Some Characteristic Results on Multivalent Analytic Functions satisfying Certain Subordinations

Vivek Kumar Yadav [1] **and Poonam Sharma** [2]

[1] *Department of Mathematics and Astronomy*
University of Lucknow, Lucknow 226001 India
yvivekkumar72@gmail.com

[2] *Department of Mathematics and Astronomy*
University of Lucknow, Lucknow 226001 India
sharma_poonam@lkouniv.ac.in; poonambaba@gmail.com

Abstract

In this paper, for functions $p \in H[1,1]$, a Ma-Minda type class $\mathcal{P}(\eta, \phi)$ and an operator $V^n_{\mu,m}$ are defined. Functions $f \in \mathcal{A}_m$ satisfying the condition $V^n_{\mu,m}\left(\frac{f(z)}{z^m}\right)^{\mu} \in \mathcal{P}(\eta, \phi)$ are studied. Some characteristic results such as subordination results, best dominant of the subordinations, a coefficient result, bounds and a radius result are obtained with the use of certain key lemmas.

Subject Classification:[2020]Primary 30C45; Secondary 30C55.
Keywords: Analytic Functions, Subordination, Best Dominant, starlike Functions, Convex Functions, Coefficient Inequality, Radius Result.

1 Introduction

Let $\mathbb{C}$ be a set of complex numbers and $H(\mathbb{U})$ be the class of functions analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

For $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and $a \in \mathbb{C}$, consider a class of functions

$$H[a, n] = \{f : f \in H(\mathbb{U}) \quad \text{and} \quad f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

Let $\mathcal{A}_m$ denotes a class of functions $f \in H[0, m]$ of the form:

$$f(z) = z^m + \sum_{n=m+1}^{\infty} a_n z^n. \tag{1.1}$$

A function $f \in \mathcal{A}_m$ is said to be starlike in $\mathbb{U}$ if [2]

$$\Re e\left(\frac{zf'(z)}{f(z)}\right) > 0 \ \ (z \in \mathbb{U}).$$

We say that an analytic function p is subordinate to an analytic function q, and write $p(z) \prec q(z)$, if and only if there exists a function ω, analytic in $\mathbb{U}$ such that $\omega(0) = 0$, $|\omega(z)| < 1$ for $z \in \mathbb{U}$ and $p(z) = q(\omega(z))$. In particular, if q is univalent in $\mathbb{U}$, then we have the following equivalence:

$$p(z) \prec q(z) \iff p(0) = q(0) \text{ and } p(\mathbb{U}) \subset q(\mathbb{U}). \tag{1.2}$$

In view of the second-order differential subordination [5, p. 16], here we use the first-order differential subordination and its best dominant which is defined as follows:

Definition 1.1. *Let $\psi : \mathbb{C}^2 \to \mathbb{C}$ and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the following differential subordination*

$$\psi\left(p(z), zp'(z)\right) \prec h(z), \tag{1.3}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply, a dominant, if $p \prec q$ for all p satisfying (1.3). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1.3) is said to be the best dominant of (1.3).

Denote by $\mathcal{P}$ the class of functions $p \in H[1,1]$ and $\Re e\,(p(z)) > 0 \ (z \in \mathbb{U})$. Let $\mathcal{Q}$ be a class of functions $\phi \in \mathcal{P}$ such that $\phi(\mathbb{U})$ is convex (univalent) and symmetrical with respect to the real axis.

We define a class $\mathcal{P}(\eta, \phi)$ of functions $p \in H[1,1]$ such that

$$\frac{1}{1-\eta}\left(p(z) - \eta\right) \prec \phi(z) \qquad (\eta \neq 1, \phi \in \mathcal{Q}; z \in \mathbb{U}).$$

Note that the class $\mathcal{P}(0, \phi) = \mathcal{P}(\phi)$ is a Ma and Minda [4] type class.

In this paper, we consider functions $f \in \mathcal{A}_m$ satisfying certain Ma-Minda type subordination condition. For this we define a class $\mathcal{P}(\eta, \phi)$ of functions $p \in H[1,1]$ and an operator $V^n_{\mu,m}$ which is defined on the class $H[1,1]$. Functions satisfying the condition $V^n_{\mu,m}\left(\frac{f(z)}{z^m}\right)^{\mu} \in \mathcal{P}(\eta, \phi)$ are studied. Some characteristic results such as subordination results, best dominant of the subordinations, a coefficient result, bounds and a radius result are obtained with the use of certain key lemmas. Throughout the work only principal values of the powers are considered.

2 A set of key Lemmas

In this section, we give some lemmas that will be used in proving the desired results in the subsequent work.

Lemma 2.1. *[3] (see [5, Theorem 3.1b, p. 71])Let h be convex in $\mathbb{U}$, with $h(0) = a$, $\gamma \neq 0$ and $\Re e\,(\gamma) \geq 0$. If $p \in H[a,n]$ and*

$$p(z) + \frac{zp'(z)}{\gamma} \prec h(z), \tag{2.1}$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t) t^{(\gamma/n)-1} \mathrm{d}t.$$

The function q is convex and is the best (a,n)-dominant.

Lemma 2.2. *([5, p. 7]) Let $a, b, c \in \mathbb{C}$ $(c \neq 0, -1, -2, \ldots)$, then*
(i) ${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} \mathrm{d}t \quad (\Re e(c) > \Re e(b) > 0)$.
(ii) ${}_2F_1(a, b; c; z) = (1-z)^{-a}\ {}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right)$.

Lemma 2.3. *[1] Let $p \in \mathcal{P}$, then for $|z| = r, 0 \leq r < 1$ and for any complex number δ, $\Re e(\delta) = \beta \geq 0$:*

$$\left| \frac{zp'(z)}{p(z) + \delta} \right| \leq \frac{2r}{(1-r)[1 + \beta + (1-\beta) r]}.$$

3 Main Results

In order to prove our first main result, we define for some $n \in \mathbb{N} \cup \{0\}$, $\mu\,(0 < \mu \leq 1)$ and $m \in \mathbb{N}$, an operator $V^n_{\mu,m} : H[1,1] \to H[1,1]$ by

$$V^0_{\mu,m} p(z) = p(z), V^1_{\mu,m} p(z) = p(z) + \frac{zp'(z)}{\mu m},$$

and for any $n \in \mathbb{N}$,

$$V^n_{\mu,m} p(z) = V^1_{\mu,m}\left(V^{n-1}_{\mu,m} p(z)\right).$$

Theorem 3.1. *Let $0 < \mu \leq 1$ and $f \in \mathcal{A}_m$. If*

$$V^{n+1}_{\mu,m} \left(\frac{f(z)}{z^m}\right)^\mu \in \mathcal{P}(\eta, \phi), \tag{3.1}$$

then

$$V^{n}_{\mu,m} \left(\frac{f(z)}{z^m}\right)^\mu \in \mathcal{P}(\eta, \phi) \tag{3.2}$$

and hence,

$$\left(\frac{f(z)}{z^m}\right)^\mu \in \mathcal{P}(\eta, \phi). \tag{3.3}$$

Proof. Let

$$p(z) := \frac{1}{1-\eta}\left(V_{\mu,m}^{n}\left(\frac{f(z)}{z^m}\right)^{\mu}-\eta\right).$$

Then $p \in H[1,1]$ and

$$\eta + (1-\eta)\,p(z) = V_{\mu,m}^{n}\left(\frac{f(z)}{z^m}\right)^{\mu}. \tag{3.4}$$

From (3.4), we obtain

$$\begin{aligned}(1-\eta)\left(p(z)+\frac{zp'(z)}{\mu m}\right) &= V_{\mu,m}^{n}\left(\frac{f(z)}{z^m}\right)^{\mu}+\frac{z\left(V_{\mu,m}^{n}\left(\frac{f(z)}{z^m}\right)^{\mu}\right)'}{\mu m}-\eta \\ &= V_{\mu,m}^{n+1}\left(\frac{f(z)}{z^m}\right)^{\mu}-\eta\end{aligned}$$

which implies by (3.1) that

$$\frac{1}{1-\eta}\left(V_{\mu,m}^{n+1}\left(\frac{f(z)}{z^m}\right)^{\mu}-\eta\right) = p(z)+\frac{zp'(z)}{\mu m} \prec \phi(z). \tag{3.5}$$

On applying Lemma 2.1 to (3.5), we get $p(z) \prec \phi(z)$ which proves the result (3.2). Consequently, we may prove $V_{\mu,m}^{n-1}\left(\frac{f(z)}{z^m}\right)^{\mu} \in \mathcal{P}(\eta,\phi)$, ..., $V_{\mu,m}^{0}\left(\frac{f(z)}{z^m}\right)^{\mu} \in \mathcal{P}(\eta,\phi)$ which is the result (3.3). □

Taking $n = 0$, in Theorem 3.1, we get following result by applying Lemma 2.1.

Corollary 3.2. *Let $0 < \mu \leq 1$ and $f \in \mathcal{A}_m$. If $V_{\mu,m}^{1}\left(\frac{f(z)}{z^m}\right)^{\mu} \in \mathcal{P}(\eta,\phi)$, then*

$$\frac{1}{1-\eta}\left(\left(\frac{f(z)}{z^m}\right)^{\mu}-\eta\right) \prec q(z) \prec \phi(z) \quad (z \in \mathbb{U})$$

and

$$q(z) = \frac{\mu m}{z^{\mu m}}\int_0^z \phi(t)t^{\mu m-1}\mathrm{d}t \tag{3.6}$$

is the best dominant.

If we take $\mu = 1$ in Corollary 3.2, we obtain a simpler result:

Corollary 3.3. *If $f \in \mathcal{A}_m$ satisfy*

$$\frac{f'(z)}{mz^{m-1}} \in \mathcal{P}(\eta,\phi),$$

then

$$\frac{1}{1-\eta}\left(\frac{f(z)}{z^m}-\eta\right) \prec s(z) \prec \phi(z) \quad (z \in \mathbb{U})$$

and

$$s(z) = \frac{m}{z^m}\int_0^z \phi(t)t^{m-1}\mathrm{d}t$$

is the best dominant.

Taking some special type of convex function $\phi(z)$ in Corollary 3.2, we get following result. Note that for $0<\nu\le 1$ and $-1\le B<A\le 1$, the function $\left(\frac{1+Az}{1+Bz}\right)^\nu$ is convex in $\mathbb{U}$ [6, Theorem 1, p.475] (see also [8]).

Corollary 3.4. *Let $0\le\eta<1, 0<\mu,\nu\le 1$ and $-1\le B<A\le 1$. If $f\in\mathcal{A}_m$ satisfy*

$$\frac{1}{1-\eta}\left(V^1_{\mu,m}\left(\frac{f(z)}{z^m}\right)^\mu-\eta\right) \prec \left(\frac{1+Az}{1+Bz}\right)^\nu \quad (z\in\mathbb{U}),$$

then

$$\frac{1}{1-\eta}\left(\left(\frac{f(z)}{z^m}\right)^\mu-\eta\right) \prec \theta(z) \prec \left(\frac{1+Az}{1+Bz}\right)^\nu \quad (z\in\mathbb{U}), \tag{3.7}$$

where

$$\theta(z) \prec \begin{cases} \left(\frac{A}{B}\right)^\nu \sum\limits_{i\ge 0}\frac{(-\nu)_i}{i!}\left(\frac{A-B}{A}\right)^i(1+Bz)^{-i}\ {}_2F_1\left(i,1;1+\mu m;\frac{Bz}{1+Bz}\right) & (B\neq 0),\\ {}_2F_1\left(-\nu,\mu m;1+\mu m;-Az\right) & (B=0)\end{cases} \tag{3.8}$$

which is the best dominant. Also

$$\Re e\left(\frac{f(z)}{z^m}\right)^\mu > (1-\eta)\,\theta(-1)+\eta. \tag{3.9}$$

Proof. On choosing $\phi(z)=\left(\frac{1+Az}{1+Bz}\right)^\nu$ in Corollary 3.2, we get result (3.7), where

$$\begin{aligned}\theta(z) &= \frac{\mu m}{z^{\mu m}}\int_0^z\left(\frac{1+At}{1+Bt}\right)^\nu t^{\mu m-1}\mathrm{d}t\\ &= \mu m\int_0^1\left(\frac{1+Asz}{1+Bsz}\right)^\nu s^{\mu m-1}\mathrm{d}s. \qquad (3.10)\end{aligned}$$

If $B\neq 0$, with the use of the binomial expansion, we write

$$\begin{aligned}\left(\frac{1+Asz}{1+Bsz}\right)^\nu &= \left(\frac{A}{B}+\frac{1-A/B}{1+Bsz}\right)^\nu\\ &= \sum_{i\ge 0}\binom{\nu}{i}\left(\frac{A}{B}\right)^{\nu-i}\left(\frac{1-A/B}{1+Bsz}\right)^i\\ &= \left(\frac{A}{B}\right)^\nu\sum_{i\ge 0}\binom{\nu}{i}(-1)^i\left(\frac{A-B}{A}\right)^i(1+Bsz)^{-i},\end{aligned}$$

where

$$\left(\begin{array}{c}\nu\\ i\end{array}\right)(-1)^{i}=\frac{\nu!}{i!\,(\nu-i)!}\,(-1)^{i}=\frac{(-\nu)\,(-\nu+1)\ldots(-\nu+i-1)}{i!}=\frac{(-\nu)_{i}}{i!}.$$

Hence, by using results (i) and (ii) of Lemma 2.2, we get result (3.8) for the case when $B\neq 0$. If $B=0$, by using result (i) of Lemma 2.2, we get from (3.10) the result (3.8) for this case. Also, we get from the subordination result (3.8), for some analytic Schwarz's function $\omega(z)$,

$$\frac{1}{1-\eta}\left(\left(\frac{f(z)}{z^{m}}\right)^{\mu}-\eta\right)=\theta(\omega(z))$$

and hence, from (3.10)

$$\begin{aligned}\Re e\left\{\frac{1}{1-\eta}\left(\left(\frac{f(z)}{z^{m}}\right)^{\mu}-\eta\right)\right\} &= \mu m\int_{0}^{1}s^{\mu m-1}\Re e\left(\frac{1+As\omega(z)}{1+Bs\omega(z)}\right)^{\nu}\mathrm{d}s\\ &> \mu m\int_{0}^{1}s^{\mu m-1}\Re e\left(\frac{1-As}{1-Bs}\right)^{\nu}\mathrm{d}s=\theta(-1)\end{aligned}$$

which proves result (3.9). □

Theorem 3.5. *Let $0<\mu\leq 1$ and $f\in\mathcal{A}_m$. If for some $\lambda>0$*

$$\left(\frac{f(z)}{z^{m}}\right)^{\mu}\left[1-\lambda+\lambda\frac{zf'(z)}{mf(z)}\right]\in\mathcal{P}\left(\eta,\phi\right), \tag{3.11}$$

then

$$\left(\frac{f(z)}{z^{m}}\right)^{\mu}\in\mathcal{P}\left(\eta,\phi\right). \tag{3.12}$$

Proof. Let

$$\psi(z):=\frac{1}{1-\eta}\left(\left(\frac{f(z)}{z^{m}}\right)^{\mu}-\eta\right) \tag{3.13}$$

Then $\psi\in H[1,1]$ and

$$\eta+(1-\eta)\,\psi(z)=\left(\frac{f(z)}{z^{m}}\right)^{\mu}. \tag{3.14}$$

In view of the condition (3.11), we obtain from (3.14),

$$\psi(z)+\frac{\lambda}{\mu m}z\psi'(z)=\frac{1}{1-\eta}\left[\left(\frac{f(z)}{z^{m}}\right)^{\mu}\left\{1-\lambda+\lambda\frac{zf'(z)}{mf(z)}\right\}-\eta\right]\prec\phi(z)$$

which by Lemma 2.1, implies that $\psi(z)\prec\phi(z)$ and this proves the result (3.12). □

Taking $\phi(z) = \left(\frac{1+Az}{1+Bz}\right)^{\nu}$ $(0 < \nu \leq 1, \ -1 \leq B < A \leq 1)$ in Theorem 3.5, we get another sufficient condition for the result (3.7) as follows:

Corollary 3.6. *Let $0 \leq \eta < 1, 0 < \mu, \nu \leq 1$ and $-1 \leq B < A \leq 1$. If for some $\lambda > 0$, $f \in \mathcal{A}_m$ satisfy*

$$\frac{1}{1-\eta}\left(\left(\frac{f(z)}{z^m}\right)^{\mu}\left\{1-\lambda+\lambda\frac{zf'(z)}{mf(z)}\right\}-\eta\right) \prec \left(\frac{1+Az}{1+Bz}\right)^{\nu} \quad (z \in \mathbb{U}),$$

then we get the result (3.7).

Remark 3.7. *Taking $\eta = 0$, $\nu = 1 = m$ and $\mu = \alpha$ in Corollary 3.6, we get the result [7, Theorem 2.1, p.3]. Further, we remark that for $\lambda = 1$, Corollary 3.6 coincides with the Corollary 3.4.*

Theorem 3.8. *Let $0 < \mu \leq 1$ and $f \in \mathcal{A}_m$. If*

$$\left(\frac{zf'(z)}{mf(z)}\right)^{\mu}\left[1+\frac{1}{m}\left(1+\frac{zf''(z)}{f'(z)}\right)-\frac{zf'(z)}{mf(z)}\right] \in \mathcal{P}(\eta, \phi), \tag{3.15}$$

then

$$\left(\frac{zf'(z)}{mf(z)}\right)^{\mu} \in \mathcal{P}(\eta, \phi). \tag{3.16}$$

Proof. Let

$$r(z) := \frac{1}{1-\eta}\left[\left(\frac{zf'(z)}{mf(z)}\right)^{\mu}-\eta\right].$$

Then $r \in H[1,1]$ and

$$\eta + (1-\eta)\, r(z) = \left(\frac{zf'(z)}{mf(z)}\right)^{\mu}. \tag{3.17}$$

On differentiating logarithmically (3.17), we obtain

$$\frac{(1-\eta)\, r'(z)}{\eta+(1-\eta)\, r(z)} = \mu\left[\frac{1}{z}+\frac{f''(z)}{f'(z)}-\frac{f'(z)}{f(z)}\right]$$

and hence, by (3.15)

$$r(z)+\frac{zr'(z)}{\mu m} = \frac{1}{1-\eta}\left[\left(\frac{zf'(z)}{mf(z)}\right)^{\mu}\left(1+\frac{1}{m}\left(1+\frac{zf''(z)}{f'(z)}\right)-\frac{zf'(z)}{mf(z)}\right)-\eta\right] \prec \phi(z)$$

which by Lemma 2.1 proves the result (3.16). □

Theorem 3.9 (A Coefficient Result)**.** *Let $0 < \mu \leq 1$, $0 \leq \eta < 1$ and $m \in \mathbb{N}$. If $f \in \mathcal{A}_m$ of the form (1.1) satisfy $V^1_{\mu,m}\left(\frac{f(z)}{z^m}\right)^{\mu} \in \mathcal{P}(\eta,\phi)$, then for $\phi(z) = 1 + \phi_1 z + ...$,*

$$|a_{m+1}| \leq \frac{|\phi_1|(1-\eta)}{\mu\left(1+\frac{1}{\mu m}\right)}. \tag{3.18}$$

Proof. Let

$$\left(\frac{f(z)}{z^m}\right)^{\mu} = 1 + \sum_{n=m+1}^{\infty} c_n z^{n-m} ,$$

where

$$c_{m+1} = \mu a_{m+1}. \tag{3.19}$$

Hence,

$$\begin{aligned} & \frac{1}{1-\eta}\left[V^1_{\mu,m}\left(\frac{f(z)}{z^m}\right)^{\mu} - \eta\right] \\ = & \frac{1}{1-\eta}\left[\left(\frac{f(z)}{z^m}\right)^{\mu} + \frac{z\left(\left(\frac{f(z)}{z^m}\right)^{\mu}\right)'}{\mu m} - \eta\right] \\ = & 1 + \sum_{n=m+1}^{\infty}\frac{1}{1-\eta}\left(1+\frac{n-m}{\mu m}\right)c_n z^{n-m} \prec 1 + \phi_1 z + ... \end{aligned}$$

which by a well-known result of Rogosinski [9] on subordination shows that

$$\frac{1}{1-\eta}\left(1+\frac{n-m}{\mu m}\right)|c_n| \leq |\phi_1| \quad (n \geq m+1)\,. \tag{3.20}$$

Above estimate (3.20) with the value (3.19) yields the desired result (3.18). □

Theorem 3.10 (A Result on Bounds)**.** *Let $-1 \leq B < A \leq 1, 0 < \mu,\nu \leq 1$ and $0 \leq \eta < 1$. If $f \in \mathcal{A}_m$ satisfy*

$$\frac{1}{1-\eta}\left(\left(\frac{f(z)}{z^m}\right)^{\mu} - \eta\right) \prec \left(\frac{1+Az}{1+Bz}\right)^{\nu},$$

then for $|z| = r$ $(0 < r < 1)$

(3.21)

$$(1-\eta)\left(\frac{1-ABr^2-(A-B)r}{1-B^2r^2}\right)^{\nu} \leq \left|\left(\frac{f(z)}{z^m}\right)^{\mu} - \eta\right| \leq (1-\eta)\left(\frac{1-ABr^2+(A-B)r}{1-B^2r^2}\right)^{\nu}.$$

Proof. Let

$$\psi(z) := \frac{1}{1-\eta}\left(\left(\frac{f(z)}{z^m}\right)^{\mu} - \eta\right) \prec \left(\frac{1+Az}{1+Bz}\right)^{\nu} \tag{3.22}$$

Then, for $|z| = r\ (0 < r < 1)$, and assuming the principal values of the powers, we get

$$\left|(\psi(z))^{1/\nu} - \frac{1-ABr^2}{1-B^2r^2}\right| \leq \frac{(A-B)\,r}{1-B^2r^2}, \tag{3.23}$$

which implies that

$$\frac{1-ABr^2-(A-B)\,r}{1-B^2r^2} \leq |\psi(z)|^{1/\nu} \leq \frac{1-ABr^2+(A-B)\,r}{1-B^2r^2}. \tag{3.24}$$

The bounds in (3.24) prove for $\psi(z)$ given by (3.22) the result (3.21). This proves Theorem 3.10. □

A simpler result follows from Theorem 3.10 if we take $\eta = 0$ there:

Corollary 3.11. *Let $-1 \leq B < A \leq 1, 0 < \mu, \nu \leq 1$. If $f \in \mathcal{A}_m$ satisfy*

$$\left(\frac{f(z)}{z^m}\right)^{\mu} \prec \left(\frac{1+Az}{1+Bz}\right)^{\nu}, \tag{3.25}$$

then for $|z| = r\ (0 < r < 1)$

$$r^m\left(\frac{1-ABr^2-(A-B)\,r}{1-B^2r^2}\right)^{\nu/\mu} \leq |f(z)| \leq r^m\left(\frac{1-ABr^2+(A-B)\,r}{1-B^2r^2}\right)^{\nu/\mu}.$$

We use Lemma 2.3 of Bernardi [1, Theorem 2, p. 115] to prove our next result which gives the radius of starlikeness of the function $f \in \mathcal{A}_m$ satisfying the subordination (3.25).

Theorem 3.12. *Let $-1 \leq B < A \leq 1, 0 < \mu, \nu \leq 1$. If $f \in \mathcal{A}_m$ satisfy the subordination (3.25), then the radius of starlikeness $r_0\,(0 < r_0 < 1)$ of f is the unique positive root of the polynomial:*

$$\begin{aligned} P(\mu, m, \nu, A, B, r) &= \mu m\,(1-r)\,(1+Ar)\,(1+Br) \\ &\quad -r\nu\left[(1+A)\,(1+Br)+(1+B)\,(1+Ar)\right]. \end{aligned} \tag{3.26}$$

Proof. Let $f \in \mathcal{A}_m$ satisfy the subordination (3.25). Then there is a Schwarz function ω with $\omega(0) = 0$, and $|\omega(z)| < 1\ (z \in \mathbb{U})$ we have

$$\left(\frac{f(z)}{z^m}\right)^{\mu} = \left(\frac{1+A\omega(z)}{1+B\omega(z)}\right)^{\nu}\ (z \in \mathbb{U}).$$

For this Schwarz function ω, we define $p \in \mathcal{P}$ such that $\omega(z) = \frac{p(z)-1}{p(z)+1}$ $(z \in \mathbb{U})$. Thus, in terms of p we write

$$\left(\frac{f(z)}{z^m}\right)^{\mu} = \left(\frac{(1+A)\,p(z)+1-A}{(1+B)\,p(z)+1-B}\right)^{\nu} \quad (z \in \mathbb{U}). \tag{3.27}$$

On differentiating logarithmically, the equation (3.27), we get after some elementary calculations that

$$\frac{zf'(z)}{f(z)} = m + \frac{\nu}{\mu}\left[\frac{zp'(z)}{p(z)+\frac{1-A}{1+A}} - \frac{zp'(z)}{p(z)+\frac{1-B}{1+B}}\right]. \tag{3.28}$$

Therefore, on applying Lemma 2.3 in (3.28), we obtain

$$\pm\Re e\left(\frac{zp'(z)}{p(z)+\frac{1-A}{1+A}}\right) \geq -\frac{2r}{(1-r)\,1+\frac{1-A}{1+A}+\left(1-\frac{1-A}{1+A}\right)r}$$

and

$$\pm\Re e\left(\frac{zp'(z)}{p(z)+\frac{1-B}{1+B}}\right) \geq -\frac{2r}{(1-r)\,1+\frac{1-B}{1+B}+\left(1-\frac{1-B}{1+B}\right)r}.$$

Hence,

$$\begin{aligned}
&\Re e\left(\frac{zf'(z)}{f(z)}\right)\\
\geq\ & m - \frac{2r\nu}{\mu\,(1-r)}\left[\frac{1}{1+\frac{1-A}{1+A}+\left(1-\frac{1-A}{1+A}\right)r} + \frac{1}{1+\frac{1-B}{1+B}+\left(1-\frac{1-B}{1+B}\right)r}\right]\\
=\ & m - \frac{r\nu}{\mu\,(1-r)}\left[\frac{1+A}{1+Ar} + \frac{1+B}{1+Br}\right]\\
=\ & m - \frac{r\nu}{\mu\,(1-r)}\left[\frac{(1+A)\,(1+Br)+(1+B)\,(1+Ar)}{(1+Ar)\,(1+Br)}\right]\\
=\ & \frac{\mu m\,(1-r)\,(1+Ar)\,(1+Br) - r\nu\,[(1+A)\,(1+Br)+(1+B)\,(1+Ar)]}{\mu\,(1-r)\,(1+Ar)\,(1+Br)}\\
=\ & \frac{P(\mu,m,\nu,A,B,r)}{(1-r)\,(1+Ar)\,(1+Br)},
\end{aligned}$$

and this proves Theorem 3.12. Since, $P(\mu,m,\nu,A,B,0) = \mu m > 0$ and

$$P(\mu,m,\nu,A,B,1) = -2\nu\,(1+A)\,(1+B) \leq 0,$$

the polynomial $P(\mu,m,\nu,A,B,r)$ has a positive real root r_0 in $(0,1)$ which is the smallest of the roots. The inequality $\Re e\left(\frac{zf'(z)}{f(z)}\right) > 0$ is valid for $|z| = r < r_0$, and hence, the radius of starlikeness is r_0. □

Acknowledgment

Vivek Kumar Yadav is very thankful to CSIR, Delhi, India for financial assistance to carry out the work.

References

[1] Bernardi, S. D. (1974): New distortion theorems for functions of positive real part and applications to the partial sums of univalent convex functions, Proc. Amer. Math. Soc. Vol 45 , pp 113-118.

[2] Duren, P. L. (1983): *Univalent functions* , Grundlehren der Mathematischen Wissenschaften, 259, Springer-Verlag, New York.

[3] Hallenbeck, D. J. and Ruscheweyh, S. (1975): Subordination by convex functions, Proc. Amer. Math. Soc. Vol 52, pp 191-195.

[4] Ma, W.C. and Minda, D. (1992): A unified treatment of some special classes of univalent functions, in Proceedings of the Conference on Complex Analysis, 1992. International Press, Cambridge, MA, pp 157-169

[5] Miller, S. S. and Mocanu, P. T. (2000): *Differential subordinations*, Monographs and Textbooks in Pure and Applied Mathematics, 225, Marcel Dekker, Inc., New York.

[6] Prajapat, J. K. and Raina, R. K. (2010): Some results on certain classes of multivalently analytic functions based on differential subordination involving a convolution structure, Math. Slovaca Vol.60 , No. 4, pp 471-484.

[7] Prajapat, J. K. and Agarwal, R. (2013): Some results on certain class of analytic functions based on differential subordination, Bull. Korean Math. Soc. Vol,50 , No.1, pp 1-10.

[8] Raina, R. K., Sharma, P. and Sokół, J. (2020): A class of strongly close-to-convex functions, Bol. Soc. Parana. Mat. Vol.38, No. 6, pp 9-24.

[9] Rogosinski, W. (1943): On the coefficients of subordinate functions, Proc. London Math. Soc. Vol. 48 No. 2, pp. 48-82.