Minimal Reducing Subspaces of k^{th} order Slant Toeplitz Operators

Munmun Hazarika [1] **and Sougata Marik** [2]

[1] *Department of Mathematical Sciences*
Tezpur University, Napam, Tezpur
munmun@tezu.ernet.in

[2]*Department of Mathematical Sciences*
Tezpur University, Napam, Tezpur
sougatam@tezu.ernet.in

Abstract

In this paper we have identified the reducing and minimal reducing subspaces of the k^{th} order slant Toeplitz operator induced by $\varphi(z) = z^N$, where N is an integer.
Subject Classification:[2010]Primary 47B37, 47B20; Secondary 47B35

Keywords: k^{th} order slant Toeplitz operator, transparent function, minimal reducing subspace.

1 Introduction

An important class of non-normal operators is that of Toeplitz operators, introduced by O. Toeplitz [10]. It has many interesting properties which can be connected to different areas of Operator Theory. This class is also closely related to the class of Laurent operators in the sense that the matrix representation of both these operators are constant along diagonals parallel to the main diagonal.

If we eliminate every alternate row of a Laurent matrix then the resultant matrix represents a slant Toeplitz operator as defined by M. C. Ho [6]. This idea was further extended by Arora and Batra [1] to define a k^{th} order slant Toeplitz operators in which every k number of consecutive rows of a Laurent matrix is alternately eliminated. Here $k(> 1)$ is a positive integer . Several authors have contributed to establish the algebraic and spectral properties of this class of operators [2], [3], [5], [7], [8]. As a result of their combined effort, a rich structure theory of this class of operators has slowly begun to evolve. In view of this it is relevant to enquire about the invariant subspace lattice of a slant Toeplitz operator. However, not much is known about the invariant subspaces of slant Toeplitz operators. Recently in [4] it was shown that a slant Toeplitz operator induced by $\varphi(z) = z^N$, where N is an integer, will have infinitely many reducing subspaces. In this paper we want to find out if this is also the case for any k^{th} order slant Toeplitz operator.

2 Preliminaries

We begin by re-introducing the definition of a k^{th} order slant Toeplitz operator defined in [1]. For this we consider the space $L^2(\mathbb{T})$ of all square integrable functions defined on the unit circle $\mathbb{T}$ in the complex plane. If $L^\infty(\mathbb{T})$ consists of all the bounded elements of $L^2(\mathbb{T})$, then for $\varphi \in L^\infty(\mathbb{T})$ the multiplication operator induced by φ is denoted as M_φ and defined as $M_\varphi f = \varphi f \ \forall \ f \in L^2(\mathbb{T})$. The set $\{e_n\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\mathbb{T})$ where $e_n(z) := z^n$ for $z \in \mathbb{T}$, and $n \in \mathbb{Z}$. Note that $\mathbb{Z}$ denotes the set of integers, and $\mathbb{Z}_+$ denotes the set of non-negative integers. For an integer $k > 1$ the operator W_k on $L^2(\mathbb{T})$ is defined as $W_k e_n = \begin{cases} e_{\frac{n}{k}} & \text{if } k \text{ divides } n \\ 0 & \text{therwise} \end{cases}$. Then for integer $k > 1$ and $\varphi \in L^\infty(\mathbb{T})$, the k^{th} order slant Toeplitz operator induced by φ is denoted as $U_{k,\varphi}$ and defined as

$U_{k,\varphi} = W_k M_\varphi$. From this it follows that $U^*_{k,\varphi} = M^*_\varphi W^*_k$, where $M^*_\varphi = M_{\bar{\varphi}}$ and $W^*_k e_n = e_{kn}$ $\forall\, n \in \mathbb{Z}$. Our aim here is to identify the reducing subspaces of $U_{k,\varphi}$. Since for $k = 2$ the operator $U_{k,\varphi}$ is same as A_φ as discussed in [4], so here we mainly consider $k > 2$. We begin by introducing a few definitions first.

Definition 2.1. *For $N \in \mathbb{Z}$ and $k \geq 2$, we define the following:*
(i) $S^{(k)}_N := \cup_{t=1}^{k-1}\{km - N + t\}_{m\in\mathbb{Z}}$.
(ii) For $j \in S^{(k)}_N$, $\Lambda^{(N,k)}_j := \{jk^t - N(\frac{k^t-1}{k-1})\}_{t\in\mathbb{Z}_+}$, *and* $H^{(N,k)}_j$ *will denote the closed linear span of* $\{e_\lambda\}_{\lambda\in\Lambda^{(N,k)}_j}$.

Lemma 2.1. *Let $N \in \mathbb{Z}$ and k be a positive integer. If k divides N, then $\mu N \notin S^{(k)}_N$ $\forall\, \mu \in \mathbb{Z}$. In particular, $N \notin S^{(k)}_N$.*

Proof. Let $N = \lambda k$ for some $\lambda \in \mathbb{Z}$.
Let, if possible, there exist $\mu \in \mathbb{Z}$ such that $\mu N \in S^{(k)}_N$. Then there exists $m \in \mathbb{Z}$ and $t \in \{1, 2, \cdots, k-1\}$ such that $\mu N = km-N+t$. Thus, $t = k[(\mu+1)\lambda-m]$ which implies that k divides t, a contradiction. Hence, there does not exist $\mu \in \mathbb{Z}$ such that $\mu N \in S^{(k)}_N$. In particular, $N \notin S^{(k)}_N$. □

Lemma 2.2. *Let $N \in \mathbb{Z}$ and k be an integer such that $k > 2$. Then $N \in S^{(k)}_N$ if k does not divide N.*

Proof. Let $N = \lambda k + \mu$ for some $\lambda \in \mathbb{Z}$ and $\mu \in \{1, 2, \cdots, k-1\}$. Then $2N = 2\lambda k + 2\mu = mk + t$ where $m \in \mathbb{Z}$ and $t \in \{1, 2, \cdots, k-1\}$, which implies that $N = km - N + t \in S^{(k)}_N$. □

Remark 2.1. *If $N \in \mathbb{Z}$ and $k = 2$ then $N \notin S^{(k)}_N$ because $N \in S^{(k)}_N \implies N = 2m - N + 1$ for $m \in \mathbb{Z}$ which gives $2N = 2m + 1$, a contradiction.*

From the above, we get the following conclusion:

Theorem 2.1. *Let $N \in \mathbb{Z}$ and k be an integer such that $k \geq 2$.*
(i) For $k = 2$, $N \notin S^{(k)}_N$.
(ii) For $k > 2$, $N \in S^{(k)}_N$ if and only if k does not divide N.

Notation 2.1. *For $N \in \mathbb{Z}$, integer $k \geq 2$, $t \in \mathbb{Z}_+$ and $j \in S^{(k)}_N$ we use the notation $\varepsilon^{(j,N,k)}_t$ to denote the basis vector $e_{[jk^t - N(\frac{k^t-1}{k-1})]}$. So with respect to this notation, we can say that $H^{(N,k)}_j$ is the closed linear span of $\{\varepsilon^{(j,N,k)}_t\}_{t\in\mathbb{Z}_+}$.*

Lemma 2.3. *Let $N \in \mathbb{Z}$ and k be an integer such that $k \geq 2$. If $k-1$ divides N then $\frac{N}{k-1} \notin S^{(k)}_N$.*

Proof. Let, if possible, $\frac{N}{k-1} \in S^{(k)}_N$. Then there exists $m \in \mathbb{Z}$ and $t \in \{1, 2, \cdots, k-1\}$, such that $\frac{N}{k-1} = km - N + t$. This implies that $N - (k-1)m = \frac{t(k-1)}{k}$, which is a contradiction as the left hand side is an integer, while the right hand side is not an integer. Thus, if $k-1$ divides N then $\frac{N}{k-1} \notin S^{(k)}_N$. □

Lemma 2.4. *Let $N \in \mathbb{Z}$ and k be an integer such that $k \geq 2$. If $j, p \in S^{(k)}_N$ with $j \neq p$ then $\Lambda^{(N,k)}_j \cap \Lambda^{(N,k)}_p = \emptyset$.*

Proof. Let, if possible, $x \in \Lambda^{(N,k)}_j \cap \Lambda^{(N,k)}_p$. Now $x \in \Lambda^{(N,k)}_j$ implies $x = \frac{1}{k-1}[N + \{(k-1)j - N\}k^t]$ for

$t \in \mathbb{Z}_+$. Also $x \in \Lambda_p^{(N,k)}$ implies $x = \frac{1}{k-1}[N + \{(k-1)p - N\}k^{t_1}]$ for $t_1 \in \mathbb{Z}_+$. Thus, $\{(k-1)j - N\}k^t = \{(k-1)p - N\}k^{t_1}$. As $j \neq p$ so we cannot have $t = t_1$, so without loss of generality suppose $t > t_1$. Then $t - t_1 > 0$ and $\{(k-1)j - N\}k^{t-t_1} = (k-1)p - N$, and so we must have $(k-1)p - N$ divisible by k. But this is not possible, because $p \in S_N^{(k)}$ implies $p = km - N + t$ for some $m \in \mathbb{Z}$ and $t \in \{1, 2, \cdots, k-1\}$, and hence $(k-1)p - N = k(k-1)m - kN + (k-1)t$ where $(k-1)t$ is not divisible by k. Thus, $\Lambda_j^{(N,k)} \cap \Lambda_p^{(N,k)} = \emptyset$ for distinct elements $j,\ p \in S_N^{(k)}$. □

Lemma 2.5. *Let $N \in \mathbb{Z}$ and k be an integer $k \geq 2$. For each $m \in \mathbb{Z}$ with $m \neq \frac{N}{k-1}$, there exists a unique $j \in S_N^{(k)}$ such that $m \in \Lambda_j^{(N,k)}$. Moreover, if $(k-1)m < N$ then $(k-1)j < N$, and if $(k-1)m > N$ then $(k-1)j > N$.*

Proof. Let $m \in \mathbb{Z}$ and $m \neq \frac{N}{k-1}$. With out loss of generality let $m > \frac{N}{k-1}$.
For $j \in S_N^{(k)}$ and $t \in \mathbb{Z}_+$, let $p_{j,t} := jk^t - N(\frac{k^t-1}{k-1})$. So each $j \in S_N^{(k)}$ can be written as $p_{j,0}$. Also $\Lambda_j^{(N,k)} = \{p_{j,t}\}_{t \in \mathbb{Z}_+}$.
Step I: If $m \in S_N^{(k)}$, then $m = p_{m,0} \in \Lambda_m^{(N,k)}$ and we are done. If $m \notin S_N^{(k)}$, then we proceed further.
Step II: $m \notin S_N^{(k)} \implies m \in \{k\xi - N\}_{\xi \in \mathbb{Z}}$, say $m = km_1 - N$ for $m_1 \in \mathbb{Z}$.
Claim1: $\frac{N}{k-1} < m_1 < m$.
Proof of claim: $m_1 = \frac{m+N}{k} > \frac{1}{k}[\frac{N}{k-1} + N] = \frac{N}{k-1}$. Also, $m_1 \geq m \implies m = km_1 - N \geq km - N \implies \frac{N}{k-1} \geq m$, a contradiction. Hence claim1 is established.
If $m_1 \in S_N^{(k)}$, then $m_1 = p_{m_1,0}$ and $m = p_{m_1,1}$ so that $m \in \Lambda_{m_1}^{(N,k)}$ and we are done. If $m_1 \notin S_N^{(k)}$, then we proceed further.
Step III: $m_1 \notin S_N^{(k)} \implies m_1 = km_2 - N$ for $m_2 \in \mathbb{Z}$ such that $\frac{N}{k-1} < m_2 < m_1$. So if $m_2 \in S_N^{(k)}$ then $m = p_{m_2,2} \in \Lambda_{m_2}^{(N,k)}$ and we are done. Otherwise the process continues.
However, since for each integer m_i we have $\frac{N}{k-1} < m_{i+1} < m_i < m$, so this process can not continue infinitely. That is, there must exist $n \in \mathbb{N}$ such that $m_n \in S_N^{(k)}$, and $m = p_{m_n,n} \in \Lambda_{m_n}^{(N,k)}$.
Uniqueness follows from Lemma 2.4. It also follows from Claim1 that if $(k-1)m > N$ then $(k-1)m_n > N$. The case where $m < \frac{N}{k-1}$ can be proved similarly. □

Definition 2.2. *Let $N \in \mathbb{Z}$ and k be an integer $k \geq 2$. A non-zero function $f \in L^2(\mathbb{T})$ is said to be $(N,\ k)$ transparent if f can be expressed as $f = \sum_{i \in S_N^{(k)}} c_i e_i$, where $c_i \in \mathbb{C}$. Here $\mathbb{C}$ denotes the complex plane.*

3 Reducing subspaces of $U_{k,\varphi}$

At the beginning of the section we recall the definition of a reducing subspace of an operator. For a bounded linear operator T on a complex Hilbert space H, a subspace M of H is said to be invariant for T if $T(M) \subseteq M$; and if M is invariant for both T and T^* then it is said to be reducing for T. Moreover, if the only reducing subspaces contained in M are zero and itself, then M is said to be a minimal reducing subspace for T.

Definition 3.1. *For an integer $k \geq 2$ and $\varphi \in L^\infty(\mathbb{T})$, let $\mathbb{S}_{k,\varphi}$ be the vector space consisting of all finite linear combination of finite products of $U_{k,\varphi}$ and $U_{k,\varphi}^*$. For $f \in L^2(\mathbb{T})$, let $\mathbb{S}_{k,\varphi} f = \{Tf : T \in \mathbb{S}_{k,\varphi}\}$. Then closure of $\mathbb{S}_{k,\varphi} f$ in $L^2(\mathbb{T})$ is a reducing subspace of $U_{k,\varphi}$ denoted by X_f. It is also called the subspace generated by f. It is clearly the smallest reducing subspace of $L^2(\mathbb{T})$ containing f.*

Lemma 3.1. *Let $\varphi(z) = z^N$ for $N \in \mathbb{Z}$ and $k (\geq 2)$ be an integer. Then the following must hold*
(i) *For $j \in S_N^{(k)}$ and $t \in \mathbb{Z}_+$, $U_{k,\varphi}\varepsilon_t^{(j,N,k)} = \begin{cases} \varepsilon_{t-1}^{(j,N,k)} & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases}$*

(ii) $U^*_{k,\varphi}\varepsilon_t^{(j,N,k)} = \varepsilon_{t+1}^{(j,N,k)}$.

(iii) *If* $k-1$ *divides* N*, then* $U_{k,\varphi} e_{\frac{N}{k-1}} = e_{\frac{N}{k-1}} = U^*_{k,\varphi} e_{\frac{N}{k-1}}$.

Proof. (i) Since $\varphi(z) = z^N$, so $M_\varphi \varepsilon_t^{(j,N,k)}(z) = z^{[jk^t - N(\frac{k^t-1}{k-1}) + N]}$.
If $t = 0$ then $M_\varphi \varepsilon_t^{(j,N,k)}(z) = z^{j+N}$. But since $j \in S_N^{(k)} = \cup_{t=1}^{k-1}\{km - N + t\}_{m\in\mathbb{Z}}$, so $j+N$ is not divisible by k, which implies that $W_k z^{j+N} = 0$. Thus, $U_{k,\varphi}\varepsilon_t^{(j,N,k)} = 0$ if $t = 0$.
If $t > 0$, then $M_\varphi \varepsilon_t^{(j,N,k)}(z) = z^{[jk^t - Nk(\frac{k^{t-1}-1}{k-1})]}$, and so
$U_{k,\varphi}\varepsilon_t^{(j,N,k)}(z) = z^{[jk^{t-1} - N(\frac{k^{t-1}-1}{k-1})]} = \varepsilon_{t-1}^{(j,N,k)}(z)$.
Thus, $U_{k,\varphi}\varepsilon_t^{(j,N,k)} = \begin{cases} \varepsilon_{t-1}^{(j,N,k)} & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases}$
(ii) Let $m,\ t \in \mathbb{Z}_+$. Now for $j \in S_N^{(k)}$ we have,
$\langle U_{k,\varphi}\varepsilon_0^{(j,N,k)}, \varepsilon_t^{(j,N,k)}\rangle = 0 = \langle \varepsilon_0^{(j,N,k)}, \varepsilon_{t+1}^{(j,N,k)}\rangle$ and for $m > 0$
we have $\langle U_{k,\varphi}\varepsilon_m^{(j,N,k)}, \varepsilon_t^{(j,N,k)}\rangle = \langle \varepsilon_{m-1}^{(j,N,k)}, \varepsilon_t^{(j,N,k)}\rangle = \langle \varepsilon_m^{(j,N,k)}, \varepsilon_{t+1}^{(j,N,k)}\rangle$.
So for $f = \sum_m \alpha_m \varepsilon_m^{(j,N,k)} \in L^2(\mathbb{T})$, we have
$\langle U_{k,\varphi} f, \varepsilon_t^{(j,N,k)}\rangle = \sum_m \alpha_m \langle U_{k,\varphi}\varepsilon_m^{(j,N,k)}, \varepsilon_t^{(j,N,k)}\rangle$
$= \sum_m \alpha_m \langle \varepsilon_m^{(j,N,k)}, \varepsilon_{t+1}^{(j,N,k)}\rangle = \langle f, \varepsilon_{t+1}^{(j,N,k)}\rangle$
Thus, $U^*_{k,\varphi}\varepsilon_t^{(j,N,k)} = \varepsilon_{t+1}^{(j,N,k)}\ \forall\ t \in \mathbb{Z}_+$.

(iii) If $k-1$ divides N, then $\frac{N}{k-1} \in \mathbb{Z}$, and by Lemma 2.3, $\frac{N}{k-1} \notin S_N^{(k)}$. Let $m = \frac{N}{k-1}$, then $U_{k,\varphi} e_m(z) = W_k z^{m+N}$. Since $m \notin S_N^{(k)}$, so $m = kp - N$ for some $p \in \mathbb{Z}$, which implies that $m + N$ is divisible by k, and $\frac{m+N}{k} = \frac{N}{k-1} = m$. Thus, $U_{k,\varphi} e_m(z) = W_k z^{m+N} = z^m$.
Similarly, for $m = \frac{N}{k-1}$, we have $U^*_{k,\varphi} e_m(z) = M^*_\varphi W^*_k z^m = z^{km-N} = z^m$. Thus, $U_{k,\varphi} e_m = e_m = U^*_{k,\varphi} e_m$ for $m = \frac{N}{k-1}$. □

Definition 3.2. *Let* $\varphi(z) = z^N$ *for* $N \in \mathbb{Z}$*, and* k *be an integer* ≥ 2*. If* $k-1$ *divides* N *then* $\frac{N}{k-1} \notin S_N^{(k)}$ *and we define* $H_{\frac{N}{k-1}} := \{\lambda e_{\frac{N}{k-1}} : \lambda \in \mathbb{C}\}$.

Theorem 3.1. *Let* $\varphi(z) = z^N$ *for* $N \in \mathbb{Z}$*, and* k *be an integer* ≥ 2*. If* $k-1$ *divides* N*, then* $H_{\frac{N}{k-1}}$ *is a minimal reducing subspace for* $U_{k,\varphi}$.

The proof follows immediately from Lemma 3.1(iii).

Theorem 3.2. *Let* $\varphi(z) = z^N$ *for* $N \in \mathbb{Z}$*, and* k *be an integer* ≥ 2*. Then for each* $j \in S_N^{(k)}$*,* $H_j^{(N,k)}$ *is a minimal reducing subspace for* $U_{k,\varphi}$*. Also* $U_{k,\varphi}$ *is the backward unilateral shift on* $H_j^{(N,k)}$.

Proof. That $H_j^{(N,k)}$ is a reducing subspace for $U_{k,\varphi}$ follows immediately by from Notation 2.1 and Lemma 3.1. Moreover, $U_{k,\varphi}\varepsilon_t^{(j,N,k)} = \begin{cases} \varepsilon_{t-1}^{(j,N,k)} & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$ shows that $U_{k,\varphi}$ is the backward unilateral shift on $H_j^{(N,k)}$. Since the unilateral shift is irreducible [9], hence $H_j^{(N,k)}$ is minimal reducing for $U_{k,\varphi}$. □

Lemma 3.2. *If* f *and* g *are two* (N,k) *transparent functions in* $L^2(\mathbb{T})$ *such that* $g \in X_f$*, then* $g = \lambda f$ *for some scalar* λ.

The result follows by applying Lemma 3.1 to Definition 1.2 for (N,k) transparent functions.

Lemma 3.3. *Let $\varphi(z) = z^N$ for $N \in \mathbb{Z}$, and k be an integer ≥ 2. Let M be a non-zero reducing subspace for $U_{k,\varphi}$. Moreover, if $k-1$ divides N, then consider $M \neq H_{\frac{N}{k-1}}$. Then M must contain a (N, K) transparent function.*

Proof. Let $0 \neq x \in M$. In view of Lemma 2.5, we can express x as $x = \sum_{t=0}^{\infty} x_t$, where $x_t \in Span\{\varepsilon_t^{(j,N,k)}\}_{j \in S_N^{(k)}}$. Let ξ be the smallest non-negative integer such that $x_\xi \neq 0$. If $y := x - (U_{k,\varphi}^*)^{\xi+1} U_{k,\varphi}^{\xi+1} x$, then applying Lemma 3.1, we see that $y = x_\xi \in M$. If $\xi = 0$ then $y = x_0 \in Span\{\varepsilon_0^{(j,N,k)}\}_{j \in S_N^{(k)}} = Span\{e_j\}_{j \in S_N^{(k)}}$ and is therefore (N, k) transparent. On the other hand, if $\xi > 0$ then $z = U_{k,\varphi}^{\xi} y \in Span\{\varepsilon_0^{(j,N,k)}\}_{j \in S_N^{(k)}}$ and is hence (N, k) transparent in M. □

Lemma 3.4. *Let $\varphi(z) = z^N$ for $N \in \mathbb{Z}$, and k be an integer ≥ 2. Then for each minimal reducing subspace of $U_{k,\varphi}$ other than $H_{\frac{N}{k-1}}$, there must exist a (N, K) transparent function $f \in L^2(\mathbb{T})$ such that $M = X_f$, where f is unique upto a scalar multiple.*

Proof. Since M is reducing for $U_{k,\varphi}$, so by Lemma 3.3 there exists a (N, k) transparent function $f \in M$. As in Definition 3.1, X_f is a reducing subspace contained in M, and since M is minimal so we must have $M = X_f$. Moreover, if there exists another (N, k) transparent function g such that $M = X_g$ then by Lemma 3.2, $g = \lambda f$ for some scalar λ. □

Theorem 3.3. *Let $\varphi(z) = z^N$ for $N \in \mathbb{Z}$, and k be an integer ≥ 2. If $f \in L^2(\mathbb{T})$ is (N, k) transparent, then X_f is a minimal reducing subspace for $U_{k,\varphi}$.*

Proof. Since X_f by Definition 3.1, is reducing for $U_{k,\varphi}$, so all we need to show is that it is minimal. For this let us consider a non-zero reducing subspace M in X_f. Clearly $M \neq H_{\frac{N}{k-1}}$, since $e_{\frac{N}{k-1}} \notin X_f$. So by Lemma 3.3, M must contain a (N, k) transparent function g. Thus, $g \in M \subset X_f$ and so by Lemma 3.2, $g = \lambda f$ for non-zero scalar λ. This in turn implies that $f \in M$, so that $M = X_f$. This shows that X_f is minimal reducing. □

We can now summarize our findings in the following statement:

Theorem 3.4. *Let $\varphi(z) = z^N$ for $n \in \mathbb{Z}$, and k be an integer ≥ 2. Then the following must hold:*
(i) *There are infinitely many minimal reducing subspaces of $U_{k,\varphi}$ in $L^2(\mathbb{T})$, each generated by a (N, k) transparent function in $L^2(\mathbb{T})$.*
(ii) *For each minimal reducing subspace M of $U_{k,\varphi}$ other than $H_{\frac{N}{k-1}}$, there exists a (N, k) transparent function f such that $M = X_f$, where f is unique upto a scalar multiple.*

References

[1] Arora S.C. and Batra R.(2003): On generalized slant Toeplitz operators. Indian J. Math. 45(2), 121–134.

[2] Arora S.C. and Batra R.(2005): Generalised slant Toeplitz operators on H^2. Math. Nachr. 278(4), 347–355.

[3] Datt G. and Ohri N.(2018): Properties of slant Toeplitz operators on the Torus. Malays. J. Math. Sci. 12(2), 195–209.

[4] Hazarika M. and Marik S. (2020): Reducing and minimal reducing subspaces of slant Toeplitz operators. Adv. Oper. Theory, 5, 336–346.

[5] Hazarika M. and Marik S. (2021): Toeplitz and slant Toeplitz operators on the polydisk. Arab J. Math. Sci., 27(1), 73–93.

[6] Ho M.C. (1996): Properties of slant Toeplitz operators. Indiana Univ. Math. J. 45, 843–862.

[7] Liu C. and Lu Y. (2013): Product and commutativity of slant Toeplitz operators. J. Math. Res. Appl. 33(1), 122–126.

[8] Lu Y., Liu C. and Yang J. (2010): Commutativity of k^{th} order slant Toeplitz operators. Math. Nachr. 283(9), 1304–1313.

[9] Shields A.L. (1974): Weighted shift operators and analytic function theory, Topics in Operator Theory, Math. Surveys 13, Amer. Math. Soc. Providence, R. I.

[10] Toeplitz O. (1911): Zur theorie der quadralishen and bilinearan Formen Von unendlichevielen, Math. Ann. 70, 351–376.