb}(\xi(\lambda,\mu),\lambda_n) \\
&= \zeta_{cb}(\lambda,\lambda_{n+1}) + \alpha\zeta_{cb}(\lambda_{n+1},\lambda) + \beta\zeta_{cb}(\xi(\lambda,\mu),\lambda_n)
\end{aligned}$$

which implies that

$$|\alpha_1| \le |\zeta_{cb}(\lambda,\lambda_{n+1})| + \alpha|\zeta_{cb}(\lambda_n,\lambda)| + \beta|\zeta_{cb}(\xi(\lambda,\mu),\lambda_n)|$$

As $n\to\infty$, $|\alpha_1| \le 0$. Which is a contradiction, therefore $|\zeta_{cb}(\lambda,\xi(\lambda,\mu))| = 0 \implies \lambda = \xi(\lambda,\mu)$. Similarly we can prove that $\mu = \xi(\mu,\lambda)$. Thus (λ,μ) is a coupled fixed point of ξ. Now, if (u,v) is another coupled fixed point of ξ, then

$$\zeta_{cb}(\lambda,\mu) = \zeta_{cb}(\xi(\lambda,\mu),\xi(u,v)) \preceq \alpha\zeta_{cb}(\xi(\lambda,\mu),u) + \beta\zeta_{cb}(\xi(u,v),\lambda),$$

Thus,

$$(1-(\alpha+\beta))\zeta_{cb}(\lambda,\mu) \preceq 0 \tag{3.6}$$

which implies that

$$(1-(\alpha+\beta))|\zeta_{cb}(\lambda,u)| \le 0 \tag{3.7}$$

Similarly,

$$(1-(\alpha+\beta))|\zeta_{cb}(\mu,v)| \le 0 \tag{3.8}$$

From (3.7) and (3.8), since $\alpha+\beta<1$. Therefore $\lambda = u$ and $\mu = v$
$\implies (\lambda,\mu) = (u,v)$.
Thus, ξ has a unique coupled fixed point. □

From Theorems (3.1) with $\alpha = \beta$, we get the following corollary.

Corollary 3.2. *Let (X, ζ_{cb}) be a complete complex partial b-metric space. Suppose that the mapping $\xi : X \times X \to X$ satisfies the following contractive condition for all $\lambda, \mu, \kappa, \nu \in X$*

$$\zeta_{cb}(\xi(\lambda, \mu), \xi(\kappa, \nu)) \preceq \alpha(\zeta_{cb}(\xi(\lambda, \mu), \kappa) + \zeta_{cb}(\xi(\kappa, \nu), \lambda)), \tag{3.9}$$

where α are nonnegative constants with $\alpha < \frac{1}{3}$. Then, ξ has a unique coupled fixed point.

Let $X = [0, \infty)$ endowed with the usual complex partial $b-$metric $\zeta_{cb} \colon X \times X \to [0, \infty)$ defined by $\zeta_{cb}(\lambda, \mu) = [\max\{\lambda, \mu\}]^2 (1+i)$. The complex partial $b-$metric space (X, ζ_{cb}) is complete because (X, ζ^t_{cb}) is complete with coefficient $s = 2$. Indeed, for any $\lambda, \mu \in X$,

$$\begin{aligned} \zeta^t_{cb} &= 2\zeta_{cb}(\lambda, \kappa) - \zeta_{cb}(\lambda, \lambda) - \zeta_{cb}(\kappa, \kappa) \\ &= 2[\max\{\lambda, \mu\}]^2(1 + i) - (\lambda + i\lambda) - (\mu + i\mu) \\ &= |\lambda - \mu|^2 + i|\lambda - \mu|^2. \end{aligned}$$

Thus, (X, ζ_{cb}) is the Euclidean complex metric space which is complete. Consider the mapping $\xi \colon X \times X \to X$ defined by $\xi(\lambda, \mu) = \frac{[\lambda+\mu]^2}{24}$. For any $\lambda, \mu, u, v \in X$, we have

$$\begin{aligned} \zeta_{cb}(\xi(\lambda, \mu), \xi(u, v)) &= \frac{1}{24}[\max\{\lambda + u, \xi(\lambda, \mu) + \xi(u, v)\}]^2(1 + i) \\ &\le \frac{1}{24}[\max\{\xi(\lambda, \mu), \lambda\} + \max\{\xi(u, v), v\}]^2(1 + i) \\ &= \frac{1}{24}[\zeta_{cb}((\lambda, \mu), \lambda) + \zeta_{cb}(\xi(u, v), u)]. \end{aligned}$$

which is the contractive condition (3.9) for $\alpha = \frac{1}{12}$. Therefore, by Corollary 3.2, ξ has a unique coupled fixed point, which is $(0, 0)$. Note that if the mapping $\xi \colon X \times X \to X$ is given by $\xi(\lambda, \mu) = \frac{[\lambda+\mu]^2}{2}$, then ξ satisfies the contractive condition (3.9) for $\alpha = 1$, that is,

$$\begin{aligned} \zeta_{cb}(\xi(\lambda, \mu), \xi(u, v)) &= \frac{1}{2}[\max\{\lambda + u, \xi(\lambda, \mu) + \xi(u, v)\}]^2(1 + i) \\ &\le \frac{1}{2}[\max\{\xi(\lambda, \mu), \lambda\} + \max\{\xi(u, v), u\}]^2(1 + i) \\ &= \frac{1}{2}[\zeta_{cb}(\lambda, u) + \zeta_{cb}(\mu, v)]. \end{aligned}$$

In this case, $(0, 0)$ and $(1, 1)$ are both coupled fixed points of ξ, and hence, the coupled fixed point of ξ is not unique. This shows that the condition $\alpha < 1$ in Corollary 3.2, and hence $\alpha + \beta < 1$ in Theorem 2 cannot be omitted in the statement of the aforesaid results.

Theorem 3.3. *Let (X, ζ_{cb}) be a complete complex partial b−metric space. Suppose that the mapping $\xi : X \times X \to X$ satisfies*

$$\zeta_{cb}(\xi(\lambda,\mu),\xi(\kappa,\nu)) \preceq r\max\{\zeta_{cb}(\lambda,\kappa),\zeta_{cb}(\mu,\nu),\zeta_{cb}(\xi(\lambda,\mu),\lambda),\zeta_{cb}(\xi(\kappa,\nu),\kappa)\},$$

for all $\lambda,\mu,\kappa,\nu \in X$. If $r \in [0,1)$, then ξ has a unique coupled fixed point.

Proof. Choose $\lambda_0, \mu_0 \in X$ and set $\lambda_1 = \xi(\lambda_0, \mu_0)$ and $\mu_1 = \xi(\mu_0, \lambda_0)$. Continuing this process, set $\lambda_{n+1} = \xi(\lambda_n, \mu_n)$ and $\mu_{n+1} = \xi(\mu_n, \lambda_n)$.
Then,

$$\begin{aligned}\zeta_{cb}(\lambda_{n+1},\lambda_{n+2}) &= \zeta_{cb}(\xi(\lambda_n,\mu_n),\xi(\lambda_{n+1},\mu_{n+1}))\\ &\preceq r\max\{\zeta_{cb}(\lambda_n,\lambda_{n+1}),\zeta_{cb}(\mu_n,\mu_{n+1}),\zeta_{cb}(\xi(\lambda_n,\mu_n),\lambda_n),\\ &\qquad \zeta_{cb}(\xi(\lambda_{n+1},\mu_{n+1}),\lambda_{n+1})\}\\ &= r\max\{\zeta_{cb}(\lambda_n,\lambda_{n+1}),\zeta_{cb}(\mu_n,\mu_{n+1}),\zeta_{cb}(\lambda_{n+1},\lambda_n),\\ &\qquad \zeta_{cb}(\lambda_{n+2},\lambda_{n+1})\}\\ &\preceq r\max\{\zeta_{cb}(\lambda_n,\lambda_{n+1}),\zeta_{cb}(\mu_n,\mu_{n+1})\}\end{aligned}$$

which implies that

$$|\zeta_{cb}(\lambda_{n+1},\lambda_{n+2})| \le r\max\{|\zeta_{cb}(\lambda_n,\lambda_{n+1})|,|\zeta_{cb}(\mu_n,\mu_{n+1})|\}. \tag{3.10}$$

Similarly, one can prove that

$$|\zeta_{cb}(\mu_{n+1},\mu_{n+2})| \le r\max\{|\zeta_{cb}(\mu_n,\mu_{n+1})|,|\zeta_{cb}(\lambda_n,\lambda_{n+1})|\}. \tag{3.11}$$

From (3.10) and (3.11), we get

$$\max\{|\zeta_{cb}(\lambda_{n+1},\lambda_{n+2})|,|\zeta_{cb}(\mu_{n+1},\mu_{n+2})|\} \le r\max\{|\zeta_{cb}(\mu_n,\mu_{n+1})|,|\zeta_{cb}(\lambda_n,\lambda_{n+1})|\}. \tag{3.12}$$

Continuing this process,.we get

$$\begin{aligned}\max\{|\zeta_{cb}(\lambda_n,\lambda_{n+1})|,|\zeta_{cb}(\mu_n,\mu_{n+1})|\} &\le r\max\{|\zeta_{cb}(\mu_{n-1},\mu_n)|,|\zeta_{cb}(\lambda_{n-1},\lambda_n)|\}\\ &\le r^2\max\{|\zeta_{cb}(\mu_{n-2},\mu_{n-1})|,|\zeta_{cb}(\lambda_{n-2},\lambda_{n-1})|\}\\ &\ \vdots\\ &\le r^n\max\{|\zeta_{cb}(\mu_0,\mu_1)|,|\zeta_{cb}(\lambda_0,\lambda_1)|\}.\end{aligned}$$

As $n \to \infty$,

$$\lim_{n\to\infty}\max\{|\zeta_{cb}(\lambda_n,\lambda_{n+1})|,|\zeta_{cb}(\mu_n,\mu_{n+1})|\} = 0.$$

Therefore,

$$\lim_{n\to\infty}|\zeta_{cb}(\lambda_n,\lambda_{n+1})| = 0, \tag{3.13}$$

$$\lim_{n\to\infty}|\zeta_{cb}(\mu_n,\mu_{n+1})| = 0 \tag{3.14}$$

For each $n > m$, we have

$$\begin{aligned}\zeta_{cb}(\lambda_n, \lambda_m) &\preceq s\left[\zeta_{cb}(\lambda_n, \lambda_{n-1}) + \zeta_{cb}(\lambda_{n-1}, \lambda_{n-2})\right] - \zeta_{cb}(\lambda_{n-1}, \lambda_{n-1})\\ &+ s^2\left[\zeta_{cb}(\lambda_{n-2}, \lambda_{n-3}) + \zeta_{cb}(\lambda_{n-3}, \lambda_{n-4})\right] - \zeta_{cb}(\lambda_{n-3}, \lambda_{n-3})\\ &+ \cdots + s^m\left[\zeta_{cb}(\lambda_{m+2}, \lambda_{m+1}) + \zeta_{cb}(\lambda_{m+1}, \lambda_m)\right] - \zeta_{cb}(\lambda_{m+1}, \lambda_{m+1})\\ &\preceq s\zeta_{cb}(\lambda_n, \lambda_{n-1}) + s^2\zeta_{cb}(\lambda_{n-1}, \lambda_{n-2}) + \cdots + s^m\zeta_{cb}(\lambda_{m+1}, \lambda_m)\end{aligned}$$

which implies that

$$|\zeta_{cb}(\lambda_n, \lambda_m)| \leq s|\zeta_{cb}(\lambda_n, \lambda_{n-1})| + s^2|\zeta_{cb}(\lambda_{n-1}, \lambda_{n-2})| + \cdots + s^m|\zeta_{cb}(\lambda_{m+1}, \lambda_m)|.$$

Therefore,

$$|\zeta_{cb}(\lambda_n, \lambda_m)| \leq r^n \max\{|\zeta_{cb}(\mu_0, \mu_1)|, |\zeta_{cb}(\lambda_0, \lambda_1)|\}.$$

As $n, m \to \infty$,

$$\lim_{n\to\infty} |\zeta_{cb}(\lambda_n, \lambda_m)| = 0.$$

Similarly, one can prove that

$$|\zeta_{cb}(\mu_n, \mu_m)| \leq s|\zeta_{cb}(\mu_n, \mu_{n-1})| + s^2|\zeta_{cb}(\mu_{n-1}, \mu_{n-2})| + \cdots + s^m|\zeta_{cb}(\mu_{m+1}, \mu_m)|,$$

$$\zeta_{cb}(\mu_n, \mu_m)| \leq r^n \max\{|\zeta_{cb}(\mu_0, \mu_1)|, |\zeta_{cb}(\lambda_0, \lambda_1)|\},$$

$$\lim_{n\to\infty} |\zeta_{cb}(\mu_n, \mu_m)| = 0.$$

which implies that $\{\lambda_n\}$ and $\{\mu_n\}$ are Cauchy sequence in (X, ζ_{cb}). Since the partial $b-$metric space (X, ζ_{cb}) is complete, there exists $\lambda, \mu \in X$ such that $\{\lambda_n\} \to \lambda$ and $\{\mu_n\} \to \mu$ as $n \to \infty$ and $\zeta_{cb}(\lambda, \lambda) = \lim_{n\to\infty} \zeta_{cb}(\lambda, \lambda_n) = \lim_{n,m\to\infty} \zeta_{cb}(\lambda_n, \lambda_m) = 0,\ \zeta_{cb}(\mu, \mu) = \lim_{n\to\infty} \zeta_{cb}(\mu, \mu_n) = \lim_{n,m\to\infty} \zeta_{cb}(\mu_n, \mu_m) = 0.$
Now,

$$\begin{aligned}\zeta_{cb}(\lambda, \xi(\lambda, \mu)) &\preceq \zeta_{cb}(\lambda, \lambda_{n+1}) + \zeta_{cb}(\lambda_{n+1}, \xi(\lambda, \mu))\\ &= \zeta_{cb}(\lambda, \lambda_{n+1}) + \zeta_{cb}(\xi(\lambda_n, \mu_n), \xi(\lambda, \mu))\\ &\preceq \zeta_{cb}(\lambda, \lambda_{n+1})\\ &\quad + r\max\{\zeta_{cb}(\lambda_n, \lambda), \zeta_{cb}(\mu_n, \mu), \zeta_{cb}(\xi(\lambda_n, \mu_n), \lambda_n), \zeta_{cb}(\xi(\lambda, \mu), \lambda)\}\\ &= \zeta_{cb}(\lambda, \lambda_{n+1})\\ &\quad + r\max\{\zeta_{cb}(\lambda_n, \lambda), \zeta_{cb}(\mu_n, \mu), \zeta_{cb}(\lambda_{n+1}, \lambda_n), \zeta_{cb}(\xi(\lambda, \mu), \lambda)\},\end{aligned}$$

which implies that

$$\begin{aligned}|\zeta_{cb}(\lambda, \xi(\lambda, \mu))| &\leq |\zeta_{cb}(\lambda, \lambda_{n+1})|\\ &\quad + r\max\{|\zeta_{cb}(\lambda_n, \lambda)|, |\zeta_{cb}(\mu_n, \mu)|, |\zeta_{cb}(\lambda_{n+1}, \lambda_n)|, |\zeta_{cb}(\xi(\lambda, \mu), \lambda)|\}\end{aligned}$$

As $n \to \infty$, $|\zeta_{cb}(\lambda, \xi(\lambda,\mu))| \leq r|\zeta_{cb}(\xi(\lambda,\mu),\lambda)|$.
Since $[0,1)$, therefore $|\zeta_{cb}(\lambda, \xi(\lambda,\mu))| = 0 \implies \lambda = \xi(\lambda,\mu)$. Similarly we can prove that $\mu = \xi(\mu,\lambda)$. Thus (λ,μ) is a coupled fixed point of ξ. Now, if (u,v) is another coupled fixed point of ξ, then

$$\begin{aligned}\zeta_{cb}(\lambda,u) = \zeta_{cb}(\xi(\lambda,\mu),\xi(u,v)) &\preceq r\max\{\zeta_{cb}(\lambda,u),\zeta_{cb}(\mu,v),\zeta_{cb}(\xi(\lambda,\mu),\lambda),\\ &\qquad \zeta_{cb}(\xi(u,v),u)\}\\ &\preceq r\max\{\zeta_{cb}(\lambda,u),\zeta_{cb}(\mu,v),\zeta_{cb}(\lambda,\lambda),\zeta_{cb}(u,u)\},\end{aligned}$$

Since $\zeta_{cb}(\lambda,\lambda) \preceq \zeta_{cb}(\lambda,u)$ and $\zeta_{cb}(u,u) \preceq \zeta_{cb}(\lambda,u)$, we have

$$\zeta_{cb}(\lambda,u) \preceq r\max\{\zeta_{cb}(\lambda,u),\zeta_{cb}(\mu,v)\}$$

$$|\zeta_{cb}(\lambda,u)| \leq r\max\{|\zeta_{cb}(\lambda,u)|,|\zeta_{cb}(\mu,v)|\}. \tag{3.15}$$

Similarly, we can prove

$$|\zeta_{cb}(\mu,v)| \leq r\max\{|\zeta_{cb}(\lambda,u)|,|\zeta_{cb}(\mu,v)|\}. \tag{3.16}$$

From (3.15) and (3.16), we have

$$\max\{|\zeta_{cb}(\lambda,u)|,|\zeta_{cb}(\mu,v)|\} \leq r\max\{|\zeta_{cb}(\lambda,u)|,|\zeta_{cb}(\mu,v)|\} \tag{3.17}$$

Since $r < 1$, we have $\max\{|\zeta_{cb}(\lambda,u)|,|\zeta_{cb}(\mu,v)|\} = 0$. Which implies that $\zeta_{cb}(\lambda,u) = 0$ and $\zeta_{cb}(\mu,v) = 0$. Therefore $\lambda = u$ and $\mu = v$
$\implies (\lambda,\mu) = (u,v)$.
Thus, ξ has a unique coupled fixed point. □

Corollary 3.4. *Let (X,ζ_{cb}) be a complete complex $b-$partial metric space. Suppose that the mapping $\xi : X \times X \to X$ satisfies*

$$\zeta_{cb}(\xi(\lambda,\mu),\xi(\kappa,\nu)) \preceq a_1\zeta_{cb}(\lambda,\kappa) + a_2\zeta_{cb}(\mu,\nu) + a_3\zeta_{cb}(\xi(\lambda,\mu),\lambda) + a_4\zeta_{cb}(\xi(\kappa,\nu),\nu),$$

for all $\lambda,\mu,\kappa,\nu \in X$ with $a_1,a_2,a_3,a_4 \in [0,1)$, then ξ has a unique coupled fixed point.

Proof. The proof follows from Theorems 3.3.
Note that

$$\begin{aligned}&a_1\zeta_{cb}(\lambda,\kappa) + a_2\zeta_{cb}(\mu,\nu) + a_3\zeta_{cb}(\xi(\lambda,\mu),\lambda) + a_4\zeta_{cb}(\xi(\kappa,\nu),\kappa)\\ \leq\ &(a_1+a_2+a_3+a_4)\max\{\zeta_{cb}(\lambda,\kappa),\zeta_{cb}(\mu,\nu),\zeta_{cb}(\xi(\lambda,\mu),\lambda),\zeta_{cb}(\xi(\kappa,\nu),\kappa)\}\end{aligned}$$

□

Let $X = [0,\infty)$ endowed with the usual complex partial $b-$metric $\zeta_{cb}\colon X \times X \to [0,\infty)$ defined by $\zeta_{cb}(\lambda,\mu) = [\max\{\lambda,\mu\}]^2(1+i)$. The complex partial $b-$metric space (X,ζ_{cb}) is complete because (X,ζ^t_{cb}) is complete with coefficient $s = 2$. Indeed, for any $\lambda,\mu \in X$,

$$\begin{aligned}\zeta^t_{cb} &= 2\zeta_{cb}(\lambda,\kappa) - \zeta_{cb}(\lambda,\lambda) - \zeta_{cb}(\kappa,\kappa)\\ &= 2[\max\{\lambda,\mu\}]^2(1+i) - (\lambda + i\lambda) - (\mu + i\mu)\\ &= |\lambda-\mu|^2 + i|\lambda-\mu|^2.\end{aligned}$$

Thus, (X, ζ_{cb}) is the Euclidean complex metric space which is complete. Consider the mapping $\xi\colon X \times X \to X$ defined by $\xi(\lambda, \mu) = \frac{[\lambda-\mu]^2}{2}$. For any $\lambda, \mu, u, v \in X$, we have

$$\begin{aligned}
\zeta_{cb}(\xi(\lambda,\mu),\xi(u,v)) &= \frac{1}{2}[\max\{|\lambda-\mu|,|u-v|\}]^2(1+i)\\
&= \frac{1}{2}[\max\{\lambda-\mu,\mu-\lambda,u-v,v-u\}]^2(1+i)\\
&\preceq \frac{1}{2}[\max\{\lambda,\mu,u,v\}]^2(1+i)\\
&= \frac{1}{2}\max\{\zeta_{cb}(\lambda,u),\zeta_{cb}(\mu,v)\}\\
&\preceq \frac{1}{2}\max\{\zeta_{cb}(\lambda,u),\zeta_{cb}(\mu,v),\zeta_{cb}(\xi(\lambda,\mu),\lambda),\zeta_{cb}(\xi(u,v),u)\}.
\end{aligned}$$

Thus, ξ has a unique coupled fixed point. Here, $(0, 0)$ is the unique fixed point of ξ.

4 Conclution

We have proved a coupled fixed point results on complex partial $b-$metric space under contractive condition. The existence and uniqueness of the result is presented in this article. This article generalized and extended many existed results in the literature.

References

[1] Banach, S.: Sur les operations dans les ensembles abstraits et leur application aux equations integrales. *Fundamenta Mathematicae* 1922, 3, 133 − 181.

[2] Bakhtin, I. A.: The contraction mappings principle in quasi−metric spaces. *Functional Analysis vol.* 30, *pp.*26 − 37, 1989, *Russian.*

[3] Czerwick, S.: Contraction mappings in $b-$metric spaces. *Acta Mathematica et Informatica Universitatis Ostraviensis vol.* 1, *pp.*5 − 11, 1993.

[4] Azam, A, Fisher, B and Khan, M.: Common fixed point theorems in complex valued metric spaces. *Numer.Funct.Anal.Optim.* 32 3, 2011, 243 − 253.

[5] Gnana Bhaskar, T and Lakshmikantham, V.: Fixed point theorems in partially ordered metric spaces and applications. *Nonlinear Analysis: Theory, Methods and Applications vol.* 65, *no.* 7, *pp.* 1379 − 1393, 2006.

[6] Ciri and Lakshmikantham, V.: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces and applications. *Nonlinear Analysis: Theory, Methods and Applications vol.* 70, *no.* 12, *pp.* 4341 − 4349, 2009.

[7] Altun, I, Sola, F and Simsek, H.: Generalized contractions on partial metric spaces. *Topology and Its Applications vol.* 157, *no.* 18, *pp.* 2778 − 2785, 2010.

[8] Dhivya, P and Marudai, M.: Common fixed point theorems for mappings satisfying a contractive condition of rational expression on a ordered complex partial metric space. *Cogent Mathematics* 2017, 4 :, 1389622.

[9] Hassen Aydi: Some Coupled fixed point results on partial metric spaces. *International Journal of Mathematics and Mathematics Sciences. vol.* 2011*, Article ID* 647091*,* 11 *pages.*

[10] Mishra, L.N, Tiwari, S.K, Mishra, V.N, Khan, I.A: *Unique Fixed Point Theorems for Generalized Contractive Mappings in Partial Metric Spaces. Journal of Function Spaces, Volume 2015 (2015), Article ID 960827, 8 pages.*

[11] Mishra, L.N, Tiwari, S.K, Mishra, V.N: *Fixed point theorems for generalized weakly S-contractive mappings in partial metric spaces. Journal of Applied Analysis and Computation, Volume 5, Number 4, 2015, pp. 600-612.*

[12] Deshpande, B, Mishra, V.N, Handa, A, Mishra, L.N: *Coincidence Point Results for Generalized* (ψ, θ, ϕ)*-Contraction on Partially Ordered Metric Spaces. Thai J. Math., Vol. 19, No. 1, (2021), pp. 93-112.*

[13] Mishra, L.N, Dewangan, V, Mishra, V.N, Amrulloh, H: *Coupled best proximity point theorems for mixed* g*-monotone mappings in partially ordered metric spaces J. Math. Comput. Sci., Vol. 11, No. 5, (2021), pp. 6168-6192.*

[14] Mishra, L.N, Dewangan, V, Mishra, V.N, Karateke, S: *Best proximity points of admissible almost generalized weakly contractive mappings with rational expressions on b-metric spaces J. Math. Computer Sci., Vol. 22, Issue 2, (2021), pp. 97109.*

[15] Deepmala, M. Jain, L.N. Mishra, V.N. Mishra, *A note on the paper "Hu et al., Common coupled fixed point theorems for weakly compatible mappings in fuzzy metric spaces Fixed Point Theory and Applications 2013, 2013:220, Int. J. Adv. Appl. Math. and Mech. 5(2) (2017), pp. 51 52.*

[16] Sharma, N, Mishra,L.N, Mishra, S.N, Mishra, V.N: *Empirical study of new iterative algorithm for generalized nonexpansive operators. Journal of Mathematics and Computer Science, Vol. 25, Issue 3, (2022), pp. 284-295.*