# Some Results on Modified Conformable Fractional Differential Equations

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**Abstract**

Recently, we introduced new concept of modified conformable fractional derivative which is based on limit. In this paper, we derive Abels formula and Wronskian formula. Moreover, we will discuss some results on existence and uniqueness theorems for modified conformable $\alpha$-fractional differential equations.

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## 1 Introduction

The theory of fractional derivation has known great importance in mathematical research from last few decades. Fractional differential equations do not have any known method to get exact solution but there are methods which gives the approximate and numerical solutions of fractional order differential equations. Fractional derivative [9, 12] was discovered in a discussion between L. Hospital and Leibnitz through a letter many mathematicians like Hadmard, Erdelyi-Kobe, Fourier, Euler, Mittag- Leffler, Laplace, Riemann, Grunewald etc. tried to develop the definition of fractional derivative.The definition of fractional derivative dont have a standard form. But the many of researchers are used to definitions of fractional derivative are Riemann-Liouville and Caputo [12].

Recently, many researcher introduce new definitions on fractional derivative and fractional integral which can be found in [1, 2, 3, 4, 5, 8, 10, 12].

## 2 Modified $\alpha$-Fractional Derivative and Integral

In [11], we introduced the concept of modified conformable fractional derivative and integral by doing some appropriate modification in the classical definition of an ordinary derivative, which is defined as.

**Definition 2.1.** **$\alpha$-Fractional Derivative[11]** *Let $\Phi\colon \mathbb{R}^+ \to \mathbb{R}$ and $r > 0$, then $\alpha$-fractional derivative of $\Phi$ is given by*

$$T_\alpha(\Phi) = \lim_{h\to 0} \frac{\Phi\left(r + \mu r^{\frac{1-\alpha}{\alpha}}\right) - \Phi(r)}{\mu} \tag{2.1}$$

*for all $r > 0$, $\alpha \in (0, 1]$.*

**Theorem 2.2.** *[11] If $\Phi$ is a $\alpha$-differential function at point $r > 0$, then*

$$T_\alpha\Phi(r) = r^{\frac{1-\alpha}{\alpha}} \frac{d\Phi(r)}{dx} \tag{2.2}$$

*Proof.* Let $\Phi$ be the $\alpha$-differential function at $r > 0$, then we have

$$\begin{aligned} T_\alpha\Phi(r) &= \lim_{\mu\to 0} \left( \frac{\Phi\left(r + \mu r^{\frac{1-\alpha}{\alpha}}\right) - \Phi(r)}{\mu} \right) \\ T_\alpha\Phi(r) &= r^{\frac{1-\alpha}{\alpha}} \frac{d\Phi(r)}{dx}. \end{aligned}$$

□

**Definition 2.3.** *Let , $0 \le \gamma \le r$ and $\Phi$ be a function defined on $(\gamma, r]$, then New $\alpha$-fractional integral is defined by*

$$I_\alpha^\gamma \Phi(r) = \int_\gamma^r \frac{\Phi(\mu)}{\mu^{\frac{1-\alpha}{\alpha}}} d\mu, \tag{2.3}$$

*provided integral exists. It is interesting to note that for $\alpha = 1$, the definition coincides with the classical definition of first order derivative.*

**Definition 2.4.** *We assume that $\gamma_1(\xi), \gamma_2(\xi), \ldots, \gamma_n(\xi) \in C^{n-1}[I]$ are $(n-1)\alpha$-differentiable functions and $\alpha \in (0, 1]$, then*

$$W_\alpha(\gamma_1, \gamma_2, , \gamma_n) = \begin{vmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ D_\alpha\gamma_1 & D_\alpha\gamma_2 & \cdots & D_\alpha\gamma_n \\ \vdots & \vdots & \vdots & \vdots \\ D_\alpha^{n-1}\gamma_1 & D_\alpha^{n-1}\gamma_2 & \cdots & D_\alpha^{n-1}\gamma_n \end{vmatrix} \tag{2.4}$$

is called $\alpha$-Wronskian of the functions $\gamma_1(\xi), \gamma_2(\xi), \ldots, \gamma_n(\xi)$.
**Note:** Throughout in this paper $I$ is an interval $(a, b)$ and $a \ge \xi_0 > 0$.

## 3 Existence and Uniqueness Theorems on Modified Conformable $\alpha$-Fractional Derivative

Consider $\alpha$-fractional differential equation of order $n\alpha$ is as follows

$$D_\alpha^n\gamma + p_{n-1}(\xi)D_\alpha^{n-1}\gamma + \cdots + p_2(\xi)D_\alpha^2\gamma + p_1(\xi)D_\alpha\gamma + p_0(\xi)\gamma = 0. \tag{3.1}$$

Where $D_\alpha^n\gamma = D_\alpha D_\alpha \dots D_\alpha\gamma$. Corresponding, non-homogeneous case is as follows

$$D_\alpha^n\gamma + p_{n-1}(\xi)D_\alpha^{n-1}\gamma + \cdots + p_2(\xi)D_\alpha^2\gamma + p_1(\xi)D_\alpha\gamma + p_0(t)\gamma = f(t). \tag{3.2}$$

We define an $n^{th}$ order differential operator as follows:

$$L_\alpha[\gamma] = D_\alpha^n\gamma + p_{n-1}(\xi)D_\alpha^{n-1}\gamma + \cdots + p_2(\xi)D_\alpha^2 y + p_1(\xi)D_\alpha\gamma + p_0(\xi)\gamma = 0 \tag{3.3}$$

**Theorem 3.1.** *Suppose $\xi^{\frac{\alpha-1}{\alpha}}p(\xi), \xi^{\frac{\alpha-1}{\alpha}}q(\xi) \in C(I)$ and $\gamma$ be $\alpha$-differentiable with $\alpha \in (0,1]$, then solution of an I.V.P.*

$$D_\alpha\gamma + p(\xi)y = q(\xi) \tag{3.4}$$

$$\gamma(\xi_0) = \gamma_0,$$

*is exist and unique on $I$, where $\xi_0 \in I$.*

*Proof.* The given I.V.P is

$$\begin{aligned} D_\alpha\gamma + p(\xi)\gamma &= q(\xi) \\ \Rightarrow \xi^{\frac{1-\alpha}{\alpha}}\gamma' + p(\xi)\gamma &= q(\xi) \end{aligned}$$

$$\Rightarrow \gamma' + \xi^{\frac{\alpha-1}{\alpha}}p(\xi)\gamma = \xi^{\frac{\alpha-1}{\alpha}}q(\xi). \tag{3.5}$$

Hence it has unique solution on the interval $I$. □

**Theorem 3.2.** *Suppose $\xi^{\frac{\alpha-1}{\alpha}}p_{n-1}(\xi), \dots, \xi^{\frac{\alpha-1}{\alpha}}p_1(\xi), \xi^{\frac{\alpha-1}{\alpha}}p_0(\xi), \xi^{\frac{\alpha-1}{\alpha}}q(\xi) \in C[I]$ and $\gamma \in C^n[I]$, then a solution $\gamma(\xi)$ of the I.V.P.*

$$D_\alpha^n\gamma + p_{n-1}(\xi)D_\alpha^{n-1}\gamma + \cdots + p_2(\xi)D_\alpha^2\gamma + p_1(\xi)D_\alpha\gamma + p_0(t)\gamma = q(\xi) \tag{3.6}$$

$$\gamma(\xi_0) = \gamma_0, D_\alpha\gamma(\xi_0) = \gamma_1, \dots, D_\alpha^{n-1}\gamma(\xi_0) = \gamma_{n-1}, \quad \xi_0 \in I$$

*is exists and unique on the interval $I$.*

*Proof.* Let $\gamma(\xi)$ be the solution of the (3.6).
**Claim:** The equation (3.6 ) has unique solution.
Firstly, we will define,

$$\nu_1 = \gamma, \nu_2 = D_\alpha\gamma, \nu_3 = D_\alpha^2\gamma, \dots, \nu_n = D_\alpha^n\gamma.$$

Therefore, we have

$$\begin{aligned} D_\alpha \nu_1 &= \nu_2 \\ D_\alpha \nu_2 &= \nu_3 \\ &\vdots \\ D_\alpha \nu_{n-1} &= \nu_{n-1} \end{aligned}$$

$D_\alpha \nu_n = -p_{n-1}\nu_n - \cdots - p_2\nu_3 - p_1\nu_2 - p_0\nu_1 + q(\xi)$.
Now, above system can be written as follows

$$T_\alpha \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_{n-1} \\ \nu_n \end{bmatrix} + \begin{bmatrix} 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \\ p_0 & p_1 & p_2 & p_3 & \cdots & p_{n-1} \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_{n-1} \\ \nu_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ q(\xi) \end{bmatrix}$$

$$D_\alpha \vartheta(\xi) + P(\xi)\vartheta(\xi) = Q(\xi),$$

$$\vartheta'(\xi) + \xi^{\frac{\alpha-1}{\alpha}} P(\xi)\vartheta(\xi) = Q(\xi)\xi^{\frac{\alpha-1}{\alpha}}. \tag{3.7}$$

By classical Theorem [6], the I.V.P (3.6) has unique solution. □

**Theorem 3.3.** *If $\gamma_1, \gamma_2 \in C^n[I]$ with $c_1, c_2$ are arbitrary numbers, then*

$$L_\alpha[c_1\gamma_1 + c_2\gamma_2] = c_1 L_\alpha[\gamma_1] + c_2 L_\alpha[\gamma_2]. \tag{3.8}$$

*Proof.* As $\gamma_1, \gamma_2 \in C^n[I]$ with $c_1, c_2$ are arbitrary numbers, then we have

$$\begin{aligned} L_\alpha[c_1\gamma_1 + c_2\gamma_2] &= D^n_\alpha(c_1\gamma_1 + c_2\gamma_2) + p_{n-1}(\xi)D^{n-1}_\alpha(c_1\gamma_1 + c_2\gamma_2) + \cdots \\ &+ p_1(\xi)D_\alpha(c_1\gamma_1 + c_2\gamma_2) + p_0(\xi)(c_1\gamma_1 + c_2\gamma_2) \\ &= c_1(D^n_\alpha\gamma_1 + p_{(n-1)}(\xi)D^{n-1}_\alpha\gamma_1 + \cdots + p_0(\xi)\gamma_1) \\ &+ c_2(D^n_\alpha\gamma_2 + p_{(n-1)}(\xi)D^{n-1}_\alpha\gamma_2 + \cdots + p_0(\xi)\gamma_2). \end{aligned}$$

□

**Theorem 3.4.** *If $\gamma_k, k = 1, \ldots, n.$ be the solutions of $L_\alpha[\gamma] = 0$, then there linear combination*

$$\gamma = c_1\gamma_1 + c_2\gamma_2 + \cdots + c_n\gamma_n \tag{3.9}$$

*is also solution of $L_\alpha[\gamma] = 0$, for the arbitrary constants $c_k, k = 1, \ldots, n$.*

*Proof.* As

$$\gamma = c_1\gamma_1 + c_2\gamma_2 + \cdots + c_n\gamma_n.$$

be the linear combination of $\gamma_1(\xi), \gamma_2(\xi), \ldots, \gamma_n(\xi)$, where $c_k$, for $k = 1, \ldots, n$, are arbitrary constants then by Theorem 3.3, $L_\alpha(\gamma) = 0$ . □

**Definition 3.5.** *If* $\{\gamma_k\}_{k=1}^{n}$ *be the set of* $n$ *solutions of* $L_\alpha(\gamma) = 0$, *then the solution* $\gamma = c_1\gamma_1 + c_2\gamma_2 + \cdots + c_n\gamma_n$ *of* $L_\alpha(\gamma) = 0$ *is called as a fundamental set of solutions.*

**Theorem 3.6.** *Let* $\gamma_k$, *where* $k = 1, 2, ..., n$, *be* $n$ *solutions of* $L_\alpha(\gamma) = 0$. *If there exists* $\xi_0 \in I$, *such that*

$$W_\alpha(\gamma_1, \gamma_2, \ldots, \gamma_n)(\xi_0) \neq 0, \tag{3.10}$$

*then* $\{\gamma_k\}_{k=1}^{n}$ *is a fundamental set of solutions.*

*Proof.* If $\gamma(\xi)$ is solution of $L_\alpha[\gamma] = 0$, then

$$y = c_1\gamma_1 + c_2\gamma_2 + \cdots + c_n\gamma_n.$$

So, it can be written as,

$$\begin{aligned} c_1\gamma_1(\xi_0) + c_2\gamma_2(\xi_0) + \cdots + c_n\gamma_n(\xi_0) &= \gamma(\xi_0) \\ c_1 D_\alpha\gamma_1(\xi_0) + c_2 D_\alpha\gamma_2(\xi_0) + \cdots + c_n D_\alpha\gamma_n(\xi_0) &= D_\alpha\gamma(\xi_0) \\ \vdots \quad \vdots \quad \vdots \end{aligned}$$

$$c_1 D_\alpha^{n-1}\gamma_1(\xi_0) + c_2 D_\alpha^{n-1}\gamma_2(\xi_0) + \cdots + c_n D_\alpha^{n-1}\gamma_n(\xi_0) = D_\alpha^{n-1}\gamma(\xi_0) \tag{3.11}$$

By applying Cramers rule, we have

$$c_k = \frac{W_\alpha^k(\xi_0)}{W_\alpha(\xi_0)}, \quad 1 \leq k \leq n. \tag{3.12}$$

As $W_\alpha(\xi_0) \neq 0$ then it follows that, the constants $c_1, c_2, \ldots, c_n$ are exist. □

**Theorem 3.7.** *If* $\xi^{\frac{\alpha-1}{\alpha}} p_{n-1}(\xi), \ldots, \xi^{\frac{\alpha-1}{\alpha}} p_1(\xi), t^{\frac{\alpha-1}{\alpha}} p_0(\xi) \in C(I)$, *then the set* $\{\gamma_k\}_{k=1}^{n}$ *be a fundamental set of solutions of the* $L_\alpha[\gamma] = 0$.

*Proof.* Let $\xi^{\frac{\alpha-1}{\alpha}} p_{n-1}(\xi), \ldots, \xi^{\frac{\alpha-1}{\alpha}} p_1(\xi), \xi^{\frac{\alpha-1}{\alpha}} p_0(\xi) \in C(I)$ and $\xi_0 \in I$.
Now, consider the following I.V.P,

$$\begin{gathered} L_\alpha[\gamma] = 0, \gamma(\xi_0) = 1, D_\alpha\gamma(\xi_0) = 0, \ldots, D_\alpha^{n-1}\gamma(\xi_0) = 0 \\ L_\alpha[\gamma] = 0, \gamma(\xi_0) = 0, D_\alpha\gamma(\xi_0) = 1, \ldots, D_\alpha^{n-1}\gamma(\xi_0) = 0 \\ \vdots \\ L_\alpha[\gamma] = 0, \gamma(\xi_0) = 0, D_\alpha\gamma(\xi_0) = 0, \ldots, D_\alpha^{n-1}\gamma(\xi_0) = 1. \end{gathered}$$

From[6], it gives that there is the solution $\gamma_k$ of $k^{th}$ problem for all $k$. As

$$W_\alpha(\xi) = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{vmatrix} = 1 \neq 0,$$

then by the Theorem 3.6, we are through. □

**Theorem 3.8.** *If $\gamma_k$, where $k \in \{1,2,...,n\}$, are $n$ solutions of equation $L_\alpha[\gamma] = 0$, then following are holds*

1. *$W_\alpha(\xi)$ is satisfies the differential equation*

$$D_\alpha W + p_{n-1}(\xi)W = 0.$$

2. *If $\xi_0 \in I$ then,*

$$W_\alpha(\xi) = W_\alpha(\xi_0)e^{\int_{\xi_0}^{\xi} \nu^{\frac{\alpha-1}{\alpha}}(p_{n-1}(\nu))}d\nu.$$

*Moreover, if $W_\alpha(\xi_0) \neq 0$ then $W_\alpha(\xi) \neq 0$, for all $\xi \in I$.*

*Proof.* We introduce new variables

$$\nu_1 = \gamma, \nu_2 = D_\alpha\gamma, \nu_3 = D_\alpha^2\gamma, \cdots, \nu_n = D_\alpha^n\gamma. \tag{3.14}$$

then it can rewriten as,

$$D_\alpha \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu_{n-1} \\ \nu_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 1 & 0 & 0 & \cdots & 1 \\ -p_0 & -p_1 & -p_2 & -p_3 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \\ \vdots \\ \nu n-1 \\ \nu_n \end{bmatrix} \tag{3.14}$$

$$D_\alpha\vartheta(\xi) = P(\xi)\vartheta(\xi)$$

$$\Rightarrow D_\alpha W_\alpha(\xi) = (p_{11} + p_{22} + \cdots + p_{nn})W_\alpha(\xi).$$

Really, if $\alpha$-differentiable of $W_\alpha(\xi)$.
Hence

$$\frac{D_\alpha W_\alpha(\xi)}{W_\alpha(\xi)} = -p_{n-1}(\xi)$$

$$\Rightarrow ln(W_\alpha(\xi)) - ln(W_\alpha(\xi_0)) = e^{\int_{\xi_0}^{\xi} \nu^{\frac{\alpha-1}{\alpha}}}(p_{n-1}(\nu))d\nu$$

$$\Rightarrow W_\alpha(\xi) = W_\alpha(\xi_0)e^{\int_{\xi_0}^{\xi} \nu^{\frac{\alpha-1}{\alpha}}}(p_{n-1}(\nu))d\nu \tag{3.15}$$

□

**Theorem 3.9.** *Let $\xi^{\frac{\alpha-1}{\alpha}}p_{n-1}(\xi), \ldots, \xi^{\frac{\alpha-1}{\alpha}}p_1(\xi), \xi^{\frac{\alpha-1}{\alpha}}p_0(\xi) \in C(I)$ , if $\{\gamma_k\}_{k=1}^n$ is the set of fundamental solutions of $L_\alpha[\gamma] = 0$, then $W_\alpha(\xi) \neq 0$ for all $\xi \in I$.*

*Proof.* Let $\xi^{\frac{\alpha-1}{\alpha}}p_{n-1}(\xi), \ldots, \xi^{\frac{\alpha-1}{\alpha}}p_1(\xi), \xi^{\frac{\alpha-1}{\alpha}}p_0(\xi) \in C(I)$ and suppose that $\xi_0 \in I$. By Theorem 3.2, $\exists$ a unique solution $\gamma(\xi)$ of the I.V.P

$$L_\alpha[\gamma] = 0, \gamma(\xi_0) = 1, D_\alpha\gamma(\xi_0) = 0, \cdots, D_\alpha^{n-1}\gamma(\xi_0) = 0. \tag{3.16}$$

As $\{\gamma_k\}_{k=1}^n$ is a set of solutions then $\exists$ unique constants $c_k$, $k \in \{1, 2, ..., n\}$ such that

$$\begin{aligned} c_1\gamma_1(\xi) + c_2\gamma_2(\xi) + \cdots + c_n\gamma_n(\xi) &= \gamma(\xi) \\ c_1 D_\alpha\gamma_1(\xi) + c_2 D_\alpha\gamma_2(\xi) + \cdots + c_n D_\alpha\gamma_n(\xi) &= D_\alpha\gamma(\xi) \\ \vdots \quad \vdots \quad \vdots \end{aligned}$$

$$c_1 D_\alpha^{n-1}\gamma_1(\xi) + c_2 D_\alpha^{n-1}\gamma_2(\xi) + \cdots + c_n D_\alpha^{n-1}\gamma_n(\xi) = D_\alpha^{n-1}\gamma(\xi) \tag{3.17}$$

for all $\xi \in I$. In particular, if $\xi = \xi_0$, then we obtain the system

$$\begin{aligned} c_1\gamma_1(\xi_0) + c_2\gamma_2(\xi_0) + \cdots + c_n\gamma_n(\xi_0) = 1 \\ c_1 D_\alpha\gamma_1(\xi_0) + c_2 D_\alpha\gamma_2(\xi_0) + \cdots + c_n D_\alpha\gamma_n(\xi_0) = 0 \\ \vdots \quad \vdots \quad \vdots \end{aligned}$$

$$c_1 D_\alpha^{n-1}\gamma_1(\xi_0) + c_2 D_\alpha^{n-1}\gamma_2(\xi_0) + \cdots + c_n D_\alpha^{n-1}\gamma_n.(\xi_0) = 0. \tag{3.18}$$

This system has a unique solution

$$c_k = \frac{W_\alpha^k}{W_\alpha(t_0)}, \quad 1 \le k \le n. \tag{3.19}$$

Here, for every $k$

$$W_\alpha^k = \begin{bmatrix} \gamma_1(\xi_0) & \cdots & \gamma_{k-1}(\xi_0) & 1 & \gamma_k(\xi_0) & \cdots & \gamma_n(\xi_0) \\ D_\alpha\gamma_1(\xi_0) & \cdots & D_\alpha\gamma_{k-1}(\xi_0) & 0 & D_\alpha\gamma_k(\xi_0) & \cdots & D_\alpha\gamma_n(\xi_0) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ D_\alpha^{n-1}\gamma_1(\xi_0) & \cdots & D_\alpha^{n-1}\gamma_{k-1}(\xi_0) & 0 & D_\alpha^{n-1}\gamma_k(\xi_0) & \cdots & D_\alpha^{n-1}\gamma_n(\xi_0) \end{bmatrix}.$$

For the existence of $c_1, c_2, \ldots, c_n$, it should be $W_\alpha(\xi_0) \neq 0$. By Theorem 3.8, we conclude that $W_\alpha(\xi_0) \neq 0$, for all $\xi_0 \in I$. □

**Theorem 3.10.** *If $\xi^{\frac{\alpha-1}{\alpha}} p_{n-1}(\xi), \ldots, \xi^{\frac{\alpha-1}{\alpha}} p_1(\xi), \xi^{\frac{\alpha-1}{\alpha}} p_0(\xi) \in C(I)$, then the solution set $\{\gamma_k\}_{k=1}^n$ is a fundamental set of solutions of $L_\alpha[\gamma] = 0$ iff the functions $\gamma_k$, $k \in \{1, 2, ..., n\}$, are L.I.*

*Proof.* By Theorem 3.9, $\exists\ \xi_0 \in I$ such that $W_\alpha(\xi_0) \neq 0$ and hence we have

$$c_1\gamma_1(\xi) + c_2\gamma_2(\xi) + \cdots + c_n\gamma_n(\xi) = 0 \tag{3.20}$$

for all $\xi \in I$. Here we can find unique constants $c_i, i = 1, 2, ..., n$, such that

$$\begin{aligned} c_1\gamma_1(\xi) + c_2\gamma_2(\xi) + \cdots + c_n\gamma_n(\xi) &= 0 \\ c_1 D_\alpha\gamma_1(\xi) + c_2 D_\alpha\gamma_2(\xi) + \cdots + c_n D_\alpha\gamma_n(\xi) &= 0 \end{aligned}$$

$$\vdots \quad \vdots \quad \vdots$$

$$c_1 D_\alpha^{n-1}\gamma_1(\xi) + c_2 D_\alpha^{n-1}\gamma_2(\xi) + \cdots + c_n D_\alpha^{n-1}\gamma_n(\xi) = 0.$$

By using Cramers rule, we have, $c_k = 0, k \in \{1, 2, ..., n\}$ Thus, set of functions $\{\gamma_k\}_{k=1}^n$ is L.I.
Conversely, let $\{\gamma_k\}_{k=1}^n$ is a L.I set.
**Claim:** $\{\gamma_k\}_{k=1}^n$ is a fundamental set.
If possible, $\{\gamma_k\}_{k=1}^n$ is not a fundamental set then, by Theorem 3.6, we get $W_\alpha(\xi) = 0$, for all $\xi \in I$. If we choose any $\xi_0 \in I$ then $W_\alpha(\xi_0) = 0$. But as $W_\alpha(\xi_0) \neq 0$, then the matrix

$$\begin{bmatrix} \gamma_1(\xi_0) & \gamma_2(\xi_0) & \cdots & \gamma_n(\xi_0) \\ D_\alpha\gamma_1(\xi_0) & D_\alpha\gamma_2(\xi_0) & \cdots & D_\alpha\gamma_n(\xi_0) \\ \vdots & \vdots & \vdots & \vdots \\ D_\alpha^{n-1}\gamma_1(\xi_0) & D_\alpha^{n-1}\gamma_2(\xi_0) & \cdots & D_\alpha^{n-1}\gamma_n(\xi_0) \end{bmatrix} \tag{3.21}$$

is singular then $\exists\ c_k, k \in \{1, 2, \ldots, n\}$ with $c_1^2 + c_2^2 + \cdots + c_n^2 \neq 0$, such that

$$c_1\gamma_1(\xi_0) + c_2\gamma_2(\xi_0) + \cdots + c_n\gamma_n(\xi_0) = 0$$

$$c_1 D_\alpha\gamma_1(\xi_0) + c_2 D_\alpha\gamma_2(\xi_0) + \cdots + c_n D_\alpha\gamma_n(\xi_0) = 0$$

$$\vdots \quad \vdots \quad \vdots$$

$$c_1 D_\alpha^{n-1}\gamma_1(\xi) + c_2 D_\alpha^{n-1}\gamma_2(\xi) + \cdots + c_n D_\alpha^{n-1}\gamma_n(\xi) = 0.$$

Now, let

$$\gamma(\xi) = c_1\gamma_1(\xi) + c_2\gamma_2(\xi) + \cdots + c_n\gamma_n(\xi) \tag{3.22}$$

for all $\xi \in I$, then $\gamma(\xi)$ is the solution of the differential equation subjected to the initial conditions

$$\gamma(\xi_0) = D_\alpha\gamma(\xi_0) = \cdots = D_\alpha^{n-1}\gamma(\xi_0) = 0.$$

By Theorem 3.2, as

$$c_1\gamma_1(\xi) + c_2\gamma_2(\xi) + \cdots + c_n\gamma_n(\xi) = 0$$

for all $\xi \in I$, with $c_1, c_2, \ldots, c_n$ not all equal to zero then $\gamma_1, \gamma_2, \ldots, \gamma_n$ are linearly dependent, a contradiction. □

**Theorem 3.11.** *If $\gamma_k, k = 1, 2, ..., n$, be the L.I solutions of $L_\alpha[\gamma] = 0$, then $y = \sum_{k=1}^n c_k\gamma_k$, where $c_k$, for $k = 1, \ldots, n$, are arbitrary constants, be the general solution of $L_\alpha[\gamma] = 0$.*

*Proof.* If $\xi = \xi_0$, then particular solution is obtained by using initial conditions as follows,

$$\gamma(\xi_0) = \lambda_0, D_\alpha\gamma(\xi_0) = \lambda_1, \ldots, D_\alpha^{n-1}\gamma(\xi_0) = \lambda_{n-1} \tag{3.23}$$

where $\xi_0 \in I$ and $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ are arbitrary constants. If we choose constants $c_1, c_2, \ldots, c_n$, which satisfy the conditions (3.23), then proof is completed. Now consider

$$c_1\gamma_1(\xi_0) + c_2\gamma_2(\xi_0) + \cdots + c_n\gamma_n(\xi_0) = \lambda_0$$

$$c_1 D_\alpha \gamma_1(\xi_0) + c_2 D_\alpha \gamma_2(\xi_0) + \cdots + c_n D_\alpha \gamma_n(\xi_0) = \lambda_1$$

$$\vdots \quad \vdots \quad \vdots$$

$$c_1 D_\alpha^{n-1} \gamma_1(\xi_0) + c_2 D_\alpha^{n-1} \gamma_2(\xi_0) + \cdots + c_n D_\alpha^{n-1} \gamma_n(\xi_0) = \lambda_{n-1}. \tag{3.24}$$

Above system of equation can be written as,

$$\begin{bmatrix} \gamma_1(\xi_0) & \gamma_2(\xi_0) & \cdots & \gamma_n(\xi_0) \\ D_\alpha \gamma_1(\xi_0) & D_\alpha \gamma_2(\xi_0) & \cdots & D_\alpha \gamma_n(\xi_0) \\ \vdots & \vdots & \vdots & \vdots \\ D_\alpha^{n-1} \gamma_1(\xi_0) & D_\alpha^{n-1} \gamma_2(\xi_0) & \cdots & D_\alpha^{n-1} \gamma_n(\xi_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_{n-1} \end{bmatrix}. \tag{3.25}$$

As $\gamma_i, i = 1, 2, ..., n$, are L.I. solutions then $W_\alpha(\xi_0) \neq 0$ and hence system (3.24) has a unique solution. □

**Theorem 3.12.** *If $\gamma_p$ is any particular solution of* (3.2) *and $\{\gamma_k\}_{k=1}^n$ be the set of fundamental solutions of* (3.1), *then the general solution of* (3.1), *is*

$$\gamma = \sum_{k=1}^{n} c_k \gamma_k + \gamma_p \tag{3.26}$$

*where $c_k, k = 1, \ldots, n$, are arbitrary constants.*

*Proof.* Let $\Upsilon(\xi)$ and $\gamma_p(\xi)$ are the general and particular solutions of $L_\alpha[\gamma] = q(\xi)$.
If $u(\xi) = \Upsilon(\xi) - \gamma_p(\xi)$, then

$$L_\alpha[u] = q(\xi) - q(\xi) = 0. \tag{3.27}$$

Therefore $u(\xi)$ is a solution of $L_\alpha[\gamma] = 0$ and hence by Theorem 3.4, we have

$$u(\xi) = c_1 \gamma_1(\xi) + c_2 \gamma_2(\xi) + \cdots + c_n \gamma_n(\xi)$$

And so

$$\Upsilon(\xi) - \gamma_p(\xi) = \sum_{k=1}^{n} c_k \gamma_k(\xi)$$

$$\Rightarrow \Upsilon(\xi) = \sum_{k=1}^{n} c_k \gamma_k(\xi) + \gamma_p(\xi) \tag{3.28}$$

be the general solution of $L_\alpha[\gamma] = q(\xi)$. □

## 4 Conclusion

In this paper, we obtained an Abels & Wronskian formula. We also discussed existence and uniqueness theorems for sequential linear modified $\alpha$-fractional differential equations.

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