Quasi Bi-Slant Lightlike Submanifolds of Indefinite Sasakian Manifolds

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Abstract

The purpose of this paper is to study the notion of quasi-bi-slant lightlike submanifolds of indefinite Sasakian manifolds. We also provide some non-trivial examples to signify that the structure introduced in this paper is valid. Integrability conditions of distributions $RadTM$ and D which are associated with definition of such submanifolds have been obtained. Furthermore, we also studied some necessary and sufficient conditions for foliations determined by the above distributions to be totally geodesic. Moreover, we characterized some results for totally umbilical and minimal quasi-bi-slant lightlike submanifolds of indefinite Sasakian manifolds.

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1 Introduction

The notion of the geometry of submanifolds is developed for ambient space with time. First it begins with the idea of geometry of the surface. We study manifolds with positive definite metric in Riemannian geometry. In [7] Duggal and Bejancu introduced the geometry of lightlike submanifolds of a semi-Riemannian manifold in 1996. As we know the properties of a manifold depend on the metric which defined on it. A lightlike submanifold M of a semi-Riemannian manifold $\bar{M}$ is a submanifold on which the induced metric is degenerate. First, in 1990, B. Y. Chen introduced the concept of slant immersions in complex geometry as a natural generalization of holomorphic and totally real submanifolds ([5], [6]). Further, the notion of semi-slant submanifolds of Kaehler manifolds was introduced by N. Papaghuic in 1994 ([13]). This motivates many geometeres to study such submanifolds. In ([16], [17]), Sahin studied the geometry of slant and screen-slant lightlike submanifolds of indefinite Hermitian manifolds. The theory of slant, CR lightlike submanifolds and SCR lightlike submanifolds of indefinite Kaehler manifolds has been studied in ([7], [8]). On the other hand, Chen et al. [21] introduced bi-slant submanifolds in Kaehler manifolds. In ([3], [4]) the geometry of slant and semi-slant submanifolds of Sasakian manifolds was studied by Cabrerizo and his authors. They also given several examples of such submanifolds.

This paper is organized as follows: In section 2, we review the basic definition and some formulas for lightlike submanifold and almost contact metric manifolds. In section 3, we introduce the notion of quasi-bi-slant lightlike submanifolds of an indefinite Sasakian manifold with some examples. Section 4 deals with the study of geometry of foliations determined by distributions on quasi-bi-slant lightlike submanifolds of indefinite Sasakian manifolds. And finally, we discuss the minimal quasi-bi-slant lightlike submanifolds and totally umbilical quasi-bi-slant lightlike submanifolds in section 5 and 6 respectively along with some non-trivial examples.

2 Preliminaries

The notation and formulas used in this paper are followed by [7]. A lightlike submanifold (M^m, g) immersed in a semi-Riemannian manifold $(\bar{M}^{m+n}, \bar{g})$ is a submanifold in which induced metric g from $\bar{g}$ is degenerate and the rank of radical distribution $Rad(TM)$ is r, where $1 \leq r \leq m$. Assume that $S(TM)$ be a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, which also called screen distribution, i.e.

$$TM = Rad(TM) \oplus_{orth} S(TM). \tag{2.1}$$

Now suppose $S(TM^{\perp})$ be a semi-Riemannian complementary vector bundle of $Rad(TM)$ in $TM^{\perp}$, called screen transversal vector bundle. Since for any local basis $\{\xi_i\}$ of $Rad(TM)$ there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\bar{g}(\xi_i, N_j) = \delta_{ij}$ and $\bar{g}(N_i, N_j) = 0$, it follows that there exists a lightlike transversal vector bundle $ltr(TM)$ locally spanned by $\{N_i\}$. Let

$$tr(TM) = ltr(TM) \oplus_{orth} S(TM^{\perp}). \tag{2.2}$$

Here, $tr(TM)$ is a complementary (but not orthogonal) vector bundle to TM in $T\bar{M}|_M$, i.e.

$$T\bar{M}|_M = TM \oplus tr(TM) \tag{2.3}$$

and therefore

$$T\bar{M}|_M = S(TM) \oplus_{orth} [Rad(TM) \oplus ltr(TM)] \oplus_{orth} S(TM^{\perp}). \tag{2.4}$$

Although $S(TM)$ is not unique but it is canonically isomorphic to the factor vector bundle $TM/RadTM$[11].

Following result is important to this paper.

Proposition 1.([7]) The lightlike second fundamental forms of a lightlike submanifold M do not depend on $S(TM)$, $S(TM^{\perp})$ and $ltr(TM)$.

Now these are the possible four subcases for a lightlike submanifold:
Case 1: r-lightlike if $r \leq min(m, n)$,
Case 2: co-isotropic if $r = n \leq m,\ S(TM^{\perp}) = \{0\}$,
Case 3: isotropic if $r = m \leq n,\ S(TM) = \{0\}$,
Case 4: totally lightlike if $r = m = n,\ S(TM) = S(TM^{\perp}) = \{0\}$.

The Gauss and Weingarten formulae are given as

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V. \tag{2.6}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_V X\}$ belong to $\Gamma(TM)$ and $\{h(X, Y), \nabla^t_X V\}$ belong to $\Gamma(tr(TM))$. Here, ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$ respectively. The second fundamental form h is a symmetric $F(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(tr(TM))$ and the shape operator A_V is a linear endomorphism of $\Gamma(TM)$.

From (2.5) and (2.6) we have

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \forall\ X, Y \in \Gamma(TM), \tag{2.7}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \quad \forall\ N \in \Gamma(ltr(TM)), \tag{2.8}$$

$$\bar{\nabla}_X W = -A_W X + D^l(X, W) + \nabla^s_X W, \quad \forall\ W \in \Gamma(S(TM^{\perp})), \tag{2.9}$$

where $h^l(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $D^l(X, W) = L(\nabla^t_X W)$, $D^s(X, N) = S(\nabla^t_X N)$, L and S are the projection morphisms of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$ respectively. Here h^l

is a $\Gamma(ltr(TM))$-valued which is called lightlike second fundamental form and h^s is a $\Gamma(S(TM^{\perp}))$-valued, called screen second fundamental form of M. On the other hand, ∇^l and ∇^s are linear connections on $ltr(TM)$ and $S(TM^{\perp})$ called the lightlike connection and screen transversal connection on M respectively.

Now by using (2.5), (2.7)-(2.9) and metric connection $\bar{\nabla}$, we obtain

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y, D^l(X,W)) = g(A_W X), \tag{2.10}$$

$$\bar{g}(D^s(X,N),W) = g(N, A_W X). \tag{2.11}$$

Suppose $\bar{P}$ is the projection of TM on $S(TM)$. Then from the decomposition of the tangent bundle of a lightlike submanifold, we have

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y), \quad \forall\, X, Y \in \Gamma(TM), \tag{2.12}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \quad \xi \in \Gamma(Rad(TM)), \tag{2.13}$$

where $\{\nabla^*_X \bar{P}Y, -A^*_\xi X\}$ and $\{h^*(X,\bar{P}Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$ respectively. It follows that ∇^* and ∇^{*t} are linear connections on $S(TM)$ and $Rad(TM)$ respectively. On the other hand, h^* and A^* are called the second fundamental forms of distributions $S(TM)$ and $Rad(TM)$ respectively, which are $\Gamma(Rad(TM))$-valued and $\Gamma(S(TM))$-valued $F(M)$-bilinear forms on $\Gamma(TM) \times \Gamma(S(TM))$ and $\Gamma(Rad(TM)) \times \Gamma(TM)$.

Now by using the above equations, we obtain

$$\bar{g}(h^l(X,\bar{P}Y),\xi) = g(A^*_\xi X, \bar{P}Y), \tag{2.14}$$

$$\bar{g}(h^*(X,\bar{P}Y),N) = g(A_N X, \bar{P}Y), \tag{2.15}$$

$$\bar{g}(h^l(X,\xi),\xi) = 0, \ A^*_\xi \xi = 0. \tag{2.16}$$

Here, it is important to note that the induced connection ∇ on M is not a metric connection in general. Since $\bar{\nabla}$ is a metric connection, by using (2.7) we get

$$(\nabla_X g)(Y,Z) = \bar{g}(h^l(X,Y),Z) + \bar{g}(h^l(X,Z),Y). \tag{2.17}$$

An odd dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an ϵ-almost contact metric manifold if there exists a $(1,1)$ tensor field ϕ, a vector field V called characteristic vector field and a 1-form η, satisfying

$$\phi^2 X = -X + \eta(X)V, \ \eta(V) = \epsilon, \ \eta o \phi = 0, \ \phi V = 0, \tag{2.18}$$

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X,Y) - \epsilon \eta(X)\eta(Y), \tag{2.19}$$

for all $X, Y \in \Gamma(T\bar{M})$, where $\epsilon = 1$ or -1. It follows that

$$\bar{g}(V,V) = \epsilon, \tag{2.20}$$

$$\bar{g}(X,V) = \eta(X), \tag{2.21}$$

(2.22) $$\bar{g}(X, \phi Y) = \breve{}\bar{g}(\phi X, Y), \quad \forall X, Y \in \Gamma(T\bar{M}).$$

Then $(\phi, V, \eta, \bar{g})$ is called an ϵ-almost contact metric structure on $\bar{M}$. An ϵ-almost contact metric structure $(\phi, V, \eta, \bar{g})$ is called an indefinite Sasakian structure if and only if

(2.23) $$(\bar{\nabla}_X \phi)Y = \bar{g}(X, Y)V - \epsilon\eta(Y)X,$$

for all $X, Y \in \Gamma(T\bar{M})$, where $\bar{\nabla}$ is Levi-Civita connection with respect to $\bar{g}$.

An indefinite Sasakian manifold is a semi-Riemannian manifold endowed with an indefinite Sasakian structure. From (2.23), for any $X \in \Gamma(T\bar{M})$, we get

(2.24) $$\bar{\nabla}_X V = \breve{}\phi X.$$

Suppose $(\bar{M}, \bar{g}, \phi, V, \eta)$ be an ϵ-almost contact metric manifold. If $\epsilon = 1$, then $\bar{M}$ is said to be a spacelike ϵ-almost contact metric manifold and if $\epsilon = -1$, then $\bar{M}$ is called a timelike ϵ-almost contact metric manifold. In this paper, we consider indefinite Sasakian manifolds with spacelike characteristic vector field V.

3 Quasi-bi-slant Lightlike Submanifolds

The notion of quasi-bi-slant lightlike submanifolds of indefinite Sasakian manifolds is introduced in this section. At first, we state the following lemma which was proved by Sahin[16]. We shall use this lemma in defining the notion of quasi-bi-slant lightlike submanifolds of indefinite Sasakian manifolds.

Lemma 1.[17] Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$. Suppose that there exists a screen distribution $S(TM)$ such that $\phi Rad(TM) \subset S(TM)$ and $\phi ltr(TM) \subset S(TM)$. Then $\phi Rad(TM) \cap \phi ltr(TM) = \{0\}$ and any complementry distribution to $\phi Rad(TM) \oplus \phi ltr(TM)$ in $S(TM)$ is Riemannian.

DEFINITION 1. Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$ such that $2q < dim(M)$. Then we say that M is a quasi-bi-slant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:
(i) $\phi Rad(TM)$ is a distribution on M such that $Rad(TM) \cap \phi Rad(TM) = \{0\}$,
(ii) there exist non-degenerate orthogonal distributions D, D_1 and D_2 on M such that

$$S(TM) = (\phi Rad(TM) \oplus \phi ltr(TM)) \oplus_{orth} D \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\},$$

(iii) the distribution D is an invariant distribution, i.e. $\phi D = D$,
(iv) the distribution D_1 is slant with angle θ_1, i.e. for each $x \in M$ and each non-zero vector $X \in (D_1)_x$, the angle θ_1 between ϕX and the vector subspace $(D_1)_x$ is a non-zero constant, which is independent of the choice of $x \in M$ and $X \in (D_1)_x$,
(v) the distribution D_2 is slant with angle θ_2, i.e. for each $x \in M$ and each non-zero vector $X \in (D_2)_x$, the angle θ_2 between ϕX and the vector subspace $(D_2)_x$ is a non-zero constant, which is independent of the choice of $x \in M$ and $X \in (D_2)_x$.

These constant angles θ_1 and θ_2 are called the slant angles of distributions D_1 and D_2 respectively. A quasi-bi-slant lightlike submanifold is said to be proper if $D_1 \neq \{0\}$, $D_2 \neq \{0\}$ and $\theta_1 \neq \pi/2$, $\theta_2 \neq \pi/2$.

From the above definition, we have the following decomposition:

$$TM = Rad(TM) \oplus_{orth} (\phi Rad(TM) \oplus \phi ltr(TM)) \oplus_{orth} D \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}.$$

We observe that the above class of submanifolds includes slant, semi-slant, bi-slant lightlike submanifolds of indefinite Sasakian manifolds as its particular cases.

Let $(\mathbb{R}^{2m+1}_{2q}, \bar{g}, \phi, \eta, V)$ denote the manifold $\mathbb{R}^{2m+1}_{2q}$ with its usual Sasakian structure given by

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{m} y^i \partial x^i), \ \ V = 2\partial z,$$
$$\bar{g} = \eta \otimes \eta + \frac{1}{4}(-\sum_{i=1}^{q} dx^i \otimes dx^i + dy^i \otimes dy^i + \sum_{i=q+1}^{m} dx^i \otimes dx^i + dy^i \otimes dy^i),$$
$$\phi(\sum_{i=1}^{m}(X_i dx_i + Y_i dy_i) + Z\partial z) = \sum_{i=1}^{m}(Y_i \partial x_i - X_i \partial y_i) + \sum_{i=1}^{m} Y_i y^i \partial z,$$

where (x^i, y^i, z) are the cartesian co-ordinates on $\mathbb{R}^{2m+1}_{2q}$. Now, we construct some examples of quasi-bi-slant lightlike submanifolds of an indefinite Sasakian manifold.

EXAMPLE 1. Let $(\mathbb{R}^{15}_2, \bar{g}, \phi, \eta, V)$ be an indefinite Sasakian manifold, where $\bar{g}$ is of signature $(-, +, +, +, +, +, +, -, +, +, +, +, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial y_7, \partial z\}$.

Suppose M is a submanifold of $\mathbb{R}^{15}_2$ given by $-x^1 = y^2 = u_1$, $x^2 = u_2$, $y^1 = u_3$, $x^3 = u_4 \cos\beta$, $y^3 = -u_5 \cos\beta$, $x^4 = u_5 \sin\beta$, $y^4 = u_4 \sin\beta$, $x^5 = u_6 \sin u_7$, $y^5 = u_6 \cos u_7$, $x^6 = k_1 \sin u_6$, $y^6 = k_1 \cos u_6$, $x^7 = u_8$, $y^7 = u_9$, $x^8 = k_2 \cos u_9$, $y^8 = k_2 \sin u_9$, $z = u_{10}$ where k_1 and k_2 are constants.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}\}$, where

$$Z_1 = 2(-\partial x_1 + \partial y_2 - y^1 \partial z),$$
$$Z_2 = 2(\partial x_2 + y^2 \partial z), \quad Z_3 = 2(\partial y_1),$$
$$Z_4 = 2(\cos\beta \partial x_3 + \sin\beta \partial y_4 + y^3 \cos\beta \partial z),$$
$$Z_5 = 2(\sin\beta \partial x_4 - \cos\beta \partial y_3 + y^4 \sin\beta \partial z),$$
$$Z_6 = 2(\sin u_7 \partial x_5 + \cos u_7 \partial y_5 + k_1 \cos u_6 \partial x_6 - k_1 \sin u_6 \partial y_6 + y^5 \sin u_7 \partial z + y^6 k_1 \cos u_6 \partial z),$$
$$Z_7 = 2(u_6 \cos u_7 \partial x_5 - u_6 \sin u_7 \partial y_5 + y^5 u_6 \cos u_7 \partial z),$$
$$Z_8 = 2(\partial x_7 + y^7 \partial z), \quad Z_9 = 2(\partial y_7 - k_2 \sin u_9 \partial x_8 + k_2 \cos u_9 \partial y_8 + y^8 k_2 \sin u_9 \partial z)$$
$$Z_{10} = 2\partial z = V.$$

Hence $Rad(TM) = span\{Z_1\}$ and $S(TM) = span\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, V\}$. Now $ltr(TM)$ is spanned by $N = \partial x_1 + \partial y_2 + y^1 \partial z$ and $S(TM^{\perp})$ is spanned by

$$W_1 = 2(\sin\beta \partial x_3 - \cos\beta \partial y_4 + y^3 \sin\beta \partial z),$$
$$W_2 = 2(\cos\beta \partial x_4 + \sin\beta \partial y_3 + y^4 \cos\beta \partial z),$$
$$W_3 = 2(k_1^2 \sin u_7 \partial x_5 + k_1^2 \cos u_7 \partial y_5 - k_1 \cos u_6 \partial x_6 + k_1 \sin u_6 \partial y_6 + y^5 k_1^2 \sin u_7 \partial z - y^6 k_1 \cos u_6 \partial z),$$
$$W_4 = 2(u_6 \sin u_6 \partial x_6 + u_6 \cos u_6 \partial y_6 + y^6 u_6 \sin u_6 \partial z).$$

It follows that $\phi Z_1 = Z_2 + Z_3$ and $\phi N = \frac{1}{2}(Z_2 - Z_3)$, which implies that $\phi Rad(TM)$ and $\phi ltr(TM)$ are distributions on M. Further, we can see that $D = span\{Z_4, Z_5\}$ such that $\phi Z_4 = Z_5$, $\phi Z_5 = -Z_4$, which implies that D is invariant with respect to ϕ. Also $D_1 = span\{Z_6, Z_7\}$ and $D_2 = span\{Z_8, Z_9\}$ are slant distributions with slant angles $\theta_1 = \cos^{-1}(1/\sqrt{1 + k_1^2})$ and $\theta_2 = \cos^{-1}(1/\sqrt{1 + k_2^2})$ respectively. Hence M is a quasi-bi-slant 2-lightlike submanifold of $\mathbb{R}^{15}_2$.

EXAMPLE 2. Let $(\mathbb{R}^{15}_2, \bar{g}, \phi, \eta, V)$ be an indefinite Sasakian manifold, where $\bar{g}$ is of signature $(-, +, +, +, +, +, +, -, +, +, +, +, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial y_7, \partial z\}$.

Suppose M is a submanifold of $\mathbb{R}^{15}_2$ given by $x^1 = y^2 = u_1$, $x^2 = u_2$, $y^1 = u_3$, $x^3 = u_4 \sin\beta$, $y^3 = -u_5 \sin\beta$, $x^4 = u_5 \cos\beta$, $y^4 = u_4 \cos\beta$, $x^5 = u_6 \cos u_7$, $y^5 = u_6 \sin u_7$, $x^6 = k_1 \cos u_6$, $y^6 = k_1 \sin u_6$, $x^7 = u_8$, $y^7 = u_9$, $x^8 = k_2 \sin u_9$, $y^8 = k_2 \cos u_9$, $z = u_{10}$, where k_1 and k_2 are constants.

The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, V\}$, where

$$
\begin{aligned}
Z_1 &= 2(\partial x_1 + \partial y_2 + y^1 \partial z),\\
Z_2 &= 2(\partial x_2 + y^2 \partial z), \quad Z_3 = 2(\partial y_1),\\
Z_4 &= 2(\sin\beta\partial x_3 + \cos\beta\partial y_4 + y^3 \sin\beta\partial z),\\
Z_5 &= 2(\cos\beta\partial x_4 - \sin\beta\partial y_3 + y^4 \cos\beta\partial z),\\
Z_6 &= 2(\cos u_7\partial x_5 + \sin u_7\partial y_5 - k_1 \sin u_6\partial x_6 + k_1 \cos u_6\partial y_6 + y^5 \cos u_7\partial z - y^6 k_1 \sin u_6\partial z),\\
Z_7 &= 2(-u_6 \sin u_7\partial x_5 + u_6 \cos u_7\partial y_5 + y^5 - u_6 \sin u_7\partial z),\\
Z_8 &= 2(\partial x_7 + y^7 \partial z), \quad Z_9 = 2(\partial y_7 + k_2 \cos u_9\partial x_8 - k_2 \sin u_9\partial y_8 + y^8 k_2 \cos u_9\partial z),\\
Z_{10} &= 2\partial z = V.
\end{aligned}
$$

Hence $Rad(TM) = span\{Z_1\}$ and $S(TM) = span\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, V\}$. Now $ltr(TM)$ is spanned by $N = -\partial x_1 + \partial y_2 - y^1 \partial z$ and $S(TM^\perp)$ is spanned by

$$
\begin{aligned}
W_1 &= 2(\cos\beta\partial x_3 - \sin\beta\partial y_4 + y^3 \partial z),\\
W_2 &= 2(\sin\beta\partial x_4 + \cos\beta\partial y_3 + y^4 \sin\beta\partial z),\\
W_3 &= 2(k_1^2 \sin u_9\partial x_8 + k_1^2 \cos u_9\partial y_8 + y^8 k_1^2 \sin u_9\partial z),\\
W_4 &= 2(k_2^2\partial y_7 - k_2 \cos u_9\partial x_8 + k_2 \sin u_9\partial y_8 - y^8 k_2 \cos u_9\partial z).
\end{aligned}
$$

It follows that $\phi Z_1 = Z_2 - Z_3$ and $\phi N = \frac{1}{2}(Z_2 + Z_3)$, which implies that $\phi Rad(TM)$ and $\phi ltr(TM)$ are distributions on M. Further, we can see that $D = span\{Z_4, Z_5\}$ such that $\phi Z_4 = Z_5$, $\phi Z_5 = -Z_4$, which implies that D is invariant with respect to ϕ. Also $D_1 = span\{Z_6, Z_7\}$ and $D_2 = span\{Z_8, Z_9\}$ are slant distributions with slant angles $\theta_1 = \cos^{-1}(1/\sqrt{1 + k_1^2})$ and $\theta_2 = \cos^{-1}(1/\sqrt{1 + k_2^2})$ respectively. Hence M is a quasi-bi-slant 2-lightlike submanifold of $\mathbb{R}_2^{15}$.

Now, for any vector field X tangent to M, we put

$$\phi X = PX + FX \tag{3.1}$$

where PX and FX are the tangential and transversal parts of ϕX respectively. We denote the projections on $Rad(TM)$, $\phi Rad(TM)$, $\phi ltr(TM)$, D, D_1 and D_2 in TM by P_1, P_2, P_3, P_4, P_5 and P_6 respectively. Similarly, we denote the projections of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$ by Q and R respectively.

Thus, for any $X \in \Gamma(TM)$, we get

$$X = P_1X + P_2X + P_3X + P_4X + P_5X + P_6X + \eta(X)V. \tag{3.2}$$

Now applying ϕ to (3.2), we have

$$\phi X = \phi P_1X + \phi P_2X + \phi P_3X + \phi P_4X + \phi P_5X + \phi P_6X, \tag{3.3}$$

which gives

$$\phi X = \phi P_1X + \phi P_2X + \phi P_3X + \phi P_4X + fP_5X + FP_5X + fP_6X + FP_6X, \tag{3.4}$$

where fP_5X and FP_5X (resp. fP_6X and FP_6X) denotes the tangential and transversal components of ϕP_5X (resp. ϕP_6X). Thus we get $\phi P_1X \in \Gamma(\phi Rad(TM))$, $\phi P_2X \in \Gamma(Rad(TM))$, $\phi P_3X \in \Gamma(ltr(TM))$, $\phi P_4X \in \Gamma(D)$, $fP_5X \in \Gamma(D_1)$, $fP_6X \in \Gamma(D_2)$ and $FP_5X, FP_6X \in \Gamma(S(TM^\perp))$.

Also, for any $W \in \Gamma(tr(TM))$, we have

$$W = QW + RW. \tag{3.5}$$

Applying ϕ to (3.5), we obtain

$$\phi W = \phi QW + \phi RW, \tag{3.6}$$

which gives

$$\phi W = \phi QW + BR_1W + CR_1W + BR_2W + CR_2W, \tag{3.7}$$

where BR_1W and CR_1W (resp. BR_2W and CR_2W) denotes the tangential and transversal) component of ϕR_1W (resp. ϕR_2W). Thus we get $\phi QW \in \Gamma(\phi ltr(TM))$, $BR_1W \in \Gamma(D_1)$, $CR_1W \in \Gamma(S(TM^{\perp}))$, $BR_2W \in \Gamma(D_2)$ and $CR_2W \in \Gamma(S(TM^{\perp}))$.

Now, by using (2.20), (3.4), (3.7) and (2.7)-(2.9) and identifying the components on $Rad(TM)$, $\phi Rad(TM)$, $\phi ltr(TM)$, D, D_1, D_2, $ltr(TM)$, $S(TM^{\perp})$ and $\{V\}$, we obtain

$$\begin{aligned} &P_1(\nabla_X\phi P_1Y) + P_1(\nabla_X\phi P_2Y) + P_1(\nabla_X\phi P_4Y) + P_1(\nabla_X fP_5Y) + P_1(\nabla_X fP_6Y) \\ &\quad = P_1(A_{FP_5Y}X) + P_1(A_{FP_6Y}X) + P_1(A_{\phi P_3Y}X) + \phi P_2\nabla_XY - \eta(Y)P_1X, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &P_2(\nabla_X\phi P_1Y) + P_2(\nabla_X\phi P_2Y) + P_2(\nabla_X\phi P_4Y) + P_2(\nabla_X fP_5Y) + P_2(\nabla_X fP_6Y) \\ &\quad = P_2(A_{FP_5Y}X) + P_2(A_{FP_6Y}X) + P_2(A_{\phi P_3Y}X) + \phi P_1\nabla_XY - \eta(Y)P_2X, \end{aligned} \tag{3.9}$$

$$\begin{aligned} &P_3(\nabla_X\phi P_1Y) + P_3(\nabla_X\phi P_2Y) + P_3(\nabla_X\phi P_4Y) + P_3(\nabla_X fP_5Y) + P_3(\nabla_X fP_6Y) \\ &\quad = P_3(A_{FP_5Y}X) + P_3(A_{FP_6Y}X) + P_3(A_{\phi P_3Y}X) + \phi h^l(X,Y) - \eta(Y)P_3X, \end{aligned} \tag{3.10}$$

$$\begin{aligned} &P_4(\nabla_X\phi P_1Y) + P_4(\nabla_X\phi P_2Y) + P_4(\nabla_X\phi P_4Y) + P_4(\nabla_X fP_5Y) + P_4(\nabla_X fP_6Y) \\ &\quad = P_4(A_{FP_5Y}X) + P_4(A_{FP_6Y}X) + P_4(A_{\phi P_3Y}X) + \phi P_4\nabla_XY - \eta(Y)P_4X, \end{aligned} \tag{3.11}$$

$$\begin{aligned} &P_5(\nabla_X\phi P_1Y) + P_5(\nabla_X\phi P_2Y) + P_5(\nabla_X\phi P_4Y) + P_5(\nabla_X fP_5Y) + P_5(\nabla_X fP_6Y) \\ &\quad = P_5(A_{FP_5Y}X) + P_5(A_{FP_6Y}X) + P_5(A_{\phi P_3Y}X) + fP_5\nabla_XY + Bh^s(X,Y) - \eta(Y)P_5X, \end{aligned} \tag{3.12}$$

$$\begin{aligned} &P_6(\nabla_X\phi P_1Y) + P_6(\nabla_X\phi P_2Y) + P_6(\nabla_X\phi P_4Y) + P_6(\nabla_X fP_5Y) + P_6(\nabla_X fP_6Y) \\ &\quad = P_6(A_{FP_5Y}X) + P_6(A_{FP_6Y}X) + P_6(A_{\phi P_3Y}X) + fP_6\nabla_XY + Bh^s(X,Y) - \eta(Y)P_6X, \end{aligned} \tag{3.13}$$

$$\begin{aligned} &h^l(X,\phi P_1Y) + h^l(X,\phi P_2Y) + h^l(X,\phi P_4Y) + h^l(X,fP_5Y) + h^l(X,fP_6Y) \\ &\quad = \phi P_3\nabla_XY - \nabla^l_X\phi P_3Y - D^l(X,FP_5Y) - D^l(X,FP_6Y), \end{aligned} \tag{3.14}$$

$$\begin{aligned} &h^s(X,\phi P_1Y) + h^s(X,\phi P_2Y) + h^s(X,\phi P_4Y) + h^s(X,fP_5Y) + h^s(X,fP_6Y) \\ &\quad = Ch^s(X,Y) - \nabla^s_XFP_5Y - \nabla^s_XFP_6Y - D^s(X,\phi P_3Y) + FP_5\nabla_XY + FP_6\nabla_XY, \end{aligned} \tag{3.15}$$

$$\begin{aligned} &\eta(\nabla_X\phi P_1Y) + \eta(\nabla_X\phi P_2Y) + \eta(\nabla_X\phi P_4Y) + \eta(\nabla_X fP_5Y) + \eta(\nabla_X fP_6Y) \\ &\quad = \eta(A_{FP_5Y}X) + \eta(A_{FP_6Y}X) + \eta(A_{\phi P_3Y}X) + \bar{g}(X,Y)V. \end{aligned} \tag{3.16}$$

THEOREM 1. Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$. Then M is a quasi-bi-slant lightlike submanifold if and only if
(i) $\phi Rad(TM)$ is a distribution on M such that $Rad(TM) \cap \phi Rad(TM) = \{0\}$;
(ii) the screen distribution $S(TM)$ can be split as a direct sum

$$S(TM) = (\phi Rad(TM) \oplus \phi ltr(TM)) \oplus D \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}$$

such that D is an invariant distribution on M, i.e. $\phi D = D$;
(iii) there exists a constant $\lambda_1 \in [0, 1)$ such that $P^2X = -\lambda_1 X$, for all $X \in \Gamma(D_1)$, where $\lambda_1 = cos^2\theta_1$ and θ_1 is the slant angle of D_1;
(iv) there exists a constant $\lambda_2 \in [0, 1)$ such that $P^2X = -\lambda_2 X$, for all $X \in \Gamma(D_2)$, where $\lambda_2 = cos^2\theta_2$ and θ_2 is the slant angle of D_2.

Proof. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then the distribution D is invariant with respect to ϕ and $\phi Rad(TM)$ is a distribution on M such that $Rad(TM) \cap \phi Rad(TM) = \{0\}$.

For any $X \in \Gamma(D_1)$ we have $|PX| = |\phi X| \cos\theta_1$, i.e.

$$\cos\theta_1 = \frac{|PX|}{|\phi X|}. \tag{3.17}$$

In view of (3.17), we get $\cos^2\theta_1 = \frac{|PX|^2}{|\phi X|^2} = \frac{g(PX, PX)}{g(\phi X, \phi X)} = \frac{g(X, P^2X)}{g(X, \phi^2 X)}$, which gives

$$g(X, P^2X) = \cos^2\theta_1 g(X, \phi^2 X). \tag{3.18}$$

Since M is a quasi-bi-slant lightlike submanifold, $\cos^2\theta_1 = \lambda_1(\text{constant}) \in [0, 1)$ and therefore from (3.18) we get $g(X, P^2X) = \lambda_1 g(X, \phi^2X) = g(X, \lambda_1\phi^2X)$, for all $X \in \Gamma(D_1)$, which implies

$$g(X, (P^2 - \lambda_1\phi^2)X) = 0. \tag{3.19}$$

Since $(P^2 - \lambda_1\phi^2)X \in \Gamma(D_1)$ and the induced metric $g = g|_{D_1\times D_1}$ is non-degenerate (positive definite). From (3.19) we have $(P^2 - \lambda_1\phi^2)X = 0$, which implies

$$P^2X = \lambda_1\phi^2X = -\lambda_1 X,\ \forall X \in \Gamma(D_1). \tag{3.20}$$

This proves (iii).

Suppose for any $X \in \Gamma(D_2)$ we have $|PX| = |\phi X| \cos\theta_2$, i.e.

$$\cos\theta_2 = \frac{|PX|}{|\phi X|}. \tag{3.21}$$

Now the proof follows by using similar steps above of proof of (iii), which gives $\cos^2\theta_2 = \lambda_2(\text{constant})$. This proves (iv).

Conversely, suppose that conditions (i), (ii), (iii) and (iv) are satisfied. From (iii), we have $P^2X = \lambda_1\phi^2X,\ \forall\ X \in \Gamma(D_1)$, where $\lambda_1 \in [0, 1)$.

Now $\cos\theta_1 = \frac{g(\phi X, PX)}{|\phi X||PX|} = -\frac{g(X, \phi PX)}{|\phi X||PX|} = -\frac{g(X, P^2X)}{|\phi X||PX|} = -\lambda_1\frac{g(X, \phi^2X)}{|\phi X||PX|} = \lambda_1\frac{g(\phi X, \phi X)}{|\phi X||PX|}$.
From the above equation, we obtain

$$\cos\theta_1 = \lambda_1\frac{|\phi X|}{|PX|}. \tag{3.22}$$

Therefore (3.16) and (3.21) give $\cos^2\theta_1 = \lambda_1(\text{constant})$.

Furthermore, from (iv) we have $P^2X = \lambda_2\phi^2X,\ \forall\ X \in \Gamma(D_2)$, where $\lambda_2 \in [0, 1)$. Now by using the similar steps above we get $\cos^2\theta_2 = \lambda_2(\text{constant})$. This completes the proof. Hence M is a quasi-bi-slant lightlike submanifold. □

THEOREM 2. Let M be a q-lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$ of index $2q$. Then M is a quasi-bi-slant lightlike submanifold if and only if
(i) $\phi Rad(TM)$ is a distribution on M such that $Rad(TM) \cap \phi Rad(TM) = \{0\}$;
(ii) the screen distribution $S(TM)$ can be split as a direct sum

$$S(TM) = (\phi Rad(TM) \oplus \phi ltr(TM)) \oplus D \oplus_{orth} D_1 \oplus_{orth} D_2 \oplus_{orth} \{V\}$$

such that D is an invariant distribution on M, i.e. $\phi D = D$;
(iii) there exists a constant $\mu_1 \in [0, 1)$ such that $BFX = -\mu_1 X,\ \forall\ X \in \Gamma(D_1)$, where $\mu_1 = sin^2\theta_1$ and θ_1 is the slant angle of D_1;
(iv) there exists a constant $\mu_2 \in [0, 1)$ such that $BFX = -\mu_2 X,\ \forall\ X \in \Gamma(D_2)$, where $\mu_2 = sin^2\theta_2$ and θ_2 is the slant angle of D_2.

Proof. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then the distribution D is invariant with respect to ϕ and $\phi Rad(TM)$ is a distribution on M such that $Rad(TM) \cap \phi Rad(TM) = \{0\}$.

Now, for any vector field $X \in \Gamma(D_1)$, we have

$$\phi X = PX + FX. \tag{3.23}$$

Applying ϕ to (3.23) and taking the tangential component, we get

$$-X = P^2X + BFX,\ \forall\ X \in \Gamma(D_1). \tag{3.24}$$

Since M is a quasi-bi-slant lightlike submanifold, $P^2X = -\lambda_1 X,\ \forall\ X \in \Gamma(D_1)$, where $\lambda_1 \in [0, 1)$ and therefore from (3.24) we get

$$BFX = -\mu_1 X,\ \forall\ X \in \Gamma(D_1), \tag{3.25}$$

where $1 - \lambda_1 = \mu_1$(constant) $\in (0, 1]$. Now, in view of Theorem 1, we have $\lambda_1 = \cos^2\theta_1$. This proves (iii).

Suppose for any vector field $X \in \Gamma(D_2)$, we have

$$\phi X = PX + FX.$$

Now the proof follows by using similar steps above of proof of (iii), which gives $1 - \lambda_2 = \mu_2$(constant) $\in [0, 1)$, where $\lambda_2 = \cos^2\theta_2$. This proves (iv).

Conversely, assume that conditions (i), (ii), (iii) and (iv) are satisfied. From (3.24) we get

$$-X = P^2X - \mu_1 X,\ \forall\ X \in \Gamma(D_1), \tag{3.26}$$

which implies

$$P^2X = -\lambda_1 X,\ \forall\ X \in \Gamma(D_1) \tag{3.27}$$

where $1 - \mu_1 = \lambda_1$(constant) $\in [0, 1)$. Furthermore, for any $X \in \Gamma(D_2)$, by using the similar steps above we have $1 - \mu_2 = \lambda_2$(constant) $\in [0, 1)$. Now the proof follows from Theorem 1. Therefore, M is a quasi-bi-slant lightlike submanifold. □

Corollary 1. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then for any slant distribution D of M with slant angle θ, we have

$$g(PX, PY) = \cos^2\theta g(X, Y),$$
$$g(FX, FY) = \sin^2\theta g(X, Y),$$

for all $X, Y \in \Gamma(D)$.

The proof of the above corollary follows by using similar steps as in the proof of Corollary 3.1

of [16].

THEOREM 3. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then the radical distribution $Rad(TM)$ is integrable if and only if
(i) $P_1(\nabla_X\phi Y) = P_1(\nabla_Y\phi X)$ and $P_4(\nabla_X\phi Y) = P_4(\nabla_Y\phi X)$,
(ii) $P_5(\nabla_X\phi Y) = P_5(\nabla_Y\phi X)$ and $P_6(\nabla_X\phi Y) = P_6(\nabla_Y\phi X)$,
(iii) $h^l(Y, \phi X) = h^l(X, \phi Y)$ and $h^s(Y, \phi X) = h^s(X, \phi Y)$,
for all $X, Y \in \Gamma(Rad(TM))$.

Proof. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Suppose $X, Y \in \Gamma(D_1)$. From (3.8), we have $P_1(\nabla_X\phi Y) = \phi P_2\nabla_X Y$, which implies $P_1(\nabla_X\phi Y) - P_1(\nabla_Y\phi X) = \phi P_2[X, Y]$. From (3.11), we have
$P_4(\nabla_X\phi Y) = \phi P_4\nabla_X Y$, which gives $P_4(\nabla_X\phi Y) - P_4(\nabla_Y\phi X) = \phi P_4[X, Y]$. From (3.12), we have $P_5(\nabla_X\phi Y) = fP_5\nabla_X Y + Bh^s(X, Y)$, which gives $P_5(\nabla_X\phi Y) - P_5(\nabla_Y\phi X) = fP_5[X, Y]$.

From (3.13), we have $P_6(\nabla_X\phi Y) = fP_6\nabla_X Y + Bh^s(X, Y)$, which gives
$P_6(\nabla_X\phi Y) - P_6(\nabla_Y\phi X) = fP_6[X, Y]$. From (3.14), we have $h^l(X, \phi Y) = \phi P_3\nabla_X Y$, which implies $h^l(X, \phi Y) - h^l(Y, \phi X) = \phi P_3[X, Y]$. From (3.15), we have $h^s(X, \phi Y) = Ch^s(X, Y) + FP_5\nabla_X Y + FP_6\nabla_X Y$, which gives $h^s(X, \phi Y) - h^s(Y, \phi X) = FP_5[X, Y]$, which completes the proof. □

THEOREM 4. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then the distribution $D \oplus \{V\}$ is integrable if and only if
(i) $P_1(\nabla_X\phi Y) = P_1(\nabla_Y\phi X)$ and $P_2(\nabla_X\phi Y) = P_2(\nabla_Y\phi X)$;
(ii) $P_5(\nabla_X\phi Y) = P_5(\nabla_Y\phi X)$ and $P_6(\nabla_X\phi Y) = P_6(\nabla_Y\phi X)$;
(iii) $h^l(Y, \phi X) = h^l(X, \phi Y)$ and $h^s(Y, \phi X) = h^s(X, \phi Y)$,
for all $X, Y \in \Gamma(D \oplus \{V\})$.

Proof. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Suppose $X, Y \in \Gamma(D_1)$. From (3.8), we have $P_1(\nabla_X\phi Y) = \phi P_2\nabla_X Y$, which implies $P_1(\nabla_X\phi Y) - P_1(\nabla_Y\phi X) = \phi P_2[X, Y]$. From (3.9), we have
$P_2(\nabla_X\phi Y) = \phi P_1\nabla_X Y$, which gives $P_2(\nabla_X\phi Y) - P_2(\nabla_Y\phi X) = \phi P_1[X, Y]$. From (3.12), we have $P_5(\nabla_X\phi Y) = fP_5\nabla_X Y + Bh^s(X, Y)$, which gives $P_5(\nabla_X\phi Y) - P_5(\nabla_Y\phi X) = fP_5[X, Y]$.

From (3.13), we have $P_6(\nabla_X\phi Y) = fP_6\nabla_X Y + Bh^s(X, Y)$, which gives
$P_6(\nabla_X\phi Y) - P_6(\nabla_Y\phi X) = fP_6[X, Y]$. From (3.14), we have $h^l(X, \phi Y) = \phi P_3\nabla_X Y$, which implies $h^l(X, \phi Y) - h^l(Y, \phi X) = \phi P_3[X, Y]$. From (3.15), we have $h^s(X, \phi Y) = Ch^s(X, Y) + FP_5\nabla_X Y + FP_6\nabla_X Y$, which gives $h^s(X, \phi Y) - h^s(Y, \phi X) = FP_5[X, Y]$, which completes the proof. □

4 Foliations Determined By Distributions

In this section, we obtain necessary and sufficient conditions for foliations determined by distributions on a quasi-bi-slant lightlike submanifold of an indefinite Sasakian manifold to be totally geodesic.

DEFINITION 2. A quasi-bi-slant lightlike submanifold M of an indefinite Sasakian manifold $\bar{M}$ is said to be mixed geodesic if its second fundamental form h satisfies $h(X, Y) = 0$, for all $X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$. Thus M is a mixed geodesic quasi-bi-slant lightlike submanifold if $h^l(X, Y) = 0$ and $h^s(X, Y) = 0$, $\forall\, X \in \Gamma(D_1)$ and $Y \in \Gamma(D_2)$.

THEOREM 5. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $Rad(TM)$ defines a totally geodesic foliation if and only if

$$\bar{g}(\nabla_X\phi P_2 Z + \nabla_X\phi P_4 Z + \nabla_X f P_5 Z + \nabla_X f P_6 Z, \phi Y) = g(A_{\phi P_3 Z}X + A_{FP_5 Z}X + A_{FP_6 Z}X, \phi Y),$$

for all $X \in \Gamma(Rad(TM))$ and $Z \in \Gamma(S(TM))$.

Proof. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. The distribution $Rad(TM)$ defines a totally geodesic foliation if and only if $\nabla_X Y \in Rad(TM)$, $\forall X, Y \in \Gamma(Rad(TM))$. Since $\bar{\nabla}$ is a metric connection, using (2.7) and (2.19), for any $X, Y \in \Gamma(Rad(TM))$ and $Z \in \Gamma(S(TM))$, we get

$$\bar{g}(\nabla_X Y, Z) = \bar{g}((\bar{\nabla}_X \phi)Z - \bar{\nabla}_X \phi Z, \phi Y). \tag{4.1}$$

Now from (2.20), (3.4) and (4.1) we get

$$\bar{g}(\nabla_X Y, Z) = -\bar{g}(\bar{\nabla}_X(\phi P_2 Z + \phi P_3 Z + \phi P_4 Z + fP_5 Z + FP_5 Z + fP_6 Z + FP_6 Z), \phi Y). \tag{4.2}$$

In view of (2.7)-(2.9) and (4.2), for any $X, Y \in \Gamma(Rad(TM))$ and $Z \in \Gamma(S(TM))$, we obtain

$$\bar{g}(\nabla_X Y, Z) = g(A_{\phi P_3 Z}X + A_{FP_5 Z}X + A_{FP_6 Z}X - \nabla_X \phi P_2 Z - \nabla_X \phi P_4 Z - \nabla_X fP_5 Z - \nabla_X fP_6 Z, \phi Y), \tag{4.3}$$

which completes the proof. □

THEOREM 6. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then $D_2 \oplus \{V\}$ defines a totally geodesic foliation if and only if
(i) $\bar{g}(\nabla_X fZ - A_{FZ}X, fY) = \bar{g}(h^s(X, \phi Z), FY)$,
(ii) $g(fY, \nabla_X \phi N) = -\bar{g}(FY, h^s(X, \phi N))$,
(iii) $g(fY, A_{\phi W}X) = \bar{g}(FY, D^s(X, \phi W))$,
for all $X, Y \in \Gamma(D_2 \oplus \{V\})$, $Z \in \Gamma(D_1)$, $W \in \Gamma(\phi ltr(TM))$ and $N \in \Gamma(ltr(TM))$.

Proof. Let M be a quasi-bi-slant lightlike submanifold of an indefinite Sasakian manifold. To prove that the distribution $D_2 \oplus \{V\}$ defines a totally geodesic foliation, it is sufficient to show that $\nabla_X Y \in \Gamma(D_2 \oplus \{V\})$, $\forall X, Y \in \Gamma(D_2 \oplus \{V\})$. Since $\bar{\nabla}$ is a metric connection, using (2.7) and (2.19) for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $Z \in \Gamma(D_1)$ we get

$$\bar{g}(\nabla_X Y, Z) = \bar{g}(\bar{\nabla}_X \phi Y, \phi Z) = -\bar{g}(\bar{\nabla}_X \phi Z, \phi Y) \tag{4.4}$$

From (2.7), (3.1) and (4.4) we get

$$\bar{g}(\nabla_X Y, Z) = -\bar{g}(\nabla_X(fZ + FZ) + h^s(X, \phi Z), fY + FY) \tag{4.5}$$

In view of (2.7)-(2.9) and (4.5) we obtain

$$\bar{g}(\nabla_X Y, Z) = -\bar{g}(\nabla_X fZ - A_{FZ}X, fY) - \bar{g}(h^s(X, \phi Z), FY) \tag{4.6}$$

From (4.6) we get (i).

Now for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $N \in \Gamma(ltr(TM))$, we have

$$\bar{g}(\nabla_X Y, N) = \bar{g}(\bar{\nabla}_X \phi Y, \phi N) = -\bar{g}(\bar{\nabla}_X \phi N, \phi Y) \tag{4.7}$$

From (2.7), (3.1) and (4.7) we get

$$\bar{g}(\nabla_X Y, N) = -\bar{g}(\nabla_X \phi N + h^s(X, \phi N), fY + FY) \tag{4.8}$$

In view of (4.8) we obtain

$$\bar{g}(\nabla_X Y, N) = -\bar{g}(\nabla_X \phi N, fY) - \bar{g}(h^s(X, \phi N), FY) \tag{4.9}$$

Thus from (4.9) we get the result (ii).

Now for any $X, Y \in \Gamma(D_2 \oplus \{V\})$ and $W \in \Gamma(\phi ltr(TM))$, we have

$$\bar{g}(\nabla_X Y, W) = \bar{g}(\bar{\nabla}_X \phi Y, \phi W) = -\bar{g}(\bar{\nabla}_X \phi W, \phi Y) \tag{4.10}$$

From (2.8), (3.1) and (4.10) we get

$$\bar{g}(\nabla_X Y, W) = -\bar{g}(-A_{\phi W}X + D^s(X, \phi W), fY + FY) \tag{4.11}$$

In view of (4.11) we obtain

$$\bar{g}(\nabla_X Y, W) = \bar{g}(A_{\phi W}X, fY) - \bar{g}(FY, D^s(X, \phi W)) \tag{4.12}$$

Thus from (4.12) we get the result (iii), which completes the proof. □

5 Minimal Quasi-bi-slant Lightlike Submanifold

In this section, we study minimal quasi-bi-slant lightlike submanifolds of indefinite Sasakian manifolds. A general notion of a minimal lightlike submanfold in a semi-Riemannian manifold, as introduced by Bejancu and Duggal in [1] is as follows:

DEFINITION 3.[8] A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is minimal if
(i) $h^s = 0$ on $Rad(TM)$,
(ii) *trace* $h = 0$, where *trace* is written with respect to g restricted to $S(TM)$.

EXAMPLE 3. Let $(\mathbb{R}^{15}_2, \bar{g}, \phi, \eta, V)$ be an indefinite Sasakian manifold, where $\bar{g}$ is of signature $(-, +, +, +, +, +, +, -, +, +, +, +, +, +, +)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial y_7, \partial z\}$.

Suppose M is a submanifold of $\mathbb{R}^{15}_2$ given by $-x^1 = y^2 = u_1$, $x^2 = u_2$, $y^1 = u_3$, $x^3 = u_4 = y^4$, $x^4 = u_5 = -y^3$, $x^5 = u_6$, $x^6 = -u_7 = y^5$, $y^6 = -u_9$, $x^7 = u_9 = -y^6$, $y^7 = u_8$, $z = u_9$. The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}\}$, where

$$\begin{aligned}
Z_1 &= 2(-\partial x_1 + \partial y_2 - y^1 \partial z),\\
Z_2 &= 2(\partial x_2 + y^2 \partial z), \quad Z_3 = 2(\partial y_1),\\
Z_4 &= 2(\partial x_3 + \partial y_4 + y^3 \partial z), \quad Z_5 = 2(\partial x_4 - \partial y_3 + y^4 \partial z),\\
Z_6 &= 2(\partial x_5 + y^5 \partial z), \quad Z_7 = 2(-\partial x_6 - \partial y_5 - y^6 \partial z),\\
Z_8 &= 2(\partial y_7), \quad Z_9 = 2(\partial x_7 - y_6 + y^7 \partial z),\\
Z_{10} &= 2(\partial z) = V.
\end{aligned}$$

Hence $Rad(TM) = span\{Z_1\}$ and $S(TM) = span\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, V\}$. Now $ltr(TM)$ is spanned by $N = \partial x_1 + \partial y_2 + y^1 \partial z$ and $S(TM^{\perp})$ is spanned by

$$\begin{aligned}
W_1 &= 2(\partial x_3 - \partial y_4 + y^3 \partial z), \quad W_2 = 2(\partial x_4 + \partial y_3 + y^4 \partial z),\\
W_3 &= 2(\partial x_6 + \partial y_5 + y^6 \partial z), \quad W_4 = 2(\partial x_7 + \partial y_6 + y^7 \partial z),
\end{aligned}$$

It follows that $\phi Z_1 = Z_2 + Z_3$ and $\phi N = \frac{1}{2}(Z_2 - Z_3)$, which implies that $\phi Rad(TM)$ and $\phi ltr(TM)$ are distributions on M. On the other hand, we can see that $D = span\{Z_4, Z_5\}$ such that $\phi Z_4 = Z_5$, $\phi Z_5 = -Z_4$, which implies that D is invariant with respect to ϕ. Also $D_1 = span\{Z_6, Z_7\}$ and $D_2 = span\{Z_8, Z_9\}$ are slant distributions with slant angles $\theta_1 = \pi/4$ and $\theta_2 = \pi/4$ respectively. Hence M is a quasi-bi-slant 2-lightlike submanifold of $\mathbb{R}^{15}_2$.

Now by direct computation and using Gauss formula, we get for any $X \in \Gamma(TM)$ we have

$$\bar{\nabla}_{Z_i} Z_j = 0, \quad \text{where } 1 \leq i, j \leq 9$$

which implies $h^l(Z_i, Z_j) = 0$, $h^s(Z_i, Z_j) = 0$. Thus $h^s(Z_1, Z_1) = 0$, i.e. $h^s = 0$ on $Rad(TM)$. We also have $\epsilon_1 = g(Z_1, Z_1) = 0$, $\epsilon_2 = g(Z_2, Z_2) = 1$, $\epsilon_3 = g(Z_3, Z_3) = -1$, $\epsilon_4 = g(Z_4, Z_4) = 2$, $\epsilon_5 = g(Z_5, Z_5) = 2$, $\epsilon_6 = g(Z_6, Z_6) = 1$, $\epsilon_7 = g(Z_7, Z_7) = 2$, $\epsilon_8 = g(Z_8, Z_8) = 1$, $\epsilon_9 = g(Z_9, Z_9) = 2$, $\epsilon = g(V, V) = 1$. Hence we get

$$trace_{g|S(TM)} h = \epsilon_2 h(Z_2, Z_2) + \epsilon_3 h(Z_3, Z_3) + \epsilon_4 h(Z_4, Z_4) + \epsilon_5 h(Z_5, Z_5) + \epsilon_6 h(Z_6, Z_6) + \epsilon_7 h(Z_7, Z_7) + \epsilon_8 h(Z_8, Z_8) + \epsilon_9 h(Z_9, Z_9) + \epsilon h(V, V) = 0.$$

Therefore, here M is a minimal quasi-bi-slant lightlike submanifold of $\mathbb{R}^{15}_2$.

Now we prove two characterization results for minimal quasi-bi-slant lightlike submanifolds.
LEMMA 2. Let M be a proper quasi-bi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Now suppose D is any slant distribution of M such that $dim(D) = dim(S(TM^{\perp}))$. If $\{e_1, ..., e_m\}$ is a local orthonormal basis of $\Gamma(D)$, then $\{csc\theta F e_1, ..., csc\theta F e_m\}$ is an orthonormal basis of $S(TM^{\perp})$.

Proof. As $\{e_1, ..., e_m\}$ is a local orthonormal basis of D, which is Riemannian. Thus from Corollory 1, we get

$$\bar{g}(csc\theta Fe_i, csc\theta Fe_j) = csc^2\theta \sin^2\theta g(e_i, e_j) = \delta_{ij},$$

where $i, j = 1, 2, ..., m$ and which proves the assertion. □

THEOREM 7. Let M be a proper quasi-bi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Then M is minimal if and only if

$$traceA^*_{\xi_j}|_{S(TM)} = 0, \quad traceA_{W_j}|_{S(TM)} = 0,$$
$$\bar{g}(D^l(X, W), Y) = 0, \ \ \forall\, X, Y \in \Gamma(Rad(TM)).$$

where $\{\xi_j\}_{j=1}^r$ is a basis of $Rad(TM)$ and $\{W_\alpha\}_{\alpha=1}^m$ is a basis of $S(TM^\perp)$.

Proof. The proof of above theorem follows by using similar steps as in proof of theorem 7 of [19]. □

THEOREM 8. Let M be a proper quasi-bi-slant lightlike submanifold of an indefinite Sasakian manifold $\bar{M}$. Now suppose D is any slant distribution of M such that $dim(D) = dim(S(TM^\perp)$. Then M is minimal if and only if

$$traceA^*_{\xi_k}|_{S(TM)} = 0, \ \ traceA_{Fe_j}|_{S(TM)} = 0,$$
$$\bar{g}(D^l(X, Fe_j), Y) = 0,$$

for $X, Y \in \Gamma(Rad(TM))$, where $\{\xi_k\}_{k=1}^r$ is a basis of $\Gamma(Rad(TM))$ and $\{e_j\}_{j=1}^m$ is a basis of D.

Proof. From Lemma 2, $\{csc\theta Fe_1, ..., csc\theta Fe_m\}$ is an orthonormal basis of $S(TM^\perp)$. Thus we can write

$$h^s(X, X) = \textstyle\sum_{i=1}^m A_i csc\theta Fe_i, \ \ \forall\, X \in \Gamma(TM)$$

for some functions $A_i, i \in \{1, ..., m\}$. Hence we obtain

$$h^s(X, X) = \textstyle\sum_{i=1}^m csc\theta g(A_{Fe_i}X, X)Fe_i, \ \ \forall X \in \Gamma(\phi Rad(TM) \oplus \phi ltr(TM) \perp D).$$

Thus the assertion of theorem follows from Theorem 7. □

6 Totally Umbilical Quasi-bi-slant Lightlike Submanifolds

In this section, we study totally umbilical quasi-bi-slant lightlike submanifolds of indefinite Sasakian manifolds. A general notion of a totally umbilical lightlike submanfold in a semi-Riemannian manifold, as introduced by Bejancu and Duggal in [1] is as follows:

DEFINITION 4.[8] A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is totally umbilical in $\bar{M}$ if there is a smooth transversal vector field $\mathcal{H} \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that for all $X, Y \in \Gamma(TM)$,

$$h(X, Y) = \mathcal{H}\bar{g}(X, Y) \tag{6.1}$$

It is easy to see that M is totally umbilical if and only if on each co-ordinate neighbourhood U, there exist smooth vector fields $\mathcal{H}^l \in \Gamma(ltr(TM))$ and $\mathcal{H}^s \in \Gamma(S(TM^\perp)$, and smooth functions $\mathcal{H}^l_i \in F(ltr(TM))$ and $\mathcal{H}^s_i \in F(S(TM^\perp))$ such that

$$h^l(X, Y) = \mathcal{H}^l\bar{g}(X, Y), \ \ h^s(X, Y) = \mathcal{H}^s\bar{g}(X, Y), \tag{6.2}$$

$$h^l_i(X, Y) = \mathcal{H}^l_i\bar{g}(X, Y), \ \ h^s_i(X, Y) = \mathcal{H}^s_i\bar{g}(X, Y). \tag{6.3}$$

for any $X, Y \in \Gamma(TM)$.

EXAMPLE 4. Let $(\mathbb{R}^{15}_2, \bar{g}, \phi, \eta, V)$ be an indefinite Sasakian manifold, where $\bar{g}$ is of signature $(-,+,+,+,+,+,+,-,+,+,+,+,+,+,+)$ with respect to the canonical basis $\{\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5, \partial y_6, \partial y_7, \partial z\}$.

Suppose M is a submanifold of $\mathbb{R}^{14}_2$ given by $x^1 = y^2 = u_1$, $x^2 = u_2$, $y^1 = u_3$, $x^3 = u_4$, $y^3 = u_5$, $x^4 = u_5$, $y^4 = -u_4$, $x^5 = u_6$, $y^5 = u_7$, $x^6 = \cos u_7$, $y^6 = \sin u_7$, $x^7 = u_8$, $y^7 = u_9$, $x^8 = \sin u_9$, $y^8 = \cos u_9$, $z = u_{10}$. The local frame of TM is given by $\{Z_1, Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}\}$, where

$$
\begin{aligned}
Z_1 &= 2(\partial x_1 + \partial y_2 + y^1 \partial z),\\
Z_2 &= 2(\partial x_2 + y^2 \partial z), \quad Z_3 = 2(\partial y_1),\\
Z_4 &= 2(\partial x_3 - \partial y_4 + y^3 \partial z), \quad Z_5 = 2(\partial x_4 + \partial y_3 + y^4 \partial z),\\
Z_6 &= 2(\partial x_5 + y^5 \partial z), \quad Z_7 = 2(\partial y_5 - \sin u_7 \partial x_6 + \cos u_7 \partial y_6 - y^6 \sin u_7 \partial z),\\
Z_8 &= 2(\partial x_7 + y^7 \partial z), \quad Z_9 = 2(\partial y_7 + \cos u_9 \partial x_8 - \sin u_9 \partial y_8 + y^8 \cos u_9 \partial z),\\
Z_{10} &= 2(\partial z) = V.
\end{aligned}
$$

Hence $Rad(TM) = span\{Z_1\}$ and $S(TM) = span\{Z_2, Z_3, Z_4, Z_5, Z_6, Z_7, Z_8, Z_9, V\}$. Now $ltr(TM)$ is spanned by $N = -\partial x_1 + \partial y_2 - y^1 \partial z$ and $S(TM^\perp)$ is spanned by

$$
\begin{aligned}
W_1 &= 2(\cos u_7 \partial x_6 + \sin u_7 \partial y_6 + y^6 \cos u_7 \partial z),\\
W_2 &= 2(\partial y_5 - \cos u_7 \partial x_6 + \sin u_7 \partial y_6 - y^6 \cos u_7 \partial z),\\
W_3 &= 2(\sin u_9 \partial x_8 + \cos u_9 \partial y_8 + y^8 \sin u_9 \partial z),\\
W_4 &= 2(\partial y_7 - \cos u_9 \partial x_8 + \sin u_9 \partial y_8 - y^8 \cos u_9 \partial z).
\end{aligned}
$$

It follows that $\phi Z_1 = Z_2 - Z_3$ and $\phi N = \frac{1}{2}(Z_2 + Z_3)$, implies that $\phi Rad(TM)$ and $\phi ltr(TM)$ are distributions on M. Further, we can see that $D = span\{Z_4, Z_5\}$ such that $\phi Z_4 = Z_5$, $\phi Z_5 = -Z_4$, which implies that D is invariant with respect to ϕ. Also $D_1 = span\{Z_6, Z_7\}$ and $D_2 = span\{Z_8, Z_9\}$ are slant distributions with slant angles $\theta_1 = \pi/4$ and $\theta_2 = \pi/4$ respectively. Hence M is a quasi-bi-slant 2-lightlike submanifold of $\mathbb{R}^{15}_2$.

Now by direct computation and using Gauss formula, we get for every $X \in \Gamma(TM)$ we have

$$\bar{\nabla}_X Z_1 = \bar{\nabla}_X Z_2 = \bar{\nabla}_X Z_3 = \bar{\nabla}_X Z_4 = \bar{\nabla}_X Z_5 = \bar{\nabla}_X Z_6 = \bar{\nabla}_X Z_8 = 0.$$

Also we can see that $\bar{\nabla}_X Z_7 = 0$, for any $X \in \Gamma(TM)$ except $X = Z_7$ as

$$\bar{\nabla}_{Z_7} Z_7 = -8(\cos u_7 \partial x_6 + \sin u_7 \partial y_6) = -4W_1.$$

In the similar way we get that $\bar{\nabla}_X Z_9 = 0$, for any $X \in \Gamma(TM)$ except $X = Z_9$ as

$$\bar{\nabla}_{Z_9} Z_9 = -8(\sin u_9 \partial x_8 + \cos u_9 \partial y_8) = -4W_3.$$

Thus by (2.7) we have $h^l(X, Y) = 0$ for all $X, Y \in \Gamma(TM)$. Also $h^s(X, Y) = 0$ for all $X, Y \in \Gamma(TM)$ except

$$
\begin{aligned}
h^s(Z_7, Z_7) &= -4W_1 = -2\bar{g}(Z_7, Z_7)W_1,\\
h^s(Z_9, Z_9) &= -4W_3 = -2\bar{g}(Z_9, Z_9)W_3.
\end{aligned}
$$

Therefore M is a totally umbilical quasi-bi-slant lightlike submanifold of $\mathbb{R}^{15}_2$.

PROPOSITION 1.[7] Let M be a lightlike submanifold of a semi-Riemannian manifold $\bar{M}$. Then,

$h^l = 0$ on $\Gamma(Rad(TM))$.

THEOREM 9.[7] There are no minimal lightlike submanifold contained in a proper totally umbilical quasi-bi-slant lightlike submanifold of an indefinite Sasakian manifold.

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