On Special Weakly Projective Symmetric Manifolds

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Abstract

The notion of a weakly symmetric and weakly projective symmetric Riemannian manifolds have been introduced by Tamassy and Binh [11], [12] and then after studied by so many authors such as De, Shaikh and Jana, Shaikh and Hui, Shaikh, Jana and Eyasmin ([1], [3] to [8]) . Recently, Singh and Khan [10] introduced the notion of special weakly symmetric Riemannian manifolds and denoted such manilfold by $(\mathrm{SWS})_n$. ln this paper, we have studied the nature of Ricci tensor R of type (1,1) in a special weakly projective symmetric Riemannian manifold $(\mathrm{SWPS})_n$ and also investigated some interesting result on $(\mathrm{SWPS})_n$.

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1 Introduction

Let (M^n,g) be a Riemannian manifold of dimension n with the Riemannian metric g and χ(M) denote the set of differentiable vector fields on M^n. Let $K(X, Y, Z)$ be the Riemannian curvature tensor of type (1,3) for $X, Y, Z \in \chi$(M). A non-flat Riemannian manifold (M^n, g), ($n \geq 2$) is called a special weakly symmetric Riemannian manifold, if its curvature tensor K of type (1, 3) satisfies the following condition [10].

$$(1.1)\qquad \begin{aligned}(D_X K)(Y,Z,V) = \;& 2\alpha(X)K(Y,Z,V) + \alpha(Y)K(X,Z,V)\\ &+\alpha(Z)K(Y,X,V) + \alpha(V)K(Y,Z,X),\end{aligned}$$

where α is a non-zero 1-form and ρ is associated vector field such that

$$(1.2)\qquad \alpha(X) = g(X,\rho),$$

for every vector field X and D denotes the operator of covariant differentiation with respect to the metric g. Such a manifold is denoted by $(\mathrm{SWS})_n$. In case, the 1-form α is zero, then $(\mathrm{SWS})_n$ is become locally symmetric manifold [9]. If we replace K by P in (1.1), then it becomes

$$(D_X P)(Y,Z,V) = \begin{array}{l} 2\alpha(X)P(Y,Z,V) + \alpha(Y)P(X,Z,V) \\ +\alpha(Z)P(Y,X,V) + \alpha(V)P(Y,Z,X), \end{array} \tag{1.3}$$

where P is the projective curvature tensor defined (see in [2] at page 155) by

$$P(Y,Z,V) = K(Y,Z,V) - \frac{1}{n-1}[Ric(Z,V)Y - Ric(Y,V)Z]. \tag{1.4}$$

Here Ric is the Ricci tensor of type (0, 2). Such an n-dimensional Riemannian manifold shall be called a special weakly projective symmetric Riemannian manifold and such a manifold is denoted by $(\mathrm{SWPS})_n$.

If a Riemannian manifold is Einstein, then

$$Ric(X,Y) = \lambda\, g(X,Y) \tag{1.5}$$

where λ is constant. From (1.5), we have

$$R(X) = \lambda\, X, \tag{1.6}$$

where R is the Ricci tensor of type (1,1) and is defined by [2]

$$g(R(X),Y) = Ric(X,Y). \tag{1.7}$$

Contracting (1.6) with respect to X, we get

$$r = n\,\lambda. \tag{1.8}$$

where r is a scalar curvature.

The above results will have used in the next section.

2 Existence of a $(\mathrm{SWPS})_n$

Let(M^n, g) be a $(\mathrm{SWPS})_n$. Taking covariant derivative of (1.4) with respect to X and then using (1.3), we get

$$\begin{aligned} &2\alpha(X)P(Y,Z,V) + \alpha(Y)P(X,Z,V) + \alpha(Z)P(Y,X,V) + \alpha(V)P(Y,Z,X) \\ &\quad = (D_X K)(Y,Z,V) - \tfrac{1}{n-1}[(D_X Ric)(Z,V)Y - (D_X Ric)(Y,V)Z]. \end{aligned} \tag{2.1}$$

By virtue of (1.4), the relation (2.1) reduces to

$$(2.2)\quad (D_XK)(Y,Z,V) - 2\alpha(X)K(Y,Z,V) - \alpha(Y)K(X,Z,V) - \alpha(Z)K(Y,X1,V) \\ -\alpha(V)K(Y,Z,X) - \tfrac{1}{n-1}[(D_XRic)(Z,V)Y - (D_XRic)(Y,V)Z \\ -2\alpha(X)\{Ric(Z,V)Y - Ric(Y,V)Z\} - \alpha(Y)\{Ric(Z,V)X - Ric(X,V)Z\} \\ -\alpha(Z)\{Ric(X,V)Y - Ric(Y,V)X\} - a(V)\{Ric(Z,X)V - Ric(Y,X)Z\}] = 0.$$

Permuting equation (2.2) twice with respect to X, Y, Z; and then adding the three obtained equations and using Bianchi's first and second identities; symmetric property of Ricci tensor and the skew-symmetric properties of curvature tensor, we get

$$(2.3)\ (D_XRic)(Z,V)Y \quad +(D_YRic)(X,V)Z + (D_ZRic)(Y,V)X - (D_XRic)(Y,V)Z \\ -(D_YRic)(Z,V)X - (D_ZRic)(X,V)Y = 0.$$

Contracting (2.3) with respect to X, we get

$$(2.4)\qquad (D_ZRic)(Y,V) - (D_YRic)(Z,V) = 0.$$

Consequently in view of (1.7), the relation (2.4) gives

$$(2.5)\qquad (D_ZR)(Y) - (D_YR)(Z) = 0.$$

This leads us to the following result:

Theorem 2.1 - In a $(\mathrm{SWPS})_n$, the Ricci tensor of type (1,1) is closed.

Contracting (2.5) with respect to Y, we get

$$Z\mathrm{r} = 0,$$

which has shows that the scalar curvature r is constant.
Thus we have the following result:

Theorem 2.2 -In a $(\mathrm{SWPS})_n$, the scalar curvature r is constant.

By virtue of (1.5), the equation (1.4) reduces to the form

$$(2.6)\qquad P(Y,Z,V) = K(Y,Z,V) - \frac{\lambda}{n-1}[g(Z,V)Y - g(Y,V)Z].$$

Taking covariant derivative of (2.6) with respect to X, we get

$$(2.7)\qquad (D_XP)(Y,Z,V) = (D_XK)(Y,Z,V).$$

Using (1.3) in (2.7), we get

$$(D_X K)(Y,Z,V) = 2 \quad \alpha(X)P(Y,Z,V) + \alpha(Y)P(X,Z,V) + \alpha(Z)P(Y,X,V) + \alpha(V)P(Y,Z,X). \tag{2.8}$$

By virtue of (2.6), the relation (2.8) reduces to the form

$$\begin{aligned}(D_X K)(Y,Z,V) &= 2\alpha(X)[K(Y,Z,V) - \tfrac{\lambda}{n-1}g(Z,V)Y - g(Y,V)Z] \\ &+\alpha(Y)[K(X,Z,V) - \tfrac{\lambda}{n-1}g(Z,V)X - g(X,V)Z] \\ &+\alpha(Z)[K(Y,X,V) - \tfrac{\lambda}{n-1}g(X,V)Y - g(Y,V)X] \\ &+\alpha(V)[K(Y,Z,X) - \tfrac{\lambda}{n-1}g(Z,X)Y - g(Y,X)Z].\end{aligned}$$

From the above relation, we have the following result:

Theorem 2.3 - The necessary and sufficient condition for an Einstein $(\text{SWPS})_n$ to be a $(\text{SWS})_n$ is that

$$\begin{aligned}&[2\alpha(X)\ Y + \alpha(Y)\ X]\ g(Z,V) - [2\alpha(X)\ Z + \alpha(Z)X]\ g(Y,V) \\ &+[\alpha(Z)\ Y - \alpha(Y)\ Z]\ g(X,V) + \alpha(V)\ [g(Z,X)\ Y - g(Y,X)\ Z] = 0.\end{aligned}$$

3 Manifold satisfying $P(Y,Z,V) = 0$

Let (M^n, g) be a projectively flat, that is, $P(Y,Z,V) = 0$, then the relation (1.4) reduces to

$$K(Y,Z,V) = \frac{1}{n-1}[Ric(Z,V)Y - Ric(Y,V)Z]. \tag{3.1}$$

Taking covariant derivative of (3.1) with respect to X, we have

$$(D_X K)(Y,Z,V) = \frac{1}{n-1}[(D_X Ric)(Z,V)Y - (D_X Ric)(Y,V)Z]. \tag{3.2}$$

Permuting equation (3.2) twice with respect to X, Y, Z; and then adding the three obtained equations and using Bianchi's second identity, we have

$$\begin{aligned}&(D_X Ric)(Z,V)Y + (D_Y Ric)(X,V)Z + (D_Z Ric)(Y,V)X \\ &-(D_X Ric)(Y,V)Z - (D_Y Ric)(Z,V)X - (D_Z Ric)(X,V)Y = 0.\end{aligned} \tag{3.3}$$

Contracting (3.3) with respect to X, we have

$$(D_Z Ric)(Y,V) - (D_Y Ric)(Z,V) = 0. \tag{3.4}$$

Consequently in view of (1.7), the relation (3.4) gives

$$(D_Z R)(Y) - (D_Y R)(Z) = 0. \tag{3.5}$$

This leads us to the following result:

Theorem 3.1 - In a projectively flat Riemannian manifold, the Ricci tensor R of type (1,1) is closed.

An n-dimensional Riemannian manifold is called a special weakly Ricci symmetric manifold [10] if the Ricci tensor Ric of type (0, 2) satisfies the follwoing condition:

$$(D_X Ric)(Y,Z) = 2\alpha(X)Ric(Y,Z) + \alpha(Y)Ric(X,Z) + \alpha(Z)Ric(Y,X), \tag{3.6}$$

where α is a non-zero I-form. Such a manifolds will have denoted by $(\text{SWRS})_n$.

Now using (3.6) in (3.3), we have

$$\begin{aligned}\alpha(X)Ric(Z,V)Y + \alpha(Y)Ric(X,V)Z + \alpha(Z)Ric(Y,V)X \\ -\alpha(X)Ric(Y,V)Z - \alpha(Y)Ric(Z,V)X - \alpha(Z)Ric(X,V)Y = 0.\end{aligned} \tag{3.7}$$

Constracting (3.7) with respect to X, we have

$$\alpha(Z)Ric(Y,V) - \alpha(Y)Ric(Z,V) = 0. \tag{3.8}$$

which in view of (1.7) gives

$$\alpha(Z)g(R(Y),V) + \alpha(Y)g(R(Z),V) = 0$$

Consequently the above relation gives

$$\alpha(Z)R(Y) - \alpha(Y)R(Z) = 0.$$

Thus, we have the following result:

Theorem 3.2 - In a projectively flat $(\text{SWRS})_n$, the 1-from α is collinear with the Ricci tensor R.

Taking covariant derivative of (1.4) with respect to X, we have

$$(D_X P)(Y,Z,V) = (D_X K)(Y,Z,V) - \frac{1}{n-1}[(D_X Ric)(Z,V)Y - (D_X Ric)(Y,V)Z]. \tag{3.9}$$

Permutting equation (3.9) twice with respect to X, Y, Z; and then adding the three obtained equations and using Bianchi's second identity, we have

$$\begin{aligned}(D_X P)(Y,Z,V) &+ (D_Y P)(Z,X,V) + (D_Z P)(X,Y,V) \\ &= -\frac{1}{n-1}[(D_X Ric)(Z,V)Y - (D_X Ric)(Y,V)Z \\ &- (D_Y Ric)(X,V)Z - (D_Y Ric)(Z,V)X \\ &- (D_Z Ric)(Y,V)X - (D_Z Ric)(X,V)Y].\end{aligned} \tag{3.10}$$

Using (1.3) and (3.6) in (3.10) and taking in mind the skew-symmetric of $P(X,Y,Z)$, cyclic property of $P(X,Y,Z)$ and symmetric poperty of Ricci tensor of type (0,2), we ahve

$$\begin{aligned}&\alpha(X)Ric(Z,V)Y - \alpha(X)Ric(Y,V)Z + \alpha(Y)Ric(X,V)Z \\ &-\alpha(Y)Ric(Z,V)X + \alpha(Z)Ric(Y,V)X - \alpha(Z)Ric(X,V)Y = 0.\end{aligned} \tag{3.11}$$

Contracting (3.11) with respect to X, we have

$$(n-2)\alpha(Z)Ric(Y,V) - (n-2)\alpha(Y)Ric(Z,V) = 0, \tag{3.12}$$

which in view of (1.7) the relation (3.12) gives

$$\alpha(Z)\ g(R(Y),V) - \alpha(Y)\ g(R(Z),V) = 0. \tag{3.13}$$

Consequently the relation (3.13) gives

$$\alpha(Z)R(Y) - \alpha(Y)R(Z) = 0.$$

This leads us to the following result:

Theorem 3.3 - In a (SWPS)$_n$, if a Riemannian manifold is a (SWRS)$_n$, the 1-from α is collinear with the Ricci tensor R. of type (1,1).

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