Hamiltonian Soft Graphs

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Abstract

In 1999, D. Molodtsov initiated the novel concept of soft set theory. This is an approach for modeling vagueness and uncertainty. It is a classification of elements of the universe with respect to some given set of parameters. The concept of soft graph introduced by Rajesh K. Thumbakara and Bobin George is used to provide a parameterized point of view for graphs. Theory of soft graphs is a fast developing area in graph theory due to its capability to deal with the parameterization tool. In this paper, we introduce the concepts of Hamiltonian soft graph and closure of a soft graph. Also we investigate some properties of them.

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1 Introduction

In Mathematics, graph theory is the study of graphs which are mathematical structures used to model pairwise relations between objects. Graphs are one of the principal objects of study in Discrete Mathematics. The basic idea of graphs were first introduced in the 18th century by the Swiss mathematician Leonhard Euler. His work on the famous “Konigsberg bridge problem”is commonly quoted as origin of graph theory. The idea of soft sets was first given by D. Molodtsov [3] in 1999. This is a new mathematical tool to deal with the uncertainties. Many practical problem can be solved easily with the help of soft set theory rather than some well-known theories viz. fuzzy set theory, probability theory etc. since these theories have certain limitations. The problem with the fuzzy set is that it lacks parameterization tools. Many authors like P.K. Maji, A.R. Roy and R. Biswas [4], [5] have further studied the theory of soft sets and used the theory to solve some decision making problems. In 2014, Rajesh K. Thumbakara and Bobin George [6] introduced the concept of soft graph to provide a parameterized point of view for graphs. They also introduced the concepts of soft subgraph, soft tree etc. and some soft graph operations. In 2015, Muhammad Akram and Saira Nawas [7] modified the definition of soft graph. They [8] also defined certain types of soft graphs and also explained the concepts of soft bridge, soft cut vertex, soft cycle etc. J.D. Thenge, B.S. Reddy and R.S. Jain [9] contributed more to connected soft graph. Also they [10] studied about the radius, diameter and center of a soft graph and introduced the concept of degree in soft graph. In 2019, N. Sarala and K. Manju [11] introduced the concept of soft bi-partite graph and studied some properties. In 2020, J.D. Thenge, B.S. Reddy and R.S. Jain [12] discussed the concepts of adjacency matrix and incidence matrix of a soft graph. Domination over soft graphs is introduced by S. Venkatraman and R. Helen [13]. In this paper we introduce the concepts of Hamiltonian soft graph and closure of a soft graph. Also we investigate some properties of them.

2 Preliminaries

2.1 Graphs

For basic concepts of graph theory we refer [1] and [2]. A graph G consists of two finite sets: V, the vertex set of G which is a nonempty set and E, the edge set which is possibly empty. G can be represented by (V, E) or $(V(G), E(G))$. The degree of a vertex v denoted by $d(v)$ is the number of edges of G incident with v. Let H be a graph with vertex set $V(H)$ and edge set $E(H)$ and G be a graph with vertex set $V(G)$ and edge set $E(G)$. Then we say that H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A walk in a graph G is a finite sequence $W = v_0e_1v_1e_2v_2...v_{k-1}e_kv_k$ whose terms are alternately vertices and edges such that for $1 \leqslant i \leqslant k$, the edge e_i has ends v_{i-1} and v_i. We say this walk as a $v_0 - v_k$ walk. Here v_0 is called origin of the walk and v_k is called the terminus of the walk. A $u - v$ walk is called closed or open depending on whether $u = v$ or $u \neq v$. Trivial walk is one containing no edges. The number of edges in the walk is called the length of the walk. If the edges in the walk are distinct then the walk is called a trail. A non-trivial closed trail in a graph G is called a cycle if its origin and internal vertices are distinct. If the vertices of the walk are distinct then that walk W is called a path. A vertex u is said to be connected to a vertex v in a graph G if there is a path in G from u to v. A graph is said to be connected if every two of its vertices are connected. If $C(u)$ denote the set of all vertices in G that are connected to u then the subgraph of G induced by $C(u)$ is called the connected component containing u, or simply the component containing u. A Hamiltonian path in a graph G is a path which contains every vertex of G. A Hamiltonian cycle in a graph G is a cycle which contains every vertex of G. A graph is called Hamiltonian if it has a Hamiltonian cycle.

2.2 Soft Set

In 1999 D. Molodtsov [3] initiated the concept of soft sets. Let U be an initial universe set and let E be a set of parameters. A pair (F, E) is called a Soft Set (over U) if and only F is a mapping of E into the set of all subsets of the set U. That is, $F : E \to P(U)$.

2.3 Soft Graph

Rajesh K. Thumbakara and Bobin George [6] introduced the concept of soft graph as follows. Let $G = (V, E)$ be a simple graph and A be any nonempty set. Let R be an arbitrary relation between elements of A and elements of V. That is $R \subseteq A \times V$. A mapping $F : A \to P(V)$ can be defined as $F(x) = \{y \in V : xRy\}$. The pair (F, A) is a soft set over V. Then (F, A) is said to be a Soft Graph of G if the subgraph induced by $F(x)$ in G is a connected subgraph of G for all $x \in A$. Muhammad Akram and Saira Nawas [7] modified the definition of soft graph as follows. Let $G^* = (V, E)$ be a simple graph and A be any nonempty set. Let R be an arbitrary relation between elements of A and elements of V. That is $R \subseteq A \times V$. A mapping $F : A \to P(V)$ can be defined as $F(x) = \{y \in V : xRy\}$. Also define a mapping $K : A \to P(E)$ by $K(x) = \{uv \in E : \{u, v\} \subseteq F(x)\}$. The pair (F, A) is a soft set over V and the pair (K, A) is a soft set over E. Then the 4 -tuple $G = (G^*, F, K, A)$ is called a Soft Graph if it satisfies the following conditions:

1. $G^* = (V, E)$ is a simple graph,
2. A is a nonempty set of parameters,
3. (F, A) is a soft set over V,
4. (K, A) is a soft set over E,
5. $(F(a), K(a))$ is a subgraph of G^* for all $a \in A$.

If we represent $(F(x), K(x))$ by $H(x)$ then soft graph G is also given by $\{H(x) : x \in A\}$.

3 Hamiltonian Soft Graphs

Definition 3.1. *Let $G^* = (V, E)$ be a graph and $G = (G^*, F, K, A)$ be a soft graph of G^* which is also represented by $\{H(x) : x \in A\}$.Then $H(x)$ corresponding to a parameter x in A is called a part of the soft graph G.*

Definition 3.2. *Let $G = (G^*, F, K, A)$ be a soft graph and v be any vertex of the part $H(x)$ of G for some $x \in A$. Then part degree of the vertex v in $H(x)$ denoted by $d(v)[H(x)]$ is the degree of the vertex v in that part $H(x)$.*

Definition 3.3. *Let $G = (G^*, F, K, A)$ be a soft graph of G^* represented by $\{H(x) : x \in A\}$. Then any part $H(x)$ of G is called Hamiltonian if there exists a Hamiltonian cycle in that part $H(x)$ of G. That is, $H(x)$ contains a cycle that contains every vertex of $H(x)$.*

Definition 3.4. *A soft graph $G = (G^*, F, K, A)$ of a graph G^* is called Hamiltonian if all parts $H(x)$ of G are Hamiltonian.*

Consider a graph $G^* = (V, E)$ as shown in the following figure 1. Let $A = \{c, m\} \subseteq V$ be a

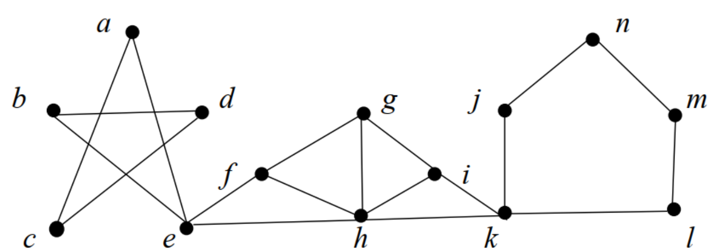

Fig. 1: Graph $G^* = (V, E)$

parameter set and (F, A) be a soft set over V with its approximate function $F : A \to P(V)$ defined by $F(x) = \{y \in V | xRy \Leftrightarrow d(x, y) \leq 2\}$ for all $x \in A$.
That is, $F(c) = \{a, b, c, d, e\}$ and $F(m) = \{j, k, l, m, n\}$.
Let (K, A) be a soft set over E with its approximate function $K : A \to P(E)$ defined by $K(x) = \{uv \in E | \{u, v\} \subseteq F(x)\}$ for all $x \in A$. That is,$K(c) = \{ac, ae, bd, be, cd\}$ and $K(m) = \{mn, nj, jk, kl, ml\}$. Thus $H(c) = (F(c), K(c))$ and $H(m) = (F(m), K(m))$ are subgraphs of G^* as shown in figure 2. Hence $G = \{H(c), H(m)\}$ is a soft graph of G^*. Here G has 2 parts $H(c)$ and

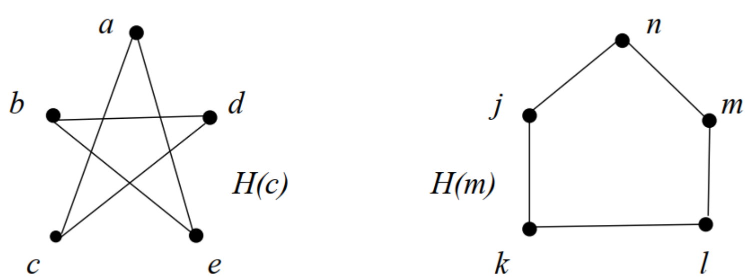

Fig. 2: Soft Graph $G = \{H(c), H(m)\}$

$H(m)$. In $H(c)$, $C_1 = aebdca$ is a Hamiltonian cycle and so $H(c)$ is a Hamiltonian part. Also in $H(m)$, $C_2 = nmlkjn$ is a Hamiltonian cycle and so $H(m)$ is a Hamiltonian part. That is, both the parts of G are Hamiltonian and hence G is a Hamiltonian soft graph.

Consider a graph $G^* = (V, E)$ as shown in the following figure 3. Let $A = \{b, g\} \subseteq V$ be a

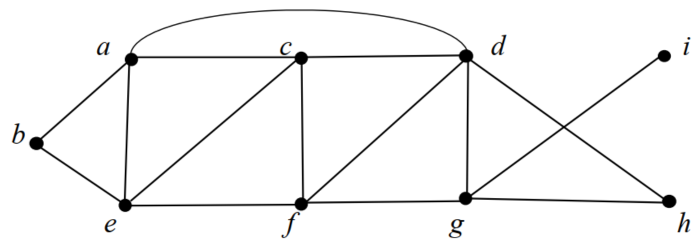

Fig. 3: Graph $G^* = (V, E)$

parameter set and (F, A) be a soft set with its approximate function $F : A \to P(V)$ defined by $F(x) = \{y \in V | xRy \Leftrightarrow d(x, y) \leq 1\}$ for all $x \in A$.
That is, $F(b) = \{a, b, e\}$ and $F(g) = \{d, f, g, h, i\}$.
Let (K, A) be a soft set over E with its approximate function $K : A \to P(E)$ defined by $K(x) = \{uv \in E | \{u, v\} \subseteq F(x)\}$ for all $x \in A$.
That is, $K(b) = \{ab, be, ae\}$ and $K(g) = \{df, dg, dh, fg, gh, gi\}$. Thus $H(b) = (F(b), K(b))$ and $H(g) = (F(g), K(g))$ are subgraphs of G^* as shown in figure 4.
Hence $G = \{H(b), H(g)\}$ is a soft graph of G^*. Here G has 2 parts $H(b)$ and $H(g)$. In $H(b)$,

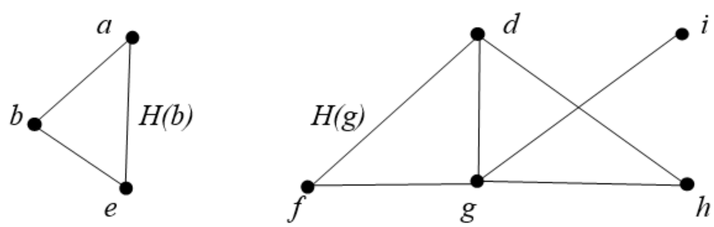

Fig. 4: Soft Graph $G = \{H(b), H(g)\}$

$C_1 = abea$ is a Hamiltonian cycle and so $H(b)$ is a Hamiltonian part. In $H(g)$, there is no Hamiltonian cycle and so $H(g)$ is not a Hamiltonian part. That is, both the parts of G are not Hamiltonian and hence G is not a Hamiltonian soft graph.

Definition 3.5. *Let $G = (G^*, F, K, A)$ be a soft graph of G^* represented by $\{H(x) : x \in A\}$. Then a part $H(x)$ of G is called a maximal non-Hamiltonian part if it is not Hamiltonian but the addition of any edge connecting any two non-adjacent vertices in that part $H(x)$ results a Hamiltonian part.*

Theorem 3.1. *Let $G^* = (V, E)$ be a graph and $G = (G^*, F, K, A)$ be a soft graph of G^* which is also represented by $\{H(x) : x \in A\}$. If $|F(x)| \geq 3$ and $d(v)[H(x)] \geq |F(x)|/2, \forall v \in F(x)$ and $\forall x \in A$ then G is a Hamiltonian soft graph where $|F(x)|$ denotes the number of vertices in the part $H(x)$ of G.*

Proof. We assume that the result is false. That is, at least one part $H(x) = (F(x), K(x))$ of the soft graph G is not a Hamiltonian part even if $|F(x)| \geq 3$ and $d(v)[H(x)] \geq |F(x)|/2, \forall v \in F(x)$. Add edges to this part connecting non-adjacent vertices of $H(x)$ until we get a maximal non-Hamiltonian part $J(x)$ having the same vertex set $F(x)$ as in $H(x)$. Here $J(x)$ has $|F(x)|$ vertices and $d(v)[J(x)] \geq |F(x)|/2, \forall v \in F(x)$. We use this maximal non-Hamiltonian part $J(x)$ to prove the result by obtaining a contradiction. $J(x)$ cannot be complete since it is not Hamiltonian. So there exist two non-adjacent vertices u and v in $J(x)$. Then $J(x) + uv$ will be Hamiltonian since $J(x)$ is maximal non-Hamiltonian. Hence there exist a Hamiltonian cycle C containing the edge uv and every vertex v of $J(x)$. So in $J(x)$ there is a Hamiltonian path $P = v_1 v_2 \cdots v_n$ where $n = |F(x)|$, $v_1 = u$ and $v_n = v$ which is obtained by removing the edge uv from C. If v_1 and v_i are adjacent in $J(x)$, $2 \leq i \leq n$, then v_{i-1} and v_n are not adjacent in $J(x)$. Otherwise $v_1 v_i v_{i+1} \cdots v_n v_{i-1} v_{i-2} \cdots v_1$ is a Hamiltonian cycle in $J(x)$ which is not possible. Hence corresponding to each vertex of $\{v_2, v_3 \cdots v_n\}$ which is adjacent to $v_1 = u$ there corresponds precisely one vertex of $\{v_1, v_2 \cdots v_{n-1}\}$ which is not adjacent to $v_n = v$. Thus $d(u)[J(x)] \leq (|F(x)| - 1) - d(v)[J(x)]$, so that $d(u)[J(x)] + d(v)[J(x)] \leq (|F(x)| - 1)$ which is a contradiction since $d(v)[J(x)] \geq |F(x)|/2, \forall v \in F(x)$. We get this contradiction due to our wrong assumption that the result is false. ◻

Theorem 3.2. *Let $G = (G^*, F, K, A)$ be a soft graph of G^* represented by $\{H(x) : x \in A\}$. Let u and v be any two non-adjacent vertices of a part $H(x)$ of G for some $x \in A$ satisfying the condition $d(u)[H(x)] + d(v)[H(x)] \geq |F(x)|$, where $|F(x)|$ denotes the number of vertices in the part $H(x)$ of G. Also let $H(x) + uv$ denote the graph obtained from the part $H(x)$ by joining u and v by an edge uv. Then $H(x)$ is Hamiltonian if and only if $H(x) + uv$ is Hamiltonian.*

Proof. Assume that $H(x)$ is a Hamiltonian part. Then $H(x)+uv$ must be Hamiltonian since $H(x)+uv$ is a supergraph of $H(x)$.
Conversely assume that $H(x) + uv$ is Hamiltonian. Then if $H(x)$ is not a Hamiltonian, as in the proof of theorem 3.1 we get the inequality $d(u)[H(x)] + d(v)[H(x)] \leq (|F(x)| - 1)$. But it is given that $d(u)[H(x)] + d(v)[H(x)] \geq |F(x)|$. Hence $H(x)$ is Hamiltonian. ◻

4 Closure of a Soft Graph

Definition 4.1. *Let $G = (G^*, F, K, A)$ be a soft graph of G^* represented by $\{H(x) : x \in A\}$. Let u_1 and v_1 be any two non-adjacent vertices of a part $H(x)$ of G for some $x \in A$ satisfying the condition $d(u_1)[H(x)] + d(v_1)[H(x)] \geq |F(x)|$, where $|F(x)|$ denotes the number of vertices in the part $H(x)$ of G. Then join u_1 and v_1 by an edge to form the supergraph $H_1(x)$. Then if there are two non-adjacent vertices u_2 and v_2 (if any) satisfying the condition $d(u_2)[H_1(x)] + d(v_2)[H_1(x)] \geq |F(x)|$, join u_2 and v_2 by an edge to form the supergraph $H_2(x)$. Continue this procedure until no such pair remains. The final supergraph thus obtained is called part closure and is denoted by $c[H(x)]$.*

Definition 4.2. *Let $G = (G^*, F, K, A)$ be a soft graph of G^* represented by $\{H(x) : x \in A\}$. Then closure of the soft graph G denoted by $c(G)$ is given by $\{c[H(x)] : x \in A\}$ where $c[H(x)]$ denotes the part closure of $H(x)$.*

Consider a graph $G^* = (V, E)$ as shown in the following figure 5. Let $A = \{f, m\} \subseteq V$ be a

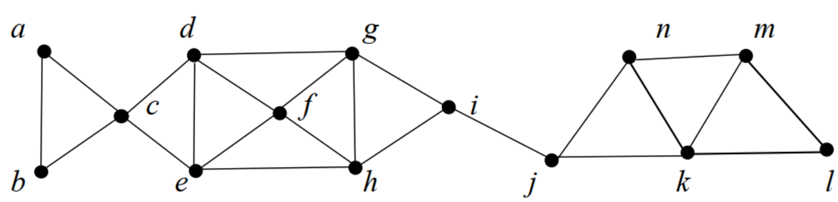

Fig. 5: Graph $G^* = (V, E)$

parameter set and (F, A) be a soft set over V with its approximate function $F : A \to P(V)$ defined by $F(x) = \{y \in V | xRy \Leftrightarrow d(x, y) \leq 2\}$ for all $x \in A$.
That is, $F(f) = \{c, d, e, f, g, h, i\}$ and $F(m) = \{n, m, j, k, l\}$.
Let (K, A) be a soft set over E with its approximate function $K : A \to P(E)$ defined by $K(x) = \{uv \in E | \{u, v\} \subseteq F(x)\}$ for all $x \in A$.

That is, $K(f) = \{cd, ce, de, dg, gh, eh, df, gf, ef, hf, gi, hi\}$ and
$K(m) = \{mn, nk, nj, kl, jk, lm, km\}$.
Thus $H(f) = (F(f), K(f))$ and $H(m) = (F(m), K(m))$ are subgraphs of G^* as shown in figure 6. Hence $G = \{H(f), H(m)\}$ is a soft graph of G^*. Then the closure $c(G) = \{c[H(f)], c[H(m)]\}$

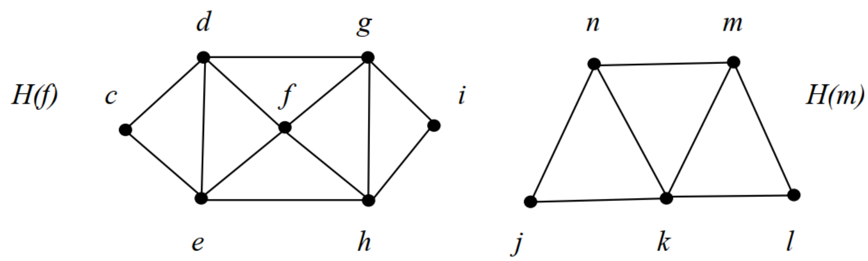

Fig. 6: Soft Graph $G = \{H(f), H(m)\}$

is give in figure 7 below.

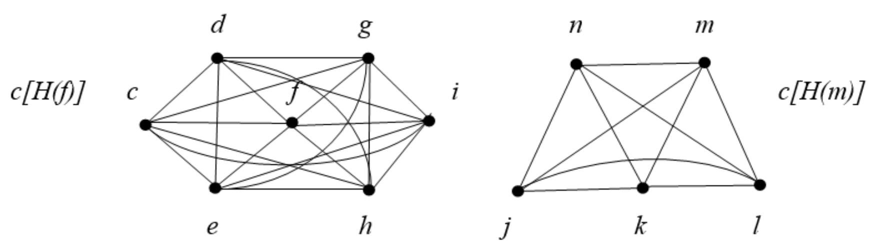

Fig. 7: $c(G) = \{c[H(f)], c[H(m)]\}$

Consider the Soft Graph $G = \{H(c), H(m)\}$ given in figure 2. Its closure $c(G) = \{c[H(c)], c[H(m)]\}$ is given in figure 8 below. This is also an example for a soft graph G having the property $c(G) = G$.

Theorem 4.1. *Let $G = (G^*, F, K, A)$ be a soft graph of G^* represented by $\{H(x) : x \in A\}$.Then G is a Hamiltonian soft graph if and only if all of its part closures are Hamiltonian.*

Proof. Assume that $G = \{H(x) : xA\}$ is a Hamiltonian soft graph. Then $H(x)$ will be a Hamiltonian part, $\forall x \in A$. So the part closure $c[H(x)]$ will be Hamiltonian $\forall x \in A$ since $c[H(x)]$ is a supergraph of $H(x)$.
Conversely suppose that all part closures $c[H(x)]$ of G are Hamiltonian. Consider a part closure $c[H(x)]$ for some $x \in A$. Let $H(x), H_1(x), H_2(x) \cdots H_m(x) = c[H(x)]$ be the sequence of graphs obtained by performing the closure operation on the part $H(x)$. Since the part closure $c[H(x)] = H_m(x)$ is obtained from $H_{m-1}(x)$ by setting $H_m(x) = H_{m-1}(x) + uv$, where u and v are two non-adjacent vertices of $H_{m-1}(x)$ satisfying the condition $d(u)[H_{m-1}(x)] + d(v)[H_{m-1}(x)] \geq |F(x)|$, $H_{m-1}(x)$ is Hamiltonian by theorem 3.2. Similarly $H_{m-2}(x)$,so $H_{m-3}(x) \cdots$ so $H_1(x)$ and so $H(x)$ must be Hamiltonian. Since we selected the part $H(x)$ arbitrarily, we can say that all parts of G are Hamiltonian part. Hence G is a Hamiltonian soft graph. $\square$

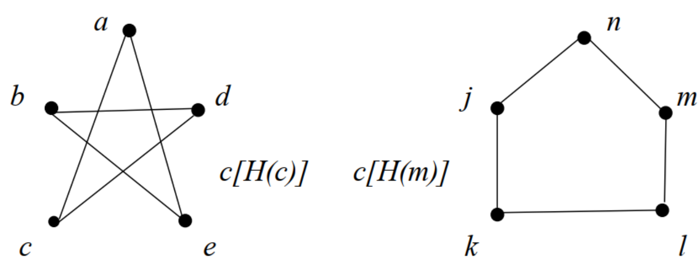

Fig. 8: $c(G) = \{c[H(c)], c[H(m)]\}$

5 Conclusion

Theory of soft graphs is a fast developing research area in graph theory due to its capability to deal with the parameterization tool. Soft graph was introduced by applying the concept of soft set in graph. By means of parameterization, soft graph produces a series of descriptions of a complicated relation described using a graph. In this paper, we introduced Hamiltonian soft graphs and closure of a soft graph and established some important properties of them.

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