C-Bochner ϕ-Symmetric (ε)-Kenmotsu Manifold admitting Semi-symmetric metric connection

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Abstract

In this paper we consider a semi-symmetric metric connection in a (ε)-Kenmotsu manifold and study locally ϕ-symmetric, locally C-Bochner ϕ-symmetric and ξ-C-Bochner flat (ε)-Kenmotsu manifold with respect to a semi-symmetric metric connection.

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1 Introduction

In 1971, Kenmotsu studied a class of contact Riemannian manifolds satisfying some special conditions [14]. We call it Kenmotsu manifold. Let ∇ be a linear connection in an n-dimensional dierentiable manifold M. The torsion tensor T and the curvature tensor R of ∇ respectively are given by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \tag{1.1}$$

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X - \nabla_{[X,Y]}Z. \tag{1.2}$$

The connection ∇ is said to be symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. A linear connection ∇ is said to be semi-symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \tag{1.3}$$

Semi-symmetric connections play an important role in the study of Riemannian manifolds. There are various physical problems involving the semi-symmetric metric connection.

In this paper, we study curvature properties in an (ε)-Kenmotsu manifold with respect to a semi-symmetric metric connection. The paper is organized as follows: In Section 2, we give a brief introduction of an (ε)-Kenmotsu manifold and define semi-symmetric metric connection. In Section 3, we study the locally ϕ-symmetric (ε)-Kenmotsu manifold with respect to a semi-symmetric metric connection. In Section 4, we study the locally C-Bochner ϕ-symmetric (ε)-Kenmotsu manifold with respect to a semi-symmetric metric connection and Section 5 is devoted ξ-C-Bochner flat (ε)-Kenmotsu manifold with respect to the semi-symmetric metric connection.

2 Preliminaries

An n-dimensional smooth manifold (M, g) is (ε)-almost contact metric manifold if

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$
$$\eta(\xi) = 1, \tag{2.2}$$
$$g(\xi, \xi) = \xi, \tag{2.3}$$
$$\eta(X) = \varepsilon g(X, \xi), \tag{2.4}$$
$$g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \tag{2.5}$$

for all vector fields X, Y on M, where ε is 1 or -1 according to which is either ξ is space like or time like vector field and rank that ϕ is (n-1). If

$$d\eta(X, Y) = g(X, \phi Y), \tag{2.6}$$

for every $X, Y \in TM$, then we say that $M(\phi, \xi, \eta, g, \varepsilon)$ is an almost contact metric manifold. Also, we have

$$\eta \circ \phi = 0, \quad \phi\xi = 0. \tag{2.7}$$

If an (ε)-contact metric manifold satisfies

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \varepsilon\eta(Y)\phi X, \tag{2.8}$$

where ∇ denotes the Riemannian connection of g, then M is called an (ε)-Kenmotsu manifold [9]. An (ε)-almost contact metric manifold is an (ε)-Kenmotsu if and only if

$$\nabla_X \xi = \varepsilon(X - \eta(X)\xi). \tag{2.9}$$

Moreover, the curvature tensor R and the Ricci tensor S in an (ε)-Kenmotsu manifold M with respect to the Levi-Civita connection satisfy [9]

$$(\nabla_X \eta)Y = g(X, Y) - \varepsilon\eta(X)\eta(Y), \tag{2.10}$$
$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.11}$$
$$R(\xi, X)Y = \eta(Y)X - \varepsilon g(X, Y)\xi, \tag{2.12}$$
$$\begin{aligned} R(X, Y)\phi Z &= \phi R(X, Y)Z + \varepsilon\{g(Y, Z)\phi X - g(X, Z)\phi Y \\ &+ g(X, \phi Z)Y - g(Y, \phi Z)X\}, \end{aligned} \tag{2.13}$$
$$\eta(R(X, Y)Z) = \varepsilon[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)], \tag{2.14}$$
$$S(X, \xi) = -(n-1)\eta(X), \tag{2.15}$$
$$S(\phi X, \phi Y) = S(X, Y) + \varepsilon(n-1)\eta(X)\eta(Y). \tag{2.16}$$

A (ε)-Kenmotsu manifold M is said to be locally C-Bochner ϕ-symmetric if

$$\phi^2((\nabla_W B)(X, Y)Z) = 0,$$

for all vector fields X, Y, Z and W orthogonal to ξ, where the C-Bochner curvature tensor B is given by

$$(2.17)\quad \begin{aligned} B(X,Y)Z &= R(X,Y)Z + \frac{1}{n+3}[S(X,Z)Y - S(Y,Z)X + g(X,Z)QY \\ &-g(Y,Z)QX + S(\phi X,Z)\phi Y - S(\phi Y,Z)\phi X + g(\phi X,Z)Q\phi Y \\ &-g(\phi Y,Z)Q\phi X + 2S(\phi X,Y)\phi Z + 2g(\phi X,Y)Q\phi Z \\ &-S(X,Z)\eta(Y)\xi + S(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)QY + \eta(Y)\eta(Z)QX] \\ &-\frac{p+n-1}{n+3}[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X + 2g(\phi X,Y)\phi Z] \\ &-\frac{p-4}{n+3}[g(X,Z)Y - g(Y,Z)X] \\ &+\frac{p}{n+3}[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X]. \end{aligned}$$

For (ε)-Kenmotsu manifold the relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ is given by

$$(2.18)\quad \tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi.$$

By the virtue of equations (2.1), (2.3), (2.9) and (2.10), equation (2.18) reduces to

$$(2.19)\quad \begin{aligned} \tilde{R}(X,Y)Z &= R(X,Y)Z + (2+\varepsilon)[g(X,Z)Y - g(Y,Z)X] \\ &+(1+\varepsilon)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi \\ &+(1+\varepsilon)\eta(Z)[\eta(Y)X - \eta(X)Y], \end{aligned}$$

where $\tilde{R}$ is the Riemannian curvature of the connection $\tilde{\nabla}$. From (2.19) it follows that

$$(2.20)\quad \begin{aligned} \tilde{S}(Y,Z) &= S(Y,Z) + [(\varepsilon+2)(\varepsilon-n)+2]g(Y,Z) \\ &+(1+\varepsilon)(n-2\varepsilon)\eta(Y)\eta(Z), \end{aligned}$$

where $\tilde{S}$ and S are the Ricci tensors of connection $\tilde{\nabla}$ and ∇, respectively.
Contracting the above equation, we get

$$(2.21)\quad \tilde{r} = r + \eta((\varepsilon+2)(\varepsilon-n)+2) + \varepsilon(1+\varepsilon)(n-2\varepsilon),$$

where $\tilde{r}$ and r are the scalar curvature of the connection $\tilde{\nabla}$ and ∇, respectively.

3 Locally ϕ-symmetric (ε)-Kenmotsu manifold with respect to a semi-symmetric metric connection

A locally ϕ-symmetric (ε)-Kenmotsu manifold with respect to a semi-symmetric metric connection is given by

$$(3.1)\quad \phi^2((\tilde{\nabla}_W\tilde{R})(X,Y)Z) = 0,$$

for any vector fields X, Y, Z and W orthogonal to ξ.
Using (2.18), we get

$$(3.2)\quad \begin{aligned} (\tilde{\nabla}_W\tilde{R})(X,Y)Z &= (\nabla_W\tilde{R})(X,Y)Z \\ &-\eta(X)\tilde{R}(W,Y)Z + g(W,X)\tilde{R}(\xi,Y)Z \\ &-\eta(Y)\tilde{R}(X,W)Z + g(W,Y)\tilde{R}(X,\xi)Z. \end{aligned}$$

Now differentiating (2.19), with respect to W and using (2.8), we obtain

$$(\nabla_W\tilde{R})(X,Y)Z = (\nabla_W R)(X,Y)Z + (1+\varepsilon)[g(Y,Z)g(W,X)\xi \\ -2\varepsilon g(Y,Z)\eta(W)\eta(X)\xi - g(X,Z)g(W,Y)\xi + 2\varepsilon g(X,Z)\eta(W)\eta(Y)\xi \\ -2\varepsilon\eta(W)\eta(Y)\eta(Z)X - \varepsilon g(X,Z)\eta(Y)W + \varepsilon g(Y,Z)\eta(X)W + g(W,Z)\eta(Y)X \\ +g(W,Y)\eta(Z)X + 2\varepsilon\eta(W)\eta(X)\eta(Z)Y - g(W,Z)\eta(X)Y - g(W,X)\eta(Z)Y]. \tag{3.3}$$

Using (2.1) and (3.3) in (3.2) and applying ϕ^2, we get

$$\phi^2((\tilde{\nabla}_W\tilde{R})(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z) \\ +(1+\varepsilon)\{\varepsilon[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\phi^2 W \\ +[g(W,Z)\eta(Y) + g(W,Y)\eta(Z) - 2\varepsilon\eta(W)\eta(Y)\eta(Z)]\phi^2 X \\ -[g(W,Z)\eta(X) + g(W,X)\eta(Z) - 2\varepsilon\eta(W)\eta(X)\eta(Z)]\phi^2 Y\} \\ +\eta(X)\phi^2(\tilde{R}(W,Y)Z) - \eta(Y)\phi^2(\tilde{R}(X,W)Z) \\ +\frac{2+\varepsilon}{\varepsilon}[g(W,X)\phi^2 Y - g(W,Y)\phi^2 X]. \tag{3.4}$$

If X, Y, Z and W orthogonal to ξ, (3.4) then becomes

$$\phi^2((\tilde{\nabla}_W\tilde{R})(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z). \tag{3.5}$$

By the above discussions we can state the following theorem

Theorem 3.1. *A (ε)-Kenmotsu manifold is locally ϕ-symmetric with respect to a semi-symmetric metric connection $\tilde{\nabla}$ if and only if it is so with respect to Levi-Civita connection ∇.*

4 Locally C-Bochner ϕ-symmetric (ε)-Kenmotsu manifold with respect to a semi-symmetric metric connection

A (ε)-Kenmotsu manifold M is said to be locally C-Bochner ϕ-symmetric with respect to a semi-symmetric metric connection if

$$\phi^2((\tilde{\nabla}_W\tilde{B})(X,Y)Z) = 0, \tag{4.1}$$

for any vector fields X, Y, Z and W orthogonal to ξ, where $\tilde{B}$ is the C-Bochner curvature tensor with respect to semi-symmetric metric connection given by

$$\tilde{B}(X,Y)Z = \tilde{R}(X,Y)Z + \frac{1}{n+3}[\tilde{S}(X,Z)Y - \tilde{S}(Y,Z)X + g(X,Z)\tilde{Q}Y \\ -g(Y,Z)\tilde{Q}X + \tilde{S}(\phi X,Z)\phi Y - \tilde{S}(\phi Y,Z)\phi X + g(\phi X,Z)\tilde{Q}\phi Y \\ -g(\phi Y,Z)\tilde{Q}\phi X + 2\tilde{S}(\phi X,Y)\phi Z + 2g(\phi X,Y)\tilde{Q}\phi Z \\ -\tilde{S}(X,Z)\eta(Y)\xi + \tilde{S}(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)\tilde{Q}Y + \eta(Y)\eta(Z)\tilde{Q}X] \\ -\frac{p+n-1}{n+3}[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X + 2g(\phi X,Y)\phi Z] \\ -\frac{p-4}{n+3}[g(X,Z)Y - g(Y,Z)X] \\ +\frac{p}{n+3}[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X], \tag{4.2}$$

Using (2.18), we obtain

$$(\tilde{\nabla}_W \tilde{B})(X,Y)Z = (\nabla_W \tilde{B})(X,Y)Z$$
$$-\eta(X)\tilde{B}(W,Y)Z + g(W,X)\tilde{B}(\xi,Y)Z$$
$$-\eta(Y)\tilde{B}(X,W)Z + g(W,Y)\tilde{B}(X,\xi)Z. \tag{4.3}$$

Now differentiating (4.2) with respect to W, we get

$$(\nabla_W \tilde{B})(X,Y)Z = (\nabla_W \tilde{R})(X,Y)Z - \frac{1}{n+3}[(\nabla_W \tilde{S})(X,Z)Y$$
$$-(\nabla_W \tilde{S})(Y,Z)X + (\nabla_W \tilde{S})(\phi X,Z)\phi Y - (\nabla_W \tilde{S})(\phi Y,Z)\phi X$$
$$+2(\nabla_W \tilde{S})(\phi X,Y)\phi Z - (\nabla_W \tilde{S})(X,Z)\eta(Y)\xi - \tilde{S}(X,Z)g(W,Y)\xi$$
$$+2\varepsilon\tilde{S}(X,Z)\eta(W)\eta(Y)\xi - \varepsilon\tilde{S}(X,Z)\eta(Y)W + (\nabla_W \tilde{S})(Y,Z)\eta(X)\xi$$
$$+\tilde{S}(Y,Z)g(W,X)\xi - 2\varepsilon\tilde{S}(Y,Z)\eta(W)\eta(X)\xi + \varepsilon\tilde{S}(Y,Z)\eta(X)W$$
$$-g(W,X)\eta(Z)\tilde{Q}Y + 2\varepsilon\eta(W)\eta(X)\eta(Z)\tilde{Q}Y - g(W,Z)\eta(X)\tilde{Q}Y$$
$$+g(W,Y)\eta(Z)\tilde{Q}X - 2\varepsilon\eta(W)\eta(Y)\eta(Z)\tilde{Q}X + g(W,Z)\eta(Y)\tilde{Q}X].$$
$$+\frac{p}{n+3}[g(X,Z)g(W,Y)\xi - 2\varepsilon g(X,Z)\eta(W)\eta(Y)\xi + \varepsilon g(X,Z)\eta(Y)W$$
$$-g(Y,Z)g(W,X)\xi + 2\varepsilon g(Y,Z)\eta(W)\eta(X)\xi - \varepsilon g(Y,Z)\eta(X)W$$
$$+g(W,X)\eta(Z)Y - 2\varepsilon\eta(W)\eta(X)\eta(Z)Y + g(W,Z)\eta(X)Y$$
$$-g(W,Y)\eta(Z)X + 2\varepsilon\eta(W)\eta(Y)\eta(Z)X - g(W,Z)\eta(Y)X]. \tag{4.4}$$

Using (3.3) and (2.20) in (4.4), we can write

$$
\begin{aligned}
(\nabla_W\tilde{B})(X,Y)Z &= (\nabla_W R)(X,Y)Z + (1+\varepsilon)[g(Y,Z)g(W,X)\xi \\
&-2\varepsilon g(Y,Z)\eta(W)\eta(X)\xi - g(X,Z)g(W,Y)\xi + 2\varepsilon g(X,Z)\eta(W)\eta(Y)\xi \\
&-2\varepsilon\eta(W)\eta(Y)\eta(Z)X - \varepsilon g(X,Z)\eta(Y)W + \varepsilon g(Y,Z)\eta(X)W \\
&+g(W,Z)\eta(Y)X + g(W,Y)\eta(Z)X + 2\varepsilon\eta(W)\eta(X)\eta(Z)Y \\
&-g(W,Z)\eta(X)Y - g(W,X)\eta(Z)Y] \\
&+\frac{1}{n+3}[(\nabla_W S)(X,Z)Y - (\nabla_W S)(Y,Z)X \\
&+(\nabla_W S)(\phi X,Z)\phi Y - (\nabla_W S)(\phi Y,Z)\phi X + 2(\nabla_W S)(\phi X,Y)\phi Z \\
&-(\nabla_W S)(X,Z)\eta(Y)\xi + (\nabla_W S)(Y,Z)\eta(X)\xi] \\
&+(1+\varepsilon)(n-2\varepsilon)[\eta(X)[g(W,Z) - \varepsilon\eta(W)\eta(Z)]Y \\
&+\eta(Z)[g(W,X) - \varepsilon\eta(W)\eta(X)]Y \\
&-\eta(Y)[g(W,Z) - \varepsilon\eta(W)\eta(Z)]X + \eta(Z)[g(W,Y) - \varepsilon\eta(W)\eta(Y)]X \\
&+\eta(\phi X)[g(W,Z) - \varepsilon\eta(W)\eta(Z)]\phi Y + \eta(Z)[g(W,\phi X) - \varepsilon\eta(W)\eta(\phi X)]Y \\
&-\eta(\phi Y)[g(W,Z) - \varepsilon\eta(W)\eta(Z)]\phi X + \eta(Z)[g(W,\phi Y) - \varepsilon\eta(W)\eta(\phi Y)]X \\
&+2[\eta(\phi X)[g(W,Y) - \varepsilon\eta(W)\eta(Y)]\phi Z + \eta(Y)[g(W,\phi X) - \varepsilon\eta(W)\eta(\phi X)]Z] \\
&-\eta(X)[g(W,Z) - \varepsilon\eta(W)\eta(Z)]\eta(Y)\xi + \eta(Z)[g(W,X) - \varepsilon\eta(W)\eta(X)]\eta(Y)\xi \\
&+\eta(Y)[g(W,Z) - \varepsilon\eta(W)\eta(Z)]\eta(X)\xi + \eta(Z)[g(W,Y) - \varepsilon\eta(W)\eta(Y)]\eta(X)\xi \\
&-[S(X,Z) + [(\varepsilon+2)(\varepsilon-n)+2]g(X,Z)]g(W,Y)\xi \\
&-(1+\varepsilon)(n-2\varepsilon)\eta(X)\eta(Z)g(W,Y)\xi \\
&+2\varepsilon[S(X,Z) + [(\varepsilon+2)(\varepsilon-n)+2]g(X,Z)]\eta(W)\eta(Y)\xi \\
&+2\varepsilon(1+\varepsilon)(n-2\varepsilon)\eta(X)\eta(Z)\eta(W)\eta(Y)\xi \\
&-\varepsilon[S(X,Z) + [(\varepsilon+2)(\varepsilon-n)+2]g(X,Z) + (1+\varepsilon)(n-2\varepsilon)\eta(X)\eta(Z)]\eta(Y)W \\
&+[S(Y,Z) + [(\varepsilon+2)(\varepsilon-n)+2]g(Y,Z) + (1+\varepsilon)(n-2\varepsilon)\eta(Y)\eta(Z)]g(W,X)\xi \\
&-2\varepsilon[S(Y,Z) + [(\varepsilon+2)(\varepsilon-n)+2]g(Y,Z)]\eta(W)\eta(X)\xi \\
&-2\varepsilon(1+\varepsilon)(n-2\varepsilon)\eta(Y)\eta(Z)\eta(W)\eta(X)\xi \\
&-\varepsilon[S(Y,Z) + [(\varepsilon+2)(\varepsilon-n)+2]g(Y,Z) + (1+\varepsilon)(n-2\varepsilon)\eta(Y)\eta(Z)]\eta(X)W \\
&-g(W,X)\eta(Z)\tilde{Q}Y + 2\varepsilon\eta(W)\eta(X)\eta(Z)\tilde{Q}Y - g(W,Z)\eta(X)\tilde{Q}Y \\
&+g(W,Y)\eta(Z)\tilde{Q}X - 2\varepsilon\eta(W)\eta(Y)\eta(Z)\tilde{Q}X + g(W,Z)\eta(Y)\tilde{Q}X] \\
&+\frac{p}{n+3}[g(X,Z)g(W,Y)\xi - 2\varepsilon g(X,Z)\eta(W)\eta(Y)\xi + \varepsilon g(X,Z)\eta(Y)W \\
&-g(Y,Z)g(W,X)\xi + 2\varepsilon g(Y,Z)\eta(W)\eta(X)\xi - \varepsilon g(Y,Z)\eta(X)W \\
&+g(W,X)\eta(Z)Y - 2\varepsilon\eta(W)\eta(X)\eta(Z)Y + g(W,Z)\eta(X)Y \\
&-g(W,Y)\eta(Z)X + 2\varepsilon\eta(W)\eta(Y)\eta(Z)X - g(W,Z)\eta(Y)X].
\end{aligned}
\tag{4.5}
$$

Next using (4.5) in (4.3) and on applying ϕ^2 and taking X, Y, Z and W orthogonal to ξ, we get

$$\phi^2((\tilde{\nabla}_W\tilde{B})(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z). \tag{4.6}$$

Thus we can state the following:

Theorem 4.1. *If M is ϕ-symmetric with respect to a semi-symmetric metric connection then a (ε)-Kenmotsu manifold is locally C-Bochner ϕ-symmetric with respect to a semi-symmetric metric connection $\tilde{\nabla}$ if and only if it is locally ϕ-symmetric with respect to Levi-Civita connection ∇.*

5 ξ-C-Bochner flat (ε)-Kenmotsu manifold with respect to the semi-symmetric metric connection

A (ε)-Kenmotsu manifold M with respect to the semi-symmetric metric connection is said to be ξ-C-Bochner flat if

$$\tilde{B}(X,Y)\xi = 0, \tag{5.1}$$

for all vector fields X, Y on M. If (5.1) holds for X, Y orthogonal to ξ, then a manifold is a horizontal ξ-projectively flat manifold.
Using (2.19) in (4.2), we get

$$\begin{aligned}\tilde{B}(X,Y)Z &= R(X,Y)Z + (2+\varepsilon)[g(X,Z)Y - g(Y,Z)X]\\ &+(1+\varepsilon)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi + (1+\varepsilon)\eta(Z)[\eta(Y)X - \eta(X)Y]\\ &+\frac{1}{n+3}[\tilde{S}(X,Z)Y - \tilde{S}(Y,Z)X + g(X,Z)\tilde{Q}Y\\ &-g(Y,Z)\tilde{Q}X + \tilde{S}(\phi X,Z)\phi Y - \tilde{S}(\phi Y,Z)\phi X + g(\phi X,Z)\tilde{Q}\phi Y\\ &-g(\phi Y,Z)\tilde{Q}\phi X + 2\tilde{S}(\phi X,Y)\phi Z + 2g(\phi X,Y)\tilde{Q}\phi Z\\ &-\tilde{S}(X,Z)\eta)(Y)\xi + \tilde{S}(Y,Z)\eta(X)\xi - \eta(X)\eta(Z)\tilde{Q}Y + \eta(Y)\eta(Z)\tilde{Q}X]\\ &-\frac{p+n-1}{n+3}[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X + 2g(\phi X,Y)\phi Z]\\ &-\frac{p-4}{n+3}[g(X,Z)Y - g(Y,Z)X]\\ &+\frac{p}{n+3}[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X].\end{aligned} \tag{5.2}$$

Putting $Z = \xi$ and using (2.1), (2.10) in (5.2), we get

$$\begin{aligned}\tilde{B}(X,Y)\xi &= [\frac{2+\varepsilon}{\varepsilon} - \varepsilon][\eta(X)Y - \eta(Y)X]\\ &+\frac{1}{n+3}\{[n-1-((\varepsilon+2)(\varepsilon-n)+2)\frac{1}{\varepsilon} - (1+\varepsilon)(n-2\varepsilon)]\eta(Y)X\\ &-[n-1-((\varepsilon+2)(\varepsilon-n)+2)\frac{1}{\varepsilon} - (1+\varepsilon)(n-2\varepsilon)]\eta(X)Y\\ &+[\frac{1}{\varepsilon} - 1]\eta(X)\tilde{Q}Y - [\frac{1}{\varepsilon} - 1]\eta(Y)\tilde{Q}X - \tilde{S}(X,\xi)\eta(Y)\xi + \tilde{S}(Y,\xi)\eta(X)\xi\}\\ &-\frac{p-4}{\varepsilon(n+3)}[\eta(X)Y - \eta(Y)X] + \frac{p}{n+3}[\eta(X)Y - \eta(Y)X].\end{aligned} \tag{5.3}$$

If we consider X, Y orthogonal to ξ, then (5.3) reduces to

$$\tilde{P}(X,Y)\xi = 0 \tag{5.4}$$

Hence (ε)-Kenmotsu manifold is horizontal ξ-C-Bochner flat with respect to the semi-symmetric metric connection.
Again using (2.20) in (5.2), we have

$$\begin{aligned}\tilde{B}(X,Y)Z &= B(X,Y)Z + (2+\varepsilon)[g(X,Z)Y - g(Y,Z)X]\\ &+(1+\varepsilon)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi\\ &+(1+\varepsilon)\eta(Z)[\eta(Y)X - \eta(X)Y]\\ &+\frac{1}{n+3}\{[(\varepsilon+2)(\varepsilon-n)+2][g(X,Z)Y - g(Y,Z)X + g(\phi X,Z)\phi Y\\ &-g(\phi Y,Z)\phi X + 2g(\phi X,Y)\phi Z - g(X,Z)\eta(Y)\xi + g(Y,Z)\eta(X)\xi]\\ &+(1+\varepsilon)(n-2\varepsilon)[\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X]\}.\end{aligned} \tag{5.5}$$

Putting $Z = \xi$ in (5.5) and using (2.1), it follows that

$$\tilde{B}(X,Y)\xi = B(X,Y)\xi + (\frac{2}{\varepsilon} - \varepsilon)[\eta(X)Y - \eta(Y)X] + \frac{1}{n+3}[\frac{(\varepsilon+2)(\varepsilon-n)+2}{\varepsilon} + (1+\varepsilon)(n-2\varepsilon)][\eta(X)Y - \eta(Y)X]. \tag{5.6}$$

If we consider X, Y orthogonal to ξ, then (5.6) reduces to

$$\tilde{B}(X,Y)\xi = B(X,Y)\xi. \tag{5.7}$$

Hence we state the following theorem:

Theorem 5.1. *A (ε)-Kenmotsu manifold is horizontal ξ-C-Bochner flat with respect to the semi-symmetric metric connection if and only if the manifold is ξ-C-Bochner flat with respect to the Levi-Civita connection.*

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