Some Results on Complement of Open Subset Inclusion Graph of a Topological Space

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Abstract

In the recent paper, authors introduced a graph topological structure, called an open subset inclusion graph $\jmath(\tau)$ of a topological space (X, τ) on a finite set X and discussed some importants properties of this graph. In this paper, the researcher discuss some properties such as diameter, girth, connectivity, maximal independent sets, different variants of domination number, clique number and chromatic number, degree of $\jmath(\tau)^c$, edge and vertex connectivity of the graph $\jmath(\tau)^c$.
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1 Introduction

If R is a commutative ring with unity then the zero divisor graph of R was firstly introduced by Beck[2], which is defined as, if R is any ring then $G(R)$ denotes the zero divisor graph of R whose vertex set is $V = R$, such that any distinct vertices x and y are adjacent if and only if $x \cdot y = 0$.

In the recent decades, graphs of several algebraic structures were defined which can be found in[1, 3]. Among these graphs, zero divisor graphs of ring and module are more attractive for many researchers. A. Das[4, 5, 6], introduced the graphs of a vector space & he also discussed some results on these graphs.

The graphs of a vector space were also studied independently by some authors which can be found in [7] and [11]. Some properties on incomparability graphs $\Gamma(L)$ of lattices L were discussed by Wasadikar, M. and Survase P[12]. They classified lattice L by using the graph $\Gamma(L)$ of a lattice L. As Graph theory has wide range of applications in various fields this motivated us to introduce new concept of graphs of topological space (X, τ) with some important properties of these graphs which can be found in [8, 9, 10]. In[8], authors indroduced the graph $\jmath(\tau)$ of τ, which is defined as follows.

Definition 1.1. *[8]***Open Subset Inclusion Graph of a Topological Space:** *Let X be a finite set and τ be a topology defined on X then a graph $\jmath(\tau) = (V(\tau), E(\tau))$ is called as an open subset* *inclusion graph of* (X, τ), *where* $V(\tau) = \{P \in \tau \mid P \neq \phi, P \neq X\}$ *and for* $P, Q \in V(\tau)$, $(P, Q) \in E(\tau)$ *iff* $P \subset Q$ *or* $Q \subset P$.

Example 1.1 Let (X, τ) be the discrete topological space with $X = \{a_1, a_2, a_3\}$ and $\tau = \{\phi, X, U_1 = \{a_1\}, U_2 = \{a_2\}, U_3 = \{a_3\}, U_{12} = \{a_1, a_2\}, U_{13} = \{a_1, a_3\}, U_{23} = \{a_2, a_3\}\}$, then an open subset inclusion graph of (X, τ) and compliment of open subset inclusion graph of a discrete topological space (X, τ) with $|X| = 3$. are shown in fig. 1 and fig. 2

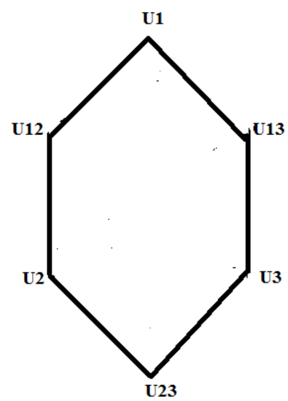

Fig. 1: Open Subset Inclusion Graph of a Discrete Topological Space (X, τ) with $|X| = 3$.

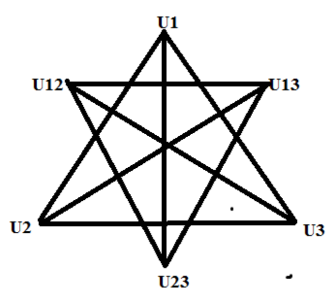

Fig. 2: Complement of Open Subset Inclusion Graph of a Discrete Topological Space (X, τ) with $|X| = 3$.

2 Some Results on Complement of Inclusion Graph of a Topological Space:

Theorem 2.1. *If* τ *is a discrete topology defined on a nonempty set* X *and* S *is any subset of* X *with cardinality greater than* 1, *then* $\jmath(\tau_S)^c$ *is a sub graph of* $\jmath(\tau)^c$.

Proof: The result follows from the definition of $\jmath(\tau)^c$ and the fact that every open subset of S is a open subset of X.

Theorem 2.2. *If* U_1 *and* U_2 *are any two open subsets of* X *such that either* $U_1 \subset U_2$ *or* $U_2 \subset U_1$, *then* $U_1 \nsim U_2$ *in* $\jmath(\tau)^c$.

Proof: If possible $U_1 \sim U_2$ in $\jmath(\tau)^c$ then $U_1 \not\subset U_2$ and $U_2 \not\subset U_2$, which is a contradiction.

Theorem 2.3. *If U_1 and U_2 are any two distinct open subsets of X of same cardinality k, then $U_1 \sim U_2$ in $\jmath(\tau)^c$.*

Proof: If U_1 and U_2 are two distinct open subsets of X of same cardinality k then $U_1 \not\subset U_2$ and $U_2 \not\subset U_1$ and hence $U_1 \sim U_2$ in $\jmath(\tau)^c$.

Corollary 2.1. *If τ is a discrete topology defined on a nonempty set X and $|X| = 2$, then $\jmath(\tau)^c$ is a complete graph.*

Proof: If τ is a discrete topology defined on a nonempty set X and $|X| = 2$, then only non-trivial proper open subsets are of cardinality 1. Now by Theorem 2.3, it follows that any two vertices of $\jmath(\tau)^c$ are adjacent. Hence graph $\jmath(\tau)^c$ is a complete graph.

Corollary 2.2. *If τ is a discrete topology defined on a nonempty set X and $|X| > 2$ then the graph $\jmath(\tau)^c$ is not complete.*

Proof: Since $|X| > 2$, there exists at least two distinct open subsets U and V such that $U \subset V$ or $V \subset U$ and hence by Theorem 2.2, $U \nsim V$ and we are through.

Theorem 2.4. *If τ is a discrete topology defined on a nonempty set X then the graph $\jmath(\tau)^c$ is complete if and only if $|X| = 2$.*

Proof: Suppose $\jmath(\tau)^c$ is a complete graph. If possible, let $|X| = n > 2$, therefore there exist $a \neq b \in X$ and $U = \{a\}$, $V = \{a, b\}$ two open subsets of X. Hence by Theorem 2.2, $U \nsim V$ in $\jmath(\tau)^c$, a contradiction. Thus $|X| \leq 2$. Converse trivially holds from Theorem 2.3.

3 Diameter and Girth of $\jmath(\tau)^c$

In this section we investigate the diameter and girth of $\jmath(\tau)^c$ and discuss when the graph $\jmath(\tau)^c$ is triangulated.

Theorem 3.1. *If τ is a discrete topology defined on a nonempty set X and $|X| \geq 3$, then $\jmath(\tau)^c$ is connected and $diam(\jmath(\tau)^c) \leq 2$.*

Proof: Let U and V are any two non empty proper open subsets of X.
Case 1: If $|X| = 2$ then by Theorem 2.4, $\jmath(\tau)^c$ is a complete graph with 2 vertices. Hence the graph $\jmath(\tau)^c \approx K_2$ is connected and $dist(U, V) = 1$ with $diam(\jmath(\tau)^c) = 1$ in this case.
Case 2: If $|X| \geq 3$ and $|U| = |V| = k$ then by Theorem 2.3, $U \sim V$ and hence $U \sim V$ be a required path from U to V .
Case 3: If $|X| \geq 3$ and $|U| \neq |V|$.
Subcase I: If $U \not\subset V$ or $V \not\subset U$ then $U \sim V$ in the graph $\jmath(\tau)^c$ and hence $U \sim V$ be a required path from U to V in the graph $\jmath(\tau)^c$ and $diam(\jmath(\tau)^c) = 1$ in this case.
Sub Case II: If $U \subset V$ or $V \subset U$ then $U \nsim V$ in the graph $\jmath(\tau)^c$. If $U \cup V \neq X$ then there exists an open set $W = (U \cap V)^c$ such that $U \not\subset V$ and $V \not\subset U$ and hence $U \sim W$, $W \sim V$. Thus, $U \sim W \sim V$ be a required path from U to V and $dist(U, V) = 2$ in the graph $\jmath(\tau)^c$.
Sub Case III: If $U \subset V$ or $V \subset U$ then $U \nsim V$ in the graph $\jmath(\tau)^c$. If $U \cup V = X$ and $U \cap V = \phi$ then $U \sim V$. Hence $U \sim V$ be a required path from U to V and $dist(U, V) = 1$ in the graph $\jmath(\tau)^c$.
Subcase IV: If $U \subset V$ or $V \subset U$ then $U \nsim V$ in the graph $\jmath(\tau)^c$. If $U \cup V = X$ and $U \cap V \neq \phi$ then there exists an open set $W = (U \cap V)^c$ such that $U \sim W$ and $V \sim W$ in the graph $\jmath(\tau)^c$. Hence $U \sim W \sim V$ be a required path from U to V and $dist(U, V) = 2$ in the graph $\jmath(\tau)^c$.
Thus, $diam(\jmath(\tau)^c) \leq 2$.

Theorem 3.2. *Let τ is a discrete topology defined on nonempty set X and $|X| = n$. If $n = 2$ then $girth(\jmath(\tau)^c)$ is ∞, and if $n \geq 3$ then $girth(\jmath(\tau)^c)$ is 3.*

Proof: We will prove the Theorem in two separate cases.
Case 1: If $n = 2$ then by Corollary 2.1, the graph $\jmath(\tau)$ is a complete graph with $\jmath(\tau)^c \approx K_2$ and hence $girth(\jmath(\tau)^c) = \infty$.
Case 2: If $n \geq 3$ and a, b, c be three distinct elements in X then there exist three open subsets U_1, U_2 and U_3 of same cardinality k, for $k \geq 1$ and none of them is equal to X. Then by Theorem 2.3, $U_1 \sim U_2 \sim U_3 \sim U_1$, which is triangle and hence $girth(\jmath(\tau)^c)$ is 3.
Note: The above theorem guarantees that there always exist at least one 3 cycle in $\jmath(\tau)^c$, when $|X| \geq 3$.

Theorem 3.3. *If τ is a discrete topology defined on a nonempty set X and $|X| = n$ and $n \geq 3$, then $\jmath(\tau)^c$ is triangulated.*

Proof: We will show that any vertex of $\jmath(\tau)^c$ is a vertex of a triangle if $|X| \geq 3$. For this we start with an open subsets U of X.
If $|U| = k$ and since $|X| \geq 3$, there exist two open subsets V and W of X with $|V| = k, |W| = k$, for $k \geq 1$ then $U \sim V \sim W \sim U$, which is triangle.
Thus, we can form a triangle with vertices U, V and W, and hence $\jmath(\tau)^c$ is triangulated.

4 Clique Number and Chromatic Number of $\jmath(\tau)^c$

Theorem 4.1. *If τ is a discrete topology defined on a nonempty set X with $|X| = n$ and Y be a clique in $\jmath(\tau)^c$, then Y is a collection of non-empty proper open subsets of X of same cardinality k.*

Proof: If $|U_1| = |U_2| = k$ then neither $U_1 \not\subset U_2$ nor $U_2 \not\subset U_1$. Therefore, $U_1 \sim U_2$, $\forall\ U_1, U_2 \in Y$ and so Y is clique in the graph $\jmath(\tau)^c$ and hence the Theorem.

Theorem 4.2. *If τ is a discrete topology defined on a nonempty set X, then $|X| = n = 2m$ if and only if $\omega(\jmath(\tau)^c) = n!/m!m!$.*

Proof: Let τ is a discrete topology defined on a nonempty set X with $X = \{a_1, a_2, ..., a_n\}$ and let $Y_k = \{U_k \mid |U_k| = k, \text{for fix } k\}$, $\forall\ k = 1, 2, ..., (n-1)$. Then by Theorem 4.1, Y_k is a clique in the graph $\jmath(\tau)^c$. By binomial theory, if $Y_m = \{U \subset X \mid |U| = m \text{ and } U \in \tau\}$, then Y is a maximal clique set in $\jmath(\tau)^c$. Hence the clique number of the graph $\jmath(\tau)^c$ is $n!/m!m!$.

Corollary 4.1. *If τ is a discrete topology defined on a nonempty set X of cardinality n, with $n \geq 4$, then $\jmath(\tau)^c$ is not planar.*

Proof: By Theorem 4.2, $\omega(\jmath(\tau)^c) = n!/m!m!$. As $n \geq 4$, the graph $\jmath(\tau)^c$ has a clique of size at least 6, i.e., a complete graph K_5 as a sub graph of $\jmath(\tau)^c$. Thus, $\jmath(\tau)^c$ is not planar, if $n \geq 4$.

Theorem 4.3. *If τ is a discrete topology defined on a nonempty set X of cardinality n, then $\chi(\jmath(\tau)^c) = n!/m!m!$.*

Proof: By Theorem 4.2, the clique number of $\jmath(\tau)^c$ is $n!/m!m!$, hence $\chi(\jmath(\tau)^c)) \geq n!/m!m!$.
To establish the equality, we demonstrate a $n!/m!m!$ colouring of $\jmath(\tau)$:
Let U be the any open subset of (X, τ) and $|U| = p$. As $U \subset X$ then $\exists$ a vertex V_k of the clique Y such that $U \subset V_k$ or $V_k \subset U$ and hence $U \nsim V_k$. In this case label this vertex U with color C_k (which is used to label the vertex V_k of the clique Y). Apply the same procedure for all vertices U of the graph $\jmath(\tau)^c$. Hence, $\chi(\jmath(\tau)^c) \geq n!/m!m!$ and therefore $\chi(\jmath(\tau)^c) = n!/m!m!$.

Theorem 4.4. *Let τ is a discrete topology defined on a nonempty set X and $|X| = n$. If Y is an ascending or a descending chain of non-empty proper open subsets of X then Y is dominating set.*

Proof: Suppose $X = \{a_1, a_2, ..., a_n\}$ and let $U_k = \{a_1, a_2, ...a_k\}$, for $k = 1, 2, ...(n-1)$. Then $U_i \subset U_j$, for $i < j$ and hence $U_i \nsim U_j$, for $i < j$.
Claim: Y is a dominating set.
Without loss of generality, we can assume that, $V = \{a_{i_1}, a_{i_2}, ..., a_{i_k}\} \in \tau\ \forall\ k = 1, 2, ..., (n-2)$, be any open subset of X of cardinality k, then there exist at least one element $U_k = \{a_1, a_2, ..., a_k\}$ of Y, for k = 1, 2, . . . , (n-1), such that $V \not\subset U_k$ or $U_k \not\subset V$ and hence $U_k \sim V$ in the graph $\jmath(\tau)^c$. By Theorem 2.2, any two open subsets from Y are not adjacent. Thus any vertex in $V \notin Y$ is adjacent to at least one elements of Y and hence Y is a dominating set in $\jmath(\tau)^c$.

Theorem 4.5. *Let τ is a discrete topology defined on a nonempty set X and $|X| = n$. If $Y = \{U_1, U_2, ...U_{n-1} \mid U_k = \{a_1, a_2, ...a_k\}\}$, then Y is a minimal dominating set. Moreover, the domination number of the graph $\jmath(\tau)^c$ is $n - 1$.*

Proof: Suppose $X = \{a_1, a_2, ..., a_n\}$ and let $Y = \{U_1, U_2, ..., U_n \mid U_k = X - \{a_k\}\}$, then by Theorem 4.4, Y is the dominating set.
Claim: Y is a minimal dominating set.
Let $Y_{n-1} = \{U_k \in Y \mid k \neq n - 1\}$, then there exists an open subset, $V = U_{n-1}$ of X such that $U_k \subset V$; $\forall\, k$, and hence $V \nsim U_k$, $\forall k$. Thus Y is a minimal dominating set and hence domination number of the graph $\jmath(\tau)^c$ is $n - 1$.

Theorem 4.6. *If τ is a discrete topology defined on a nonempty set X and $|X| = n$, if $Y = \{U_1, U_2, ...U_{n-1} \mid U_k = \{a_1, a_2, ...a_k\}\}$, then Y is not a totally dominating set in the graph $\jmath(\tau)^c$.*

Proof: Suppose $X = \{a_1, a_2, ..., a_n\}$ and let $Y = \{U_1, U_2, ...U_{n-1} \mid U_k = \{a_1, a_2, ...a_k\}\}$, then by Theorem 4.5, Y is minimal dominating set. By Theorem 2.2, no two elements of Y are adjacent and hence Y is not a clique in $\jmath(\tau)^c$. Thus Y is not a totally dominating set in the graph $\jmath(\tau)^c$.

5 Degree and Independence Number of $\jmath(\tau)^c$

Theorem 5.1. *If τ is a discrete topology defined on a nonempty set X and $|X| = n$ and U be any open subset of X with $|U| = k$, then $deg(U) = |Y| = 2^n - (2^k - 2) + (2^{(n-k)} - 2) - 3$.*

Proof: Suppose $X = \{a_1, a_2, ..., a_n\}$. Let $U = \{a_1, a_2, ..., a_k\}$ with V be any non empty proper open subset of X such that $V \not\subset U$ and $U \not\subset V$ then $V \sim U$ in the graph $\jmath(\tau)^c$.
Let $Y = \{V \in \tau \mid V \not\subset U \;\&\; U \not\subset V\}$ then $deg(U) = |Y|$. By set theory $|Y| = 2^n-2-(2^k-2)+(2^{(n-k)}-1)$ and hence $deg(U) = 2^n - (2^k - 2) + (2^{(n-k)}) - 3$.

Theorem 5.2. *If τ is a discrete topology defined on a nonempty set X and $|X| = n$, then $deg(\jmath(\tau)^c) = \Sigma_1^{(n-1)}(n!/((n - k)!k!))\{(2^k - 2) + (2^{(n-k)} - 2)\}$.*

Proof: Proof follows from Theorem 5.1.

Theorem 5.3. *If τ is a discrete topology defined on a nonempty set X, where $|X| = n$, If $Y = \{U_k \mid U_k = \{a_1, a_2, ...a_k\} \;\forall k = 1, ..., (n - 1)\}$ then Y be an independence set of $\jmath(\tau)^c$.*

Proof: Suppose $X = \{a_1, a_2, ..., a_n\}$ and let $U_k = \{a_1, a_2, ...a_k\}$, for $k = 1, 2, ...(n - 1)$. Then $U_i \subset U_j$, for $i < j$ and hence $U_i \nsim U_j$, for $i < j$.
Claim: Y is an independence set.
Without loss of generality, we can assume that,$V = \{a_{i_1}, a_{i_2}, ..., a_{i_k}\} \in \tau \;\forall\; k = 1, 2, ..., (n - 1)$, be any open subset of X of cardinality k, then there exist at least one element $U_k = \{a_{i_1}, a_{i_2}, ..., a_{i_k}\}$ of Y, for k = 1, 2, . . . , (n-1), such that $V \not\subset U_k$ or $U_k \not\subset V$ and hence $U_k \sim V$ in the graph $\jmath(\tau)^c$. By Theorem 2.2, any two open subsets from Y are not adjacent. Thus any vertex in $V \notin Y$ is adjacent to at least one elements of Y and hence Y is an independence set in $\jmath(\tau)^c$.

Theorem 5.4. *If τ is a discrete topology defined on a nonempty set X, where $|X| = n$, If $Y = \{U_k \mid U_k = \{a_1, a_2, ...a_k\} \;\forall k = 1, ..., (n-1)\}$ then Y is a maximal independence set of $\jmath(\tau)^c$. Moreover, the independence number of $\jmath(\tau)^c$ is $(n - 1)$.*

Proof: Suppose $X = \{a_1, a_2, ..., a_n\}$ and let $U_k = \{a_1, a_2, ...a_k\}$, for $k = 1, 2, ...(n - 1)$. Then $U_i \subset U_j$, for $i < j$ and hence $U_i \nsim U_j$, for $i < j$.
Claim: Y is a maximal independence set.
If possible, $Z = Y \cup \{W\}$ be an independence set in the graph $\jmath(\tau)^c$ then $|W| = k$, for some k. Hence, by Theorem 2.2, $U_i \nsim W$, $\forall i$ and therefore Z is not an independence set in the graph $\jmath(\tau)^c$. Thus Y is an maximal independence set in the graph $\jmath(\tau)^c$ and hence independence number of $\jmath(\tau)^c$ is $(n - 1)$.

6 Conclusion

In this paper, various inter-relationships among $\jmath(\tau)^c$ as a graph and (X, τ) as a topological space are studied. Moreover, some properties of topological space (X, τ) and graph theoretic properties of $\jmath(\tau)^c$ have been discussed and established the equivalence between the corresponding graph and topological space. Apart from this, some basic properties such as diameter, girth, completeness, connectedness, domination, chromaticity, maximal cliques and maximal independent sets, degree of vertex and graph $\jmath(\tau)^c$ of a graph $\jmath(\tau)^c$ have been obtained.

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