An Algorithm for solving Linear Fractional Programming Problems

Poonam Kumari

Department of Mathematics
Magadh Mahila College, Patna University, Patna
poonamkumari1865@gmail.com

Abstract

This paper describes a method for solving Linear Fractional Programming Problems. The algorithm is based on the effective selection of incoming vectors. We start with an initial basic feasible solution and keep improving basic feasible solution until the optimal solution is reached. We use separate selection criteria for incoming vectors for problems with non-negative and negative denominator in the objective function. The rule of selection of outgoing vector is the same as for the simplex method. The proposed algorithm is illustrated with numerical examples and the results are compared with the results obtained by other methods.

Subject Classification:[2010]90C05, 90C32

Keywords: Linear Fractional Programming, Optimal Solution, Incoming Vector.

1 Introduction

A linear fractional programming (LFP) problem is the problem of optimization, i.e., maximization or minimization of a fraction of two linear functions subject to linear constraints. Because of its wide range of applications in real life, LFP is of considerable research and interest. It is used to achieve the highest ratio of outcome to cost, the highest ratio of net profit to capital invested, i.e., the ratio representing the highest efficiency. A LFP problem can be written mathematically as follows:

$$\text{Max z} = \frac{z^1}{z^2} = \frac{c_0 + c_1x_1 + c_2x_2 + ... + c_nx_n}{d_0 + d_1x_1 + d_2x_2 + ... + d_nx_n} \tag{1.1}$$

subject to

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \leq b_1, \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \leq b_2, \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \leq b_m, \end{gathered}$$

where $x_1, x_2, ..., x_n \geq 0$.

Introducing the slack variables $x_{n+1}, x_{n+2}, .., x_{n+m}$, the above constraints can be written as follows:

$$\begin{gathered} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n + x_{n+1} = b_1, \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n + x_{n+2} = b_2, \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n + x_{n+m} = b_m, \end{gathered}$$

where $x_1, x_2, ..., x_n, x_{n+1}, ..., x_{n+m} \geq 0$.

Let

$$\mathbf{c} = [c_1 \quad c_2 \quad ... \quad c_n \quad 0 \quad 0 \quad ... \quad 0]_{1\times(m+n)},$$

$$\mathbf{d} = [d_1 \quad d_2 \quad ... \quad d_n \quad 0 \quad 0 \quad ... \quad 0]_{1\times(m+n)},$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} & 1 & 0 & 0 & ... & 0 \\ a_{21} & a_{22} & .... & a_{2n} & 0 & 1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & .... & 0 \\ a_{m1} & a_{m2} & ... & a_{mn} & 0 & 0 & 0 & ... & 1 \end{bmatrix}_{m\times(m+n)},$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ ... \\ x_n \\ x_{n+1} \\ ... \\ x_{n+m} \end{bmatrix}_{(n+m)\times 1}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ ... \\ b_m \end{bmatrix}_{m\times 1}$$

and it is assumed that the denominator is positive for all feasible solutions.

Then the above problem can be written as

$$\text{Max z} = \frac{z^1}{z^2} = \frac{c_0 + \mathbf{cx}}{d_0 + \mathbf{dx}} \tag{1.2}$$

subject to

$$\mathbf{Ax} = \mathbf{b}, \ \mathbf{x} \geq \mathbf{0}.$$

2 Objective

The purpose of this work is to make easy the manual computation of the algorithm for any kind of linear fractional programming problem.

3 Literature Review

Various methods have been developed to find the optimal solution to LFP problems. Charnes and Cooper [1] solved the linear fractional programming problem by converting it into an equivalent linear programming problem having one additional constraint. Hartmut [2] suggested a parametric method to solve linear fractional programming problems. Isbell and Marlow [3], Martos [4], and Dinkelbach [5] determined the optimal solution to the linear fractional programming problem using a sequence of linear programs. Kanti Swarup [6] gave an algorithm for the solution of programming problems with linear fractional functionals without reducing it to linear programming problems. Sharma and Swarup [7], Bector [8], [9], and Seshan [10] defined a dual form in which the objective function is fractional, that is,the ratio of two linear functions. The works of Bajalinov [11] and Jahan and Islam [12] in the field of the LFP problem are also highly remarkable. Tantawy [13] transformed the LFP problem into an equivalent linear programming (LP) problem and then used the dual of this LP problem to find the optimal solution of the given LFP problem. Sankaraiyer [14] proposed a ratio algorithm to solve linear fractional programming problems, in which the optimality of the solution is checked using the objective function value and the ratio of the contribution coefficients of the variables. But, in this method, so many cases arise that the process becomes tedious for manual working and hence the algorithm is suitable for computers only.

In this work, a new method has been proposed for the solution of linear fractional programming problems, which is based on the effective selection of incoming vectors. The layout of the paper is as follows. In Section 4, the proposed method is presented, considering the denominator of the objective function positive. The algorithm for the proposed method is given in Section 5. Section 6 describes the proposed method to find the optimal solution if the denominator of the objective functions is negative for any basic feasible solution. Numerical examples are given in Section 7. Validity of the proposed method is proved in Section 8 by comparing the results obtained for the numerical examples by the proposed method and the existing methods. Finally, discussion for highlighting the importance of the proposed method is given in the last section.

4 Proposed Method

Let $\mathbf{B}$ be any $m \times m$ submatrix of $\mathbf{A}$ formed from m linearly independent columns of $\mathbf{A}$ and let $\mathbf{x}_B = [x_{B_1} \quad x_{B_2} \quad ... \quad ... \quad x_{B_m}]^T$ be an initial basic feasible solution of the above LFP Problem such that

$$\mathbf{B}\mathbf{x}_B = \mathbf{b}, \text{ i.e., } \mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}. \tag{4.1}$$

Also, let $z^1 = c_0 + \mathbf{c}_B\mathbf{x}_B$ and $z^2 = d_0 + \mathbf{d}_B\mathbf{x}_B$, where $\mathbf{c}_B$ and $\mathbf{d}_B$ are the vectors having their components associated with the basic variables in the numerator and the denominator of the objective function respectively.

If the columns of matrix $\mathbf{A}$ be denoted by $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, ..., \boldsymbol{\alpha}_{n+m}$ and columns of submatrix $\mathbf{B}$ by $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, ..., \boldsymbol{\beta}_m$, then

$$\mathbf{A} = [\boldsymbol{\alpha}_1 \quad \boldsymbol{\alpha}_2 \quad ... \quad \boldsymbol{\alpha}_{n+m}] \text{ and } \mathbf{B} = [\boldsymbol{\beta}_1 \quad \boldsymbol{\beta}_2 \quad ... \quad \boldsymbol{\beta}_m].$$

Using simplex method, we obtain a new basic feasible solution by replacing one of the vectors $\boldsymbol{\beta}_i \in \mathbf{B}$ by $\boldsymbol{\alpha}_j$, which is a vector in $\mathbf{A}$ but not in $\mathbf{B}$.

Let the new basic feasible solution be given by

$$\mathbf{x}_B' = \left\{x_{B_1} - x_{B_r}\frac{y_{1j}}{y_{rj}}, \quad x_{B_2} - x_{B_r}\frac{y_{2j}}{y_{rj}}, \quad ..., \quad \frac{x_{B_r}}{y_{rj}}, \quad ..., \quad x_{B_m} - x_{B_r}\frac{y_{mj}}{y_{rj}}\right\}, \tag{4.2}$$

where

$$\frac{x_{B_r}}{y_{rj}} = \min_i \frac{x_{B_i}}{y_{ij}} \tag{4.3}$$

and other non-basic components are zero.

Now, we proceed to find the criterion to select the incoming vector $\boldsymbol{\alpha}_j \in \mathbf{A}$ such that the value of the objective function corresponding to the new basic feasible solution is improved.

The value of the objective function for the original basic feasible solution is

$$\mathrm{z} = \frac{z^1}{z^2} = \frac{c_0 + \mathbf{c}_B\mathbf{x}_B}{d_0 + \mathbf{d}_B\mathbf{x}_B} = \frac{c_0 + \sum_{i=1}^m c_{B_i}x_{B_i}}{d_0 + \sum_{i=1}^m d_{B_i}x_{B_i}}. \tag{4.4}$$

The value of the objective function for the new basic feasible solution is

$$\bar{z} = \frac{\bar{z}^1}{\bar{z}^2} = \frac{c_0 + \mathbf{c}_B'\mathbf{x}_B'}{d_0 + \mathbf{d}_B'\mathbf{x}_B'} = \frac{c_0 + \sum_{i=1}^m c_{B_i}'x_{B_i}'}{d_0 + \sum_{i=1}^m d_{B_i}'x_{Bi}'}. \tag{4.5}$$

But

$$c_{B_i}' = c_{B_i} \ (i = 1, 2, .., m, i \neq r), \ c_{B_r}' = c_j, \tag{4.6}$$

$$d_{B_i}' = d_{B_i} \ (i = 1, 2, ..., m, i \neq r), \ d_{B_r}' = d_j. \tag{4.7}$$

Substituting the values of $c_{B_i}{}'$ and $d_{B_i}{}'$ from equations (4.6) and (4.7) in equation (4.5) and using equation (4.2), we get

$$\begin{aligned}
\bar{z} = \frac{\bar{z}^1}{\bar{z}^2} &= \frac{c_0 + \sum_{i=1}^m c_{B_i}(x_{B_i} - x_{B_r}\frac{y_{ij}}{y_{rj}}) + c_j\frac{x_{B_r}}{y_{rj}}}{d_0 + \sum_{i=1}^m (d_{B_i}x_{B_i} - x_{B_r}\frac{y_{ij}}{y_{rj}}) + d_j\frac{x_{B_r}}{y_{rj}}} \\
&= \frac{c_0 + \sum_{i=1}^m c_{B_i}x_{B_i} + \frac{x_{B_r}}{y_{rj}}(c_j - \sum_{i=1}^m c_{B_i}y_{ij})}{d_0 + \sum_{i=1}^m d_{B_i}x_{B_i} + \frac{x_{B_r}}{y_{rj}}(d_j - \sum_{i=1}^m d_{B_i}y_{ij})} \\
&= \frac{z^1 + \frac{x_{B_r}}{y_{rj}}(c_j - z_j^1)}{z^2 + \frac{x_{B_r}}{y_{rj}}(d_j - z_j^2)}, \quad \text{where } z_j^1 = \sum_{i=1}^m c_{B_i}y_{ij} \text{ and } z_j^2 = \sum_{i=1}^m d_{B_i}y_{ij} \\
&= \frac{z^1 - \frac{x_{B_r}}{y_{rj}}(z_j^1 - c_j)}{z^2 - \frac{x_{B_r}}{y_{rj}}(z_j^2 - d_j)} \\
&= \frac{\frac{z^1}{z^2} - \frac{x_{B_r}}{y_{rj}}\frac{z_j^1 - c_j}{z^2}}{1 - \frac{x_{B_r}}{y_{rj}}\frac{z_j^2 - d_j}{z^2}} \\
&= \left[\frac{z^1}{z^2} - \frac{x_{B_r}}{y_{rj}}\frac{z_j^1 - c_j}{z^2}\right]\left[1 - \frac{x_{B_r}}{y_{rj}}\frac{z_j^2 - d_j}{z^2}\right]^{-1} \\
&= \left[\frac{z^1}{z^2} - \frac{x_{B_r}}{y_{rj}}\frac{z_j^1 - c_j}{z^2}\right]\left[1 + \frac{x_{B_r}}{y_{rj}}\frac{z_j^2 - d_j}{z^2}\right] \quad \text{(neglecting terms of higher powers)} \\
&= \frac{z^1}{z^2} + \frac{x_{B_r}}{y_{rj}}\frac{1}{z^2}\left[\frac{z^1}{z^2}(z_j^2 - d_j) - (z_j^1 - c_j)\right]. \quad \text{(neglecting terms of higher powers)}
\end{aligned}$$

Therefore

$$\frac{\bar{z}^1}{\bar{z}^2} = \frac{z^1}{z^2} + \frac{x_{B_r}}{y_{rj}}\frac{1}{(z^2)^2}\left[z^1(z_j^2 - d_j) - z^2(z_j^1 - c_j)\right]. \tag{4.8}$$

Equation (4.8) implies that $\frac{\bar{z}^1}{\bar{z}^2} > \frac{z^1}{z^2}$ only if $z^1(z_j^2 - d_j) - z^2(z_j^1 - c_j) > 0$.Hence it follows that as soon as $z^1(z_j^2 - d_j) - z^2(z_j^1 - c_j) \leq 0$, no further improvement is possible and the optimal solution is reached.

Also, equation (4.8) implies that $\frac{\bar{z}^1}{\bar{z}^2}$ is maximum if $z^1(z_j^2 - d_j) - z^2(z_j^1 - c_j)$ is maximum.

Therefore, we can conclude the following:

If $z^1(z_j^2 - d_j) - z^2(z_j^1 - c_j) > 0$, then the non-basic vector $\boldsymbol{\alpha}_j \in \mathbf{A}$ corresponding to Max $\left\{z^1(z_j^2 - d_j) - z^2(z_j^1 - c_j)\right\}$ is selected as the incoming vector. If $z^1(z_j^2 - d_j) - z^2(z_j^1 - c_j) \leq 0$ for all the non-basic vectors, then no further improvement is possible and the optimal solution is reached.

Now, we consider the following cases:

Case 1. $z^1 = 0$ or $(z_j^2 - d_j) = 0$.

In this case, it follows from equation (4.8) that

$$\frac{\bar{z}^1}{\bar{z}^2} > \frac{z^1}{z^2} \text{ only if } -z^2(z_j^1 - c_j) > 0, \text{ i.e., if } z_j^1 - c_j < 0.$$

Also, equation (4.8) implies that $\frac{\bar{z}^1}{\bar{z}^2}$ is maximum if $-z^2(z_j^1 - c_j)$ is maximum i.e., if $\left(z_j^1 - c_j\right)$ is minimum.

Therefore, we can conclude the following:

If $\left(z_j^1 - c_j\right) < 0$, then the non-basic vector $\boldsymbol{\alpha}_j \in \mathbf{A}$ corresponding to $\mathrm{Min}\left(z_j^1 - c_j\right)$ is selected as the incoming vector. If $\left(z_j^1 - c_j\right) \geq 0$ for all the non-basic vectors, then no further improvement is possible and the optimal solution is reached.

Case 2. $z^1 \neq 0$ and also $(z_j^2 - d_j) \neq 0$.

In this case, equation (4.8) can be written as follows:

$$\frac{\bar{z}^1}{\bar{z}^2} = \frac{z^1}{z^2} + \frac{x_{B_r}}{y_{rj}} z^2 (z_j^2 - d_j) \left[\frac{z^1}{z^2} - \frac{z_j^1 - c_j}{z_j^2 - d_j} \right],$$

i.e.,

$$\frac{\bar{z}^1}{\bar{z}^2} = \frac{z^1}{z^2} + \frac{x_{B_r}}{y_{rj}} z^2 (z_j^2 - d_j) \left[\frac{z^1}{z^2} - R_j \right], \tag{4.9}$$

where

$$R_j = \frac{z_j^1 - c_j}{z_j^2 - d_j}.$$

Now, consider the following sub-cases:

Sub-case 1. $z^1 \neq 0$ and $(z_j^2 - d_j) > 0$.

In this case, equation (4.9) implies that $\frac{\bar{z}^1}{\bar{z}^2} > \frac{z^1}{z^2}$ only if $R_j < \frac{z^1}{z^2}$.

Also, equation (4.9) implies that $\frac{\bar{z}^1}{\bar{z}^2}$ is maximum if $\left(\frac{z^1}{z^2} - R_j\right)$ is maximum, i.e., if R_j is minimum.

Therefore, we can conclude the following:

If $z^1 \neq 0$ and also $(z_j^2 - d_j) > 0$ and $R_j < \frac{z^1}{z^2}$, then the non-basic vector $\boldsymbol{\alpha}_j \in \mathbf{A}$ corresponding to $\mathrm{Min}\, R_j$ is selected as the incoming vector. If this condition is not satisfied, then no further improvement is possible and the optimal solution is reached.

Sub-case 2. $z^1 \neq 0$ and $(z_j^2 - d_j) < 0$.

In this case, equation (4.9) implies that $\frac{\bar{z}^1}{\bar{z}^2} > \frac{z^1}{z^2}$ only if $R_j > \frac{z^1}{z^2}$.

Also, equation (4.9) implies that $\frac{\overline{z}^1}{\overline{z}^2}$ is maximum if $\left(\frac{z^1}{z^2} - R_j\right)$ is minimum, i.e., if R_j is maximum.

Therefore, we can conclude the following:

If $z^1 \neq 0$ and also $(z_j^2 - d_j) < 0$ and $R_j > \frac{z^1}{z^2}$, then the non-basic vector $\boldsymbol{\alpha}_j \in \mathbf{A}$ corresponding to $\operatorname{Max} R_j$ is selected as the incoming vector. If this condition is not satisfied, then no further improvement is possible and the optimal solution is reached.

5 Algorithm for the Proposed Method

Step 1. Find an initial basic feasible solution of the given LFP problem.

Step 2. Compute the values of z^1, z^2 and $\frac{z^1}{z^2}$.

Step 3. Compute the values of $(z_j^1 - c_j)$ and $(z_j^2 - d_j)$ for all the non-basic vectors.

Step 4. Check whether $z^1 = 0$ or $(z_j^2 - d_j) = 0$ for the non-basic vectors holds or not.If yes, go to Step 5; else, Step 6.

Step 5. If either $z^1 = 0$ or $(z_j^2 - d_j) = 0$ for all the non-basic vectors holds, then calculate $(z_j^1 - c_j)$ for all the non-basic vectors. If either of the above two holds, then calculate $z_j^1 - c_j$ for all the non-basic vectors.

If $\left(z_j^1 - c_j\right) < 0$, then the non-basic vector $\boldsymbol{\alpha}_j \in \mathbf{A}$ corresponding to $\operatorname{Min}\left(z_j^1 - c_j\right)$ is selected as the incoming vector. Go to Step 7.

If $\left(z_j^1 - c_j\right) \geq 0$ for all the non-basic vectors, then no further improvement is possible and the optimal solution is reached.

Step 6. If neither $z^1 = 0$ nor $(z_j^2 - d_j) = 0$ for the non-basic vectors, then check whether $z_j^2 - d_j > 0$ or $z_j^2 - d_j < 0$.

Now, calculate

$$R_j = \frac{z_j^1 - c_j}{z_j^2 - d_j}$$

for the non-basic vectors for which$(z_j^2 - d_j) \neq 0$.

Step 6a. Check whether the conditions $(z_j^2 - d_j) < 0$ and $R_j > \frac{z^1}{z^2}$ for one or more non- basic vectors are satisfied or not. If yes, then the non-basic vector $\boldsymbol{\alpha}_j \in \mathbf{A}$ corresponding to $\operatorname{Min} R_j$ is selected as the incoming vector. Go to Step 7; else, Step 6(b).

Step 6b. Check whether the conditions $(z_j^2 - d_j) > 0$ and $R_j < \frac{z^1}{z^2}$ for one or more non- basic vectors are satisfied or not. If yes, then the non-basic vector corresponding to $\operatorname{Max} R_j$ is selected as the incoming vector. Go to Step 7; else, no further improvement is possible and the optimal solution is reached.

Step 7. Using simplex method, the outgoing vector is selected and a new basic feasible solution is obtained. The process is continued till the criterion of optimality is satisfied.

6 Solution of LFP Problems with Negative Denominator in the Objective Function

In the method of solution discussed above, it is supposed that the denominator is positive for all feasible solutions. Now we consider the case when the denominator is negative for some feasible solutions.

Let us suppose that $z^2 < 0$ for the LFP Problem (1.1) for some feasible solution. To select the incoming vector in this case, we proceed as follows:

$$\text{Equation (4.8)} \Rightarrow \frac{\bar{z}^1}{\bar{z}^2} \text{ is maximum if } z^1(z_j^2 - d_j) - z^2(z_j^1 - c_j) \text{ is maximum.}$$

$$\Rightarrow \frac{\bar{z}^1}{\bar{z}^2} \text{ is maximum if } z^2(z_j^2 - d_j)\left[\frac{z^1}{z^2} - \frac{z_j^1 - c_j}{z_j^2 - d_j}\right] \text{ is maximum.}$$

$$\Rightarrow \frac{\bar{z}^1}{\bar{z}^2} \text{ is maximum if } (z_j^2 - d_j)\left[\frac{z^1}{z^2} - \frac{z_j^1 - c_j}{z_j^2 - d_j}\right] \text{ is minimum.}$$

$$\Rightarrow \frac{\bar{z}^1}{\bar{z}^2} \text{ is maximum if } (z_j^2 - d_j)\left[\frac{z^1}{z^2} - R_j\right] \text{is minimum,}$$

$$\text{where } R_j = \frac{z_j^1 - c_j}{z_j^2 - d_j}$$

$$\Rightarrow \frac{\bar{z}^1}{\bar{z}^2} \text{ is maximum if we take } \begin{cases} \text{Max}\, R_j \text{ for } z_j^2 - d_j > 0 \text{ and } R_j > \frac{z^1}{z^2}, \\ \text{Min}\, R_j \text{ for } z_j^2 - d_j < 0 \text{ and } R_j < \frac{z^1}{z^2}. \end{cases}$$

Therefore we conclude the following:

1. If $z_j^2 - d_j > 0$ and $R_j > \frac{z^1}{z^2}$, then the non-basic vector $\boldsymbol{\alpha}_j \in \mathbf{A}$ corresponding to MaxR_j is selected as the incoming vector.
2. If $z_j^2 - d_j < 0$ and $R_j < \frac{z^1}{z^2}$, then the non-basic vector $\boldsymbol{\alpha}_j \in \mathbf{A}$ corresponding to MinR_j is selected as the incoming vector.

If neither of the above two conditions is satisfied, then no further improvement is possible and the optimal solution is reached.

The outgoing vector is selected using simplex method and a new basic feasible solution is obtained. The process is continued till the criterion of optimality is satisfied.

7 Numerical Examples

Example 1. $\text{Max}\dfrac{10x_1 + 5x_2 + x_3}{2x_1 + 2x_2 + 3x_3 + 75}$

subject to

$$2x_1 + 1.1x_2 - x_3 \leq 15, \quad 4x_1 + 6x_2 + 2x_3 \leq 60, \quad -4x_1 - 4x_2 + 3x_3 \leq 5,$$

$$x_1, \; x_2, \; x_3 \geq 0.$$

Solution: After adding slack variables, the above problem can be written in the standard form as follows:

$$\text{Max}\frac{10x_1 + 5x_2 + x_3}{2x_1 + 2x_2 + 3x_3 + 75}$$

subject to

$$2x_1 + 1.1x_2 - x_3 + x_4 = 15, \quad 4x_1 + 6x_2 + 2x_3 + x_5 = 60, \quad -4x_1 - 4x_2 + 3x_3 + x_6 = 5,$$

$$x_1, \; x_2, \; x_3, \; x_4, \; x_5, \; x_6 \geq 0.$$

Basis	c_B	d_B	X_B	y_1	y_2	y_3	y_4	y_5	y_6	$\frac{x_{B_i}}{y_{ij}}$
y_4	0	0	15	②	1.1	-1	1	0	0	$\frac{15}{2}$
y_5	0	0	60	4	6	2	0	1	0	15
y_6	0	0	5	-4	-4	3	0	0	1	-
$z^1 = 0,\ z^2 = 75$			$z_j^1 - c_j$	-10	-5	-1	0	0	0	
z = 0			$z_j^2 - d_j$	-2	-2	-3	0	0	0	
				↑			↓			

Tab. 1(a): Initial Table for Example 1

Basis	c_B	d_B	X_B	y_1	y_2	y_3	y_4	y_5	y_6	$\frac{x_{B_i}}{y_{ij}}$
y_1	10	2	15/2	1	11/20	-(1/2)	1/2	0	0	-
y_5	0	0	30	0	19/5	④	-2	1	0	$\frac{15}{2}$
y_6	0	0	35	0	-(9/5)	1	2	0	1	35
$z^1 = 75$			$z_j^1 - c_j$	0	1/2	-6	5	0	0	
$z^2 = 90$			$z_j^2 - d_j$	0	-(9/10)	-4	1	0	0	
$z = \frac{5}{6}$			R_j	-	-(5/9)	3/2	-	-	-	
						↑		↓		

Tab. 1(b): Intermediate Table for Example 1

Now, we follow the proposed method and continue till the optimal solution is reached. Thus we have the following tables:

Now the criterion of optimality is satisfied, therefore the optimal solution of the given LFPP is reached, which is given by $x_1 = \frac{45}{4},\ x_2 = 0,\ x_3 = \frac{15}{2}$ and max z = 1.

Example 2. $$\text{Max}\,\frac{3x_1 + x_2 + x_3 + 1}{x_1 + 2x_2 + 3x_3 + 2};$$

subject to

$$2x_1 + x_2 + 3x_3 \le 4, \quad x_1 + 2x_2 + x_3 \le 1, \quad x_1, x_2, x_3 \ge 0.$$

Solution: After adding slack variables, the above problem can be written in the standard form as follows:

$$\text{Max}\,\frac{3x_1 + x_2 + x_3 + 1}{x_1 + 2x_2 + 3x_3 + 2};$$

subject to

$$2x_1 + x_2 + 3x_3 + x_4 = 4, \quad x_1 + 2x_2 + x_3 + x_5 = 1,$$

Basis	c_B	d_B	X_B	y_1	y_2	y_3	y_4	y_5	y_6
y_1	10	2	45/4	1	41/40	0	1/4	1/8	0
y_3	1	3	15/2	0	19/20	1	-(1/2)	1/4	0
y_6	0	0	55/2	0	-(11/4)	0	5/2	-(1/4)	1
$z^1 = 120$			$z_j^1 - c_j$	0	31/5	0	2	3/2	0
$z^2 = 120$			$z_j^2 - d_j$	0	(29/10)	0	-1	1	0
z = 1			R_j	-	62/29	-	-2	3/2	-

Tab. 1(c): Final Table for Example 1

$x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0.$

Now, we follow the proposed method and continue till the optimal solution is reached. Thus we have the following tables:

Basis	c_B	d_B	X_B	y_1	y_2	y_3	y_4	y_5	$\frac{x_{B_i}}{y_{ij}}$
y_4	0	0	4	2	1	3	1	0	2
y_5	0	0	1	1	2	1	0	1	1
$z^1 = 1$			$z_j^1 - c_j$	-3	-1	-1	0	0	
$z^2 = 2$			$z_j^2 - d_j$	-1	-2	-3	0	0	
$z = \frac{1}{2}$			R_j	3	1/2	1/3	-	-	
				↑				↓	

Tab. 2(a): Initial Table for Example 2

Basis	c_B	d_B	X_B	y_1	y_2	y_3	y_4	y_5
y_4	0	0	2	0	-3	1	1	-2
y_1	3	1	1	1	2	1	0	1
$z^1 = 4$			$z_j^1 - c_j$	0	5	2	0	3
$z^2 = 3$			$z_j^2 - d_j$	0	0	-2	0	1
$z = \frac{4}{3}$			R_j	-	-	-1	-	3

Tab. 2(b): Final Table for Example 2

Now the criterion of optimality is satisfied, therefore the optimal solution of the given LFPP is reached, which is given by $x_1 = 1,\ x_2 = 0,\ \ x_3 = 0$ and max z $= \frac{4}{3}$.

Example 3. $\text{Max}\,\dfrac{5x_1 - 3x_2 + 7x_3 - 7x_4 - 4x_5 + 6x_6 + 5}{-2x_1 + 3x_2 - 3x_3 + x_4 + x_5 - 2x_6 - 150}$

subject to

$$x_1 + x_2 + x_3 \leq 20,\quad x_4 + x_5 + x_6 \leq 12,\quad x_1 + x_4 \leq 10,\quad x_2 + x_5 \leq 25,\quad x_3 + x_6 \leq 15,$$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$

Solution: After adding slack variables, the above problem can be written in the standard form as follows:

$$\text{Max} \frac{5x_1 - 3x_2 + 7x_3 - 7x_4 - 4x_5 + 6x_6 + 5}{-2x_1 + 3x_2 - 3x_3 + x_4 + x_5 - 2x_6 - 150}$$

subject to

$$x_1 + x_2 + x_3 + x_7 = 20, \quad x_4 + x_5 + x_6 + x_8 = 12,$$

$$x_1 + x_4 + x_9 = 10, \quad x_2 + x_5 + x_{10} = 25, \quad x_3 + x_6 + x_{11} = 15,$$

$$x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6,\ x_7,\ x_8,\ x_9,\ x_{10},\ x_{11} \geq 0.$$

Now, we follow the proposed method and continue till the optimal solution is reached. Thus we have the following tables:

Basis	c_B	d_B	X_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	$\frac{x_{B_i}}{y_{ij}}$
y_7	0	0	20	1	1	1	0	0	0	1	0	0	0	0	-
y_8	0	0	12	0	0	0	1	1	1	0	1	0	0	0	12
y_9	0	0	10	1	0	0	1	0	0	0	0	1	0	0	10
y_{10}	0	0	25	0	1	0	0	1	0	0	0	0	1	0	-
y_{11}	0	0	15	0	0	1	0	0	1	0	0	0	0	1	-
$z^1 = 5$			$z_j^1 - c_j$	-5	3	-7	7	4	-6	0	0	0	0	0	
$z^2 = -150$			$z_j^2 - d_j$	2	-3	3	-1	-1	2	0	0	0	0	0	
$z = -\frac{1}{30}$			R_j	-(5/2)	-1	-(7/3)	-7	-4	-3	-	-	-	-	-	
							↑					↓			

Tab. 3(a): Initial Table for Example 3

Basis	c_B	d_B	X_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	$\frac{x_{B_i}}{y_{ij}}$
y_7	0	0	20	1	1	1	0	0	0	1	0	0	0	0	-
y_8	0	0	2	-1	0	0	0	1	1	0	1	-1	0	0	2
y_4	-7	1	10	1	0	0	1	0	0	0	0	1	0	0	-
y_{10}	0	0	25	0	1	0	0	1	0	0	0	0	1	0	25
y_{11}	0	0	15	0	0	1	0	0	1	0	0	0	0	1	-
$z^1 = -65$			$z_j^1 - c_j$	-12	3	-7	0	4	-6	0	0	-7	0	0	
$z^2 = -140$			$z_j^2 - d_j$	3	-3	3	0	-1	2	0	0	1	0	0	
$z = -\frac{13}{28} = 0.46$			R_j	-4	-1	-(7/3)	-	-4	-3	-	-	-7	-	-	
								↑			↓				

Tab. 3(b): Intermediate Table for Example 3

Basis	c_B	d_B	X_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}	$\frac{x_{B_i}}{y_{ij}}$
y_7	0	0	20	1	1	1	0	0	0	1	0	0	0	0	20
y_5	-4	1	2	-1	0	0	0	1	1	0	1	-1	0	0	-
y_4	-7	1	10	1	0	0	1	0	0	0	0	1	0	0	-
y_{10}	0	0	23	1	1	0	0	0	-1	0	-1	1	1	0	23
y_{11}	0	0	15	0	0	1	0	0	1	0	0	0	0	1	-
$z^1 = -73$			$z_j^1 - c_j$	-8	3	-7	0	0	-10	0	-4	-3	0	0	
$z^2 = -138$			$z_j^2 - d_j$	2	-3	3	0	0	3	0	1	0	0	0	
$z = \frac{73}{138} = 0.528$			R_j	-4	-1	-(7/3)	-	-	-(10/3)	-	-4	-	-	-	
					$\uparrow$					$\downarrow$					

Tab. 3(c): Intermediate Table for Example 3

Basis	c_B	d_B	X_B	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}	y_{11}
y_2	-3	3	20	1	1	1	0	0	0	1	0	0	0	0
y_5	-4	1	2	-1	0	0	0	1	1	0	1	-1	0	0
y_4	-7	1	10	1	0	0	1	0	0	0	0	1	0	0
y_{10}	0	0	3	0	0	-1	0	0	-1	-1	-1	1	1	0
y_{11}	0	0	15	0	0	1	0	0	1	0	0	0	0	1
$z^1 = -133$			$z_j^1 - c_j$	-11	0	-10	0	0	-10	-3	-4	-3	0	0
$z^2 = -78$			$z_j^2 - d_j$	5	0	6	0	0	3	3	1	0	0	05
$z = \frac{133}{78} = 1.7051$			R_j	-(11/5)	-	-(5/3)	-	-	-(10/3)	-1	-4	-	-	-

Tab. 3(d): Final Table for Example 3

Now the criterion of optimality is satisfied, therefore the optimal solution of the given LFPP is reached, which is given by $x_1 = 0$, $x_2 = 20$, $x_3 = 0$, $x_4 = 10, x_5 = 2$, $x_6 = 0$, and max z $= \frac{133}{78} = 1.7051$.

8 Comparison of Numerical Results

The following table shows the comparison between the proposed method and other optimization methods :

Example	Reference	Optimal solution	Optimal value
Ex.1	Proposed	(11.25, 0, 7.5)	1
	Ref.[15]	(11.25, 0, 7.5)	1
	Ref.[16]	(11.25, 0, 7.5)	1
Ex.2	Proposed	(1, 0, 0)	1.3333
	Ref.[13]	(1, 0, 0)	1.3333
Ex.3	Proposed	(0, 20, 0, 10, 2, 0)	1.7051
	Ref.[16]	(0, 20, 0, 10, 2, 0)	1.7051

Tab. 4: Comparison of the Numerical Results

It can be seen that the results obtained by the proposed method are the same as those obtained by other methods for all the examples, which proves the validity of the proposed method.

It is obvious that in order to solve example 1 or 2 by the existing ratio method [14], a large number of optimality conditions has to be checked in each iteration.

The method of Tantawy [13] involves derivation of a linear programming problem from the given LFP problem, construction of the corresponding dual problem and solution of a system of linear equations for some active constraints introduced in the system.

The ratio multiplex algorithm [15] makes a search in the direction of the steepest ascent or descent of the objective function. In this algorithm, a θ matrix of intercepts with the promising variables is constructed to select a set of entering and leaving variable in each iteration. This algorithm brings a set of variables into the basis first at the initial pass and then in the subsequent passes until the optimal solution is reached. Although the computational steps of this algorithm are less than those of the existing ratio method [14], it is more effective in handling those LFP problems which have constraint matrix of high sparsity.

Moreover it is not possible to find the solution of some problems like example 3 by the existing methods [13], [14], [15]. Problems of this type can be solved by the existing modified ratio algorithm [16] , but this algorithm involves a large number of computations in order to select the entering variable. This algorithm selects all promising variables in phase 1.Out of these promising variables, it selects the entering variables in phase 2, where a large number of computations are required. For example, solution of example 3 by this algorithm involves multiplication of two square matrices of order 7 apart from other computations.

9 Conclusion

In this paper, an easy method for solving linear fractional programming problem has been developed. In order to find an improved basic feasible solution, the method of selection of entering variable is organised in a way that the total computational effort required is minimum. This makes the process easy for manual calculation and saves our time. Additionally, the proposed method can be used to find the optimal solution of all linear fractional programming problems, irrespective of the sign of the denominator of the objective function. A comparison of the proposed method with other methods for numerical examples shows that the proposed method gives identical results with those obtained by the other methods. This proves the validity of the method.

References

[1] Charnes, Abraham and Cooper, W. W. (1962): Naval Research logistics quarterly, Vol. 9, No. 3-4, pp. 181-186.

[2] Wolf, Hartmut (1985): A parametric method for solving the linear fractional programming problem, Operations Research, Vol.33, No. 4, pp. 835-841.

[3] Isbell, J. and Marlow, W. (1956): Naval Research Logistics Quarterly, Vol. 3, No. 1-2, pp. 71-94.

[4] Martos, Bela (1964): Hyperbolic programming, CARNEGIE INST OF TECH PITTSBURGH PA GRADUATE SCHOOL OF INDUSTRIAL ADMINISTRATION.

[5] Dinkelbach, Werner (1967): Management science,Vol. 13, No. 7, pp. 492-498.

[6] Swarup, Kanti (1977): On almost paracontact Riemannian manifold, TRU maths. 13(2), pp. 22-39.

[7] Sharma, I. and Swarup, Kanti (1972): Operations Research, Vol. 16, No. 3, pp. 81-100.

[8] Bector, C. (1968): ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik, Vol. 48, No. 6, pp. 418-420.

[9] Bector, C. (1973): Utilitas Mathematica, Vol. 4, pp. 155-168.

[10] Seshan, C. (1980): Proceedings of the Indian Academy of Sciences-Mathematical Sciences, Springer, Vol. 89, No. 1, pp. 35.

[11] Bajalinov, Erik B. (2003): Linear-Fractional Programming Theory, Methods, Applications

and Software, Springer, pp. 381-408.

[12] Jahan, Sohana and Islam, M. Ainul (2010): International Journal of Basic and Applied Sciences IJBAS-IJENS, Vol. 10, No. 2, pp. 39-44.

[13] Tantawy, S.F. (2007): Australian Journal of Basic and Applied Science, Vol. 1, No. 2, pp. 105-108.

[14] Sankaraiyer, P. (1994): A computer oriented ratio alogrithm to solve linear frational programming problems.

[15] Naganathan, ER and Sakthivel, S. (1998): Opsearch, Vol. 35, No. 3, pp. 214-225.

[16] Jeyaraman, K. (2001): A modified ratio algorithm to solve linear fractional programming problems.