Stability Criteria For a System Involving Delays In Growth Response And Harvesting

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Abstract

The influence of time delay on harvesting policy of stage-structured species was investigated using a non - linear mathematical model devised and tested in this work. A single species is divided into two life stages: immature and mature with a time delay. The adult population is thought to be subjected to dynamic harvesting. The existence and stability of steady points, as well as stability change criteria, are all examined. Using Pontryagin's Maximum Principle, a suitable Hamiltonian function is created for the discussion of optimal harvesting of resources used by the population. The highest sustainable production is achieved using this technique.

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1 Introduction

In these days, there has been considerable interest in the problems of finding out the optimal policy for harvest of biological populations and forestry biomass. Bioeconomics modeling has gradually emerged as an offshoot of this approach to manage exploitation of biological resources. An extensive study on the effect of harvesting and other parameters on fishery resources is done by Clark [5,6] using ecological and economic models. He obtained the optimal harvesting of two ecologically independent species whose dynamics is governed by the logistic law of growth. Bhattacharya and Begum [4] discussed the feasible bionomic equilibrium points for a logistic growth model of two ecologically independent species, two competing species and Lotka - Volterra model of one prey and predator. Pradhan and Chaudhuri [11] developed a dynamic reaction model of two species fishery with tax as control instrument and obtained its optimal harvesting policy. Later, they investigated the harvesting of a schooling fish species [12]. Dubey et.al. [8] in their paper discussed the dynamics of fish population partially dependent on a logistically growing resource with functional response and the harvest term is assumed to be proportional to both stock level and applied effort. For a specific species, its structure according to the age, size or stage of its members, is also an interesting aspect of research. Here, particularly if we structure a population according to the stages of life, time delay is to be incorporated. Aiello and Freedman [2, 3] have worked together on a population, where the individual members of the population have a life history that takes them through two stages, immature and mature. Agarwal and Devi [1] discussed the stage structured prey predator model with maturation delay. Das and Sarkar [7] studied the effect of discrete time delays in the numerical response term and recycling term in autotroph - herbivore system. They derived the conditions for asymptotic stability in both absence and presence of delay. Sang and Chen [9] studied optimal harvesting and stability for stage - structured competitive population. Zhang et.al. [13] obtained optimal harvest policy for stage - structured species without time delay and with predation effect. Singh et.al. [10] studied the optimal harvesting policy in tritrophic ratio - dependent food chain model. From the above literature, it is clear that no attempt has been made to obtain optimal harvesting of structured population with time delay. In present paper, we are considering a single species which is structured according to two stages of life – immature and mature and it is assumed that mature population is subjected to dynamic harvesting. We have in mind mammalian population and some amphibious animals which exhibit these two stages. As an example, we depict the case of Chinese alligators which can be regarded as stage - structured species since mature is more than ten years old.

In present paper, we will study many aspects of analysis that are, the existence and uniqueness of equilibriums, their stability and optimal harvesting policy for the system. A critical value of delay is also obtained after which bifurcation occurs.

2 Mathematical Model

The system introduced by Aiello and Freedman [2] is considered as a model for single stage structured species.

$$\begin{aligned} \dot{x}_i(t) &= \alpha x_m(t) - \gamma x_i(t) - \alpha e^{-\gamma\tau} x_m(t-\tau), \\ \dot{x}_m(t) &= \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t), \\ x_i(t) &= \phi_1(t) \geq 0, x_m(t) = \phi_2(t) \geq 0, -\tau \leq t \leq 0. \end{aligned} \tag{2.1}$$

The immature and mature population densities, respectively, are $x_i(t)$ and $x_m(t)$. At $t > 0$, it is assumed that the immature population birth rate is proportional to the existing mature population with a proportionality constant of $\alpha > 0$, and its death rate is proportional to the existing immature population with a proportionality constant of $\gamma > 0$. For mature population, death rate is of logistic nature and proportional to the square of population with constant $\beta > 0$. Finally, we assume that the immature born at time $(t - \tau)$, would If $N(t)$ is a given population at time t, the number that survives from $(t - \tau)$ to (t) is,

$$N(t) = N(t-\tau)e^{-\gamma\tau} \tag{2.2}$$

If $\phi_2(t)$ is the provided initial mature population and $\phi_1(t)$ is the derived initial immature population, then we should have $\phi_1(t)$ for continuity of starting circumstances.

$$x_i(0) = \int_{-\tau}^{0} \alpha x_m(t) e^{\gamma t} dt. \tag{2.3}$$

Here, it has been assumed that $x_m(t)$ is continuous and non-negative. Then solutions of (2.1) exist and are unique $\forall t \geq 0$.
By considering $E(t)$ as the effort used to harvest mature class, the system (2.1) is changed to allow for optimal harvesting of the mature population. The following is the set of differential equations that governs the model:

$$\begin{aligned} \dot{x}_i(t) &= \alpha x_m(t) - \gamma x_i(t) - \alpha e^{-\gamma\tau} x_m(t-\tau), \\ \dot{x}_m(t) &= \alpha e^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2 - qEx_m, \\ \dot{E}(t) &= \alpha_0 E[(p-s)qx_m - c], \end{aligned} \tag{2.4}$$
$$x_i(0) > 0, x_m(0) > 0, E(0) \geq 0.$$

In this case, it is assumed that the mature population is subjected to dynamic harvesting in direct proportion to its concentration and applied effort, with a constant catchability coefficient q and a constant tax $s > 0$ imposed by the regulatory agency in order to keep the mature population at the desired level. The fixed price per unit of mature population is p, while the harvesting cost per unit effort exerted is c. Constant α_0 is called stiffness parameter measuring the strength of reaction of effort to the perceived rent.

3 BASIC RESULTS

3.1 Boundedness of Solutions

We shall consider only those solutions of system (2.4) which always lie in

$$R^3_+ = \{(x_i, x_m, E) : x_i(t) > 0, x_m(t) > 0, E(t) \geq 0\}, \forall t > 0.$$

Theorem 3.1. *All solutions of system (2.4) that are initiated in R^3_+ are uniformly bounded and enter into the region,*

$$\Omega \quad = \quad \{(x_i, x_m, E) \in R^3_+ : [\alpha_0(p-s)(x_i(t) + x_m(t)) + E(t)] \leq \frac{\alpha_0(p-s)(\alpha+\epsilon)^2}{4\epsilon\beta}, \epsilon = min(\alpha_0 c, \gamma)\}$$

Proof. Let us define,

$$\mu(t) = \alpha_0(p-s)(x_i(t) + x_m(t)) + E(t), for \in > 0,$$
$$\dot{\mu}(t) + \in \mu(t) \leq -\alpha_0(p-s)\{\beta x_m^2 - (\alpha + \in)x_m\} + E(-\alpha_0 c + \in) + x_i\alpha_0(p-s)(-\gamma + \in).$$

Choosing $\in = min(\alpha_0 c, \gamma)$,

$$\dot{\mu} + \in \mu \leq \quad -\alpha_0(p-s)(\sqrt{\beta}x_m - \frac{\alpha + \in}{2\sqrt{\beta}})^2 + \frac{\alpha_0(p-s)(\alpha + \in)^2}{4\beta},$$
$$\dot{\mu} + \in \mu \leq \quad \frac{\alpha_0(p-s)(\alpha + \in)^2}{4\beta}.$$

On integrating,

$$\mu(t) \leq \frac{\alpha_0(p-s)(\alpha + \in)^2}{4\beta \in} + Ae^{-\in t}$$

as $t \to \infty$, $\alpha_0(p-s)(x_i + x_m) + E \leq \dfrac{\alpha_0(p-s)(\alpha + \in)^2}{4\beta \in}$.

Hence, theorem is proved. □

3.2 Permanence and Positivity of Structured Population

Theorem 3.2. *Let $x_m(0) > 0, x_i(t) > 0$ on $-\tau \leq t \leq 0$. Then for system (2.4) $x_i(t)$ and $x_m(t)$ are strictly positive $t > 0$.*

Proof. Since $x_m(0) > 0$, if there is a t_0 such that $x_m(t_0) = 0$. Let $t_0 > 0$ be the first time such that $x_m(t) = 0$, then

$$\dot{x}_m(t) = \{\alpha e^{-\gamma\tau} if t_0 \leq 0$$
$$\alpha e^{-\gamma\tau} if t_0 > 0\}$$

So, $\dot{x}_m(t_0) > 0$. Hence, for sufficiently small $\eta > 0$, $\dot{x}_m(t_0 - \tau) > 0$. But by definition of t_0, $\dot{x}_m(t_0 - \tau) > 0$. This contradiction, proves that $x_m(t) > 0, \forall t > 0$.
Now, let us consider the equation

$$\dot{u}(t) \quad = \quad -\gamma u(t) - \alpha e^{-\gamma\tau} x_m(t-\tau), u(0) = x_i(0)$$

On integration,

$$u(t) = e^{-\gamma\tau}[x_i(0) - \int_0^t \alpha e^{-\gamma(\eta-\tau)} x_m(\eta-\tau)d\eta],$$

$$u(\tau) = e^{-\gamma\tau}[\int_{-\tau}^0 \alpha x_m(\xi)e^{\gamma\xi}d\xi - \int_0^\tau \alpha e^{-\gamma(\eta-\tau)} x_m(\eta-\tau)d\eta],$$

From (2.3) $u(\tau) = 0$ $u(t) > 0$ for $t \in [0, \tau)$.
On camparison of $\dot{u}(t)$ and $\dot{x}_i(t)$, we can see that $x_i(t) > u(t)$ on $0 < t \leq \tau$.
By induction, we can show that $x_i(t) > 0$ $\forall t > 0$.
Hence the theorem. □

4 Existence of Equilibrium Points and Their Stability

The system (2.4) has three non-negative equilibrium points $P_0(0,0,0)$,$P_1(\bar{x}_i, \bar{x_m}, 0)$ and $P^*(x_i^*, x_m^*, E^*)$. It is self-evident that $P_0(0,0,0)$ exists. $P_1(\bar{x}_i, \bar{x_m}, 0)$ is given by

$$\bar{x}_i = \frac{\alpha(1-e^{-\gamma\tau})\bar{x_m}}{\gamma}, \bar{x_m} = \frac{\alpha e^{-\gamma\tau}}{\beta}$$

$P^*(x_i^*, x_m^*, E^*)$ is given by equations,

$$x_m^* = \frac{c}{(p-s)q}, exist for p > s, \tag{4.1}$$

$$E^* = \frac{1}{q}[\alpha e^{-\gamma\tau} - \beta x_m^*], \tag{4.2}$$

$$x_i^* = \frac{\alpha(1-e^{-\gamma\tau})x_m^*}{\gamma}. \tag{4.3}$$

For the existence of P^*

$$\alpha e^{-\gamma\tau} - \beta x_m^* > 0, \implies s < p - \frac{\beta c e^{\gamma\tau}}{\alpha q} \tag{4.4}$$

The upper bound for the tax is given by Equation (4.4), and it lowers as the delay increases. It is also evident from (4.2) that increase in delay reduces the effort level.Even effort can be lost if

$$\alpha e^{-\gamma\tau} = \frac{\beta c}{(p-s)q}.$$

Now, We shall explore variational matrices corresponding to distinct equilibrium points for linear stability analysis.:

$$M_0 = \begin{bmatrix} -\gamma & \alpha(1-e^{(-\gamma+\lambda)\tau}) & 0 \\ 0 & e^{(-\gamma+\lambda)\tau} & 0 \\ 0 & 0 & -\alpha_0 c \end{bmatrix}$$

$$M_1 = \begin{bmatrix} -\gamma & \alpha(1-e^{(-\gamma+\lambda)\tau}) & 0 \\ 0 & e^{-\gamma\tau}(e^{-\gamma\tau}-2) & -q\alpha e^{-\gamma\tau} \\ 0 & 0 & \alpha_0[\dfrac{(p-s)q\alpha e^{-\gamma\tau}}{\beta} - c] \end{bmatrix}$$

$$M^* = \begin{bmatrix} -\gamma & \alpha(1-e^{(-\gamma+\lambda)\tau}) & 0 \\ 0 & e^{(-\gamma+\lambda)\tau} - 2\beta x_m^* - qE^* & -qx_m^* \\ 0 & \alpha_0 E^*(p-s)q & -\alpha_0[(p-s)qx_m^* - c] \end{bmatrix}$$

The variational matrices M_0, M_1, and M^* correspond to the equilibria P_0, P_1, and P^*, respectively. The following findings are achieved using these variational matrices:
(1) P_0 is a saddle point that is asymptotically stable in the direction $x_i - E$ but unstable in the direction x_m.
(2) If $\alpha e^{-\gamma\tau} < \dfrac{\beta c}{(p-s)q}$, P_1 is a locally asymptotically stable point; if $\alpha e^{-\gamma\tau} > \dfrac{\beta c}{(p-s)q}$, P_1 becomes unstable, and then P^* exists.

For the interior equilibrium point P^*, using $\alpha e^{-\gamma\tau} = \beta x_m^* + qE^*$, characteristic equation can be written as

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = e^{-\lambda\tau}(a_4\lambda^2 + a_5\lambda) \tag{4.5}$$

where

$$\begin{aligned} a_1 &= \gamma + 2\beta x_m^* + qE^* \\ a_2 &= \gamma(2\beta x_m^* + qE^*) + q^2\alpha_0 m^* E^*(p-s) \\ a_3 &= \gamma q^2 \alpha_0 x_m^* E^*(p-s) \\ a_4 &= \beta x_m^* + qE^* \\ a_5 &= \gamma(\beta X_m^* + qE^*) \end{aligned}$$

The above values show that $\lambda = 0$ is a root of (4.5) if $a_3 = 0$. Because $(a_1 - a_4) > 0$, $a_2 - a_5 > 0$, $a_3 > 0$, and $(a_1 - a_4)(a_2 - a_5) - a_3 > 0$, when $\tau = 0$, all requirements for the Routh-Hurwitz criterion are met. As a result, in the absence of delay, all roots of (4.5) will have negative real parts, and P^* is locally asymptotically stable.

5 CRITERION FOR STABILITY CHANGE

In this part, we will attempt to determine the critical delay value at which the change in stability occurs. Since $a_3 \neq 0$, the only way the system's stability may alter is if there exists one root of (4.5) such that $Re\lambda = 0$.Let $\lambda = iw$ be one of them. We obtain by keeping $\lambda = iw$ in equation (4.5) and equating the real and imaginary portions of the equation.

$$-a_1 w^2 + a_3 = -a_4 w^2 cosW\tau + a_5 w sinw\tau \tag{5.1}$$

$$-w^3 + a_2 w = a_4 w^2 sinw\tau + a_5 w cosW\tau \tag{5.2}$$

On squaring and adding above equations, we get

$$w^6 + w^4(a_1 - 2a_2 - a_4^2) + w^2(-2a_1 a_3 + a_2^2 - a_5^2) + a_3^2 = 0 \tag{5.3}$$

Letting $w^2 = u$, (5.3) can be written as

$$u^3 + u^2(a_1^2 - 2a_2 - a_4^2) + u(-2a_1 a_3 + a_2^2 - a_5^2) + a_3^2 = 0 \tag{5.4}$$

In order to exist real solutions of (5.3), there must be a positive solution of (5.4). Let $P*$ be locally asymptotically stable point with $\tau = 0$ and following inequalities hold

$$a_1^2 - 2a_2 - a_4^2 > 0 \tag{5.5}$$

$$a_2^2 - 2a_1a_3 - a_5^2 > 0 \tag{5.6}$$

$$(a_1^2 - 2a_2 - a_4^2)(a_2^2 - 2a_1a_3 - a_5^2) - a_3^2 > 0. \tag{5.7}$$

Then, according to the Routh-Hurwitz criterion, equation (5.3) has only negative roots, implying that there is no real solution of (5.3), implying that $\lambda = iw$ is not a solution of characteristic equation (4.3), and P^* stays stable $\forall\tau > 0$.
Again, we have a critical value of delay for which the system becomes unstable when we solve equations (5.1) and (5.2).

$$\tau_c = \frac{1}{w} sin^{-1}[\frac{(a_1w^2 - a_3)a_5 + (w^2 - a_2)a_4w^2}{a_5^2w + a_4^2w^3}] \tag{5.8}$$

This gives the least positive value of τ for which $Re\lambda = 0$ or stability change occur.
Remark. Since P^* loses its stability, by a parameter (τ) change with a pair of complex eigen values having its real parts transecting from negative to positive, a Hopf bifurcation will occur at $\tau = \tau_c$, leading to oscillatory solutions for $\tau > \tau_c$.

6 MAXIMUM SUSTAINABLE YIELD

The maximum sustainable yield (MSY) of any biological species is the highest rate at which it can be harvested with keeping the population constant, and any higher harvest rate would lead to population decline and finally extinction. The mature population's sustainable yield is given by

$$\begin{aligned} h = qE^*x_m^* &= [\alpha e^{-\gamma\tau} - \beta x_m^*]x_m^* \\ \frac{dh}{dx_m^*} &= 0 \Rightarrow x^* = \frac{\alpha e^{-\gamma\tau}}{2\beta} \\ \frac{d^2h}{dx_m^*} &= -2\beta < 0 \end{aligned}$$

Hence

$$h_{MSY} = \frac{\alpha^2 e^{-2\gamma\tau}}{4\beta}$$

It may be noted here that increase in delay decreases the value of MSY for mature population.

7 OPTIMAL HARVESTING POLICY

Now, we will examine the optimal harvesting policy that should be implemented by regulatory agencies in order to maximise net economic revenue to society. The societal net economic revenue $pi(x_i, x_m, E, s)$ = net revenue to the regulatory agency + net revenue to harvester
$= sqx_mE + [(p - s)qx_m - c]E = (pqx_m - c)E.$
Our goal is to address the optimization challenge,

$$max \int_0^\infty e^{-\delta t}(pqx_m - c)Edt \tag{7.1}$$

subject to state equations of (2.4) and to the control constraint

$$s_{min} \leq s \leq s_{max} \tag{7.2}$$

where δ is the instantaneous annual rate of discount. In order to solve (7.1) using Pontryagin's Maximum principle, associated Hamiltonian function is given as

$$\begin{aligned} H(x_i, x_m, E, s, t) &= e^{-\delta t}(pqx_m - c) + \lambda_1(t)[\alpha x_m - x_i - e^{-\gamma t} - x_m(t-\tau) \\ &+ \lambda_2(t)[e^{\gamma t} - x_m(t-\tau) - x_m^2 - qEx_m] + \lambda_3(t)\alpha_0 E[(p-s)qx_m \\ &- c] \end{aligned}$$

where λ_1, λ_2 and λ_3 are adjoint variables.
For 'H' to be maximum, we must have

$$\frac{\partial H}{\partial s} = 0 \implies \lambda_3(t) = 0 \tag{7.3}$$

As per maximum principle, we have

$$\frac{\partial \lambda_1}{dt} = -\frac{\partial H}{\partial x_i}, \frac{\partial \lambda_2}{dt} = -\frac{\partial H}{\partial x_m}, \frac{\partial \lambda_3}{dt} = -\frac{\partial H}{\partial E}$$

Above equations can be written as

$$\frac{d\lambda_1}{dt} = \lambda_1(t)\gamma \tag{7.4}$$

$$\frac{d\lambda_2}{dt} = -e^{\delta t}pqE - \lambda_1\alpha(1 - e^{-\gamma\tau}) - \lambda_2(t)[\alpha e^{-\gamma\tau} - 2\beta x_m - qE] \tag{7.5}$$

$$\frac{d\lambda_3}{dt} = -e^{-\delta t}(pqx_m - c) + \lambda_2(t)qx_m \tag{7.6}$$

From (7.3) and (7.6)

$$\lambda_2(t) = -e^{-\delta t}(p - \frac{c}{qx_m})E \tag{7.7}$$

Considering interior equilibrium point and on eliminating λ_1, λ_2 and λ_3 from above equations, we get equation in the form

$$b_1 x_m^2 + b_2 x_m + b_3 = 0 \tag{7.8}$$

where

$$b_1 = qp\beta(\delta + \gamma), b_2 = -\beta c(\delta + \gamma) + pq\delta^2 - (\delta + \gamma)pq^2, b_3 = \delta(\gamma - c\delta).$$

If $x_{m\delta}$ be the positive root of (7.6), then $x_{m\delta}$ will be the optimal equilibrium level of mature population. Then, the optimal equilibrium levels of immature species, effort and tax are given by

$$x_{i\delta} = \frac{\alpha(1 - e^{\gamma\tau})x_{m\delta}}{} \tag{7.9}$$

$$E_\delta = \frac{\alpha e^{-\gamma\tau} - \gamma\beta x_{m\delta}}{q} \tag{7.10}$$

$$s_\delta = p - \frac{c}{qx_{m\delta}} \tag{7.11}$$

8 Numerical Simulations of Model

In this part, we employed numerical simulation to demonstrate analytic findings. Consider the parametric values utilized in the analysis: $\alpha = 0.8$, $\gamma = 0.1$, $\beta = 0.06$, $c = 10$, $p = 25$ $\alpha_0 = 1$, $q = .1, \delta = 1$

$$I : \tau = 1, s < 16.711$$
$$II : \tau = 3, s < 14.8758$$
$$III : \tau = 5, s < 12.685$$

s	x_m	$\tau = 1$		$\tau = 2$		$\tau = 3$	
		x_i	E	x_i	E	x_i	E
0	4	3.0464	4.838	8.2944	3.526	12.592	2.42
3	4.5454	3.4618	4.511	9.4252	3.199	14.3091	2.093
6	5.2632	4.0084	4.08	10.9138	2.768	16.5684	1.668
9	6.25	4.76	3.488	12.96	2.176	19.675	1.07
12	7.1429	5.44	2.952	14.8115	1.64	22.4857	0.534

Effort vanishes i.e. $E = 0$ when

τ	x_m	s
1	12.0633	16.7104
3	9.8767	14.8752
5	8.0333	12.552

From above data table, it is clear how increase in delay effects the range of tax imposed and the value of applied effort. Increase in delay increases the concentration of immature species and decreases value of mature concentration level at which harvesting should be stopped. For the optimal equilibrium level of mature species, putting values of constants in (7.8), equation takes form:

$$0.03x_m^2 - 0.145x_m - 0.03 = 0 \tag{8.1}$$

Its positive root, i.e.$x_{m\delta} = 5.39$, $s_\delta = 6.4471$

The optimal value of effort will be determined by the value of delay.

τ	1	3	5
E_δ	4.004	2.692	1.586

9 Conclusion

We studied a stage-structured single species with two life phases, immature and mature, and dynamic harvesting effort for the mature population in this study. The period of time between birth and maturity is assumed to be constant We have demonstrated fundamental results in relation to the population's long-term viability and the system's limited solutions (2.4). We explored the presence and stability of equilibria, as well as the criterion for the sake of stability change and obtaining the best harvest policy for the system. Overall, the following conclusions may be drawn:

i. Increases in time latency τ have a negative impact on applied harvesting effort and the upper bound for tax. The longer the period between birth and adulthood, the lower the system's equilibrium level.

ii. The maximum sustainable production for a mature population is greatest when $\tau = 0$ and falls as the delay increases.

iii.If the price of the mature population rises faster than the cost of harvesting, the species population will settle to a lower equilibrium level.

The final numerical computation was performed for various values of hypothetical parameters. Our model can be improved for future research by taking into account the following factors:
i. The predator's effect on a stage-structured prey population growth and optimal harvesting policy.
ii. System (2.4) can be modified to account for the dependence of on population density.
iii. Time delay for the applied effort to harvest population.

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