On Some New Class of Soft Real Sequences

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Abstract

In present paper, we define a basic metric on soft real numbers and study some basic structures, using the metric and Orlicz function we introduced a new class of soft real sequences. We also study some structures and inclusive relations on this new class. In last we define another metric on new class of sequences and study its completeness.

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1 Introduction

In our present work, we attempt to composite the Orlicz sequence space of real numbers with soft set theory. An Orlicz sequence space is a special case of Orlicz space. Lindensstrauss and Tzafriri[15], first investigated Orlicz sequence spaces in detail.

An Orlicz function is a continuous mapping M defined on the set of non-negative real numbers i.e. $[0, \infty)$ onto itself, which is non-decreasing and convex with $\mathsf{M}(x) = 0 \Leftrightarrow x = 0$ and $\lim_{x\to\infty} \mathsf{M}(x) = \infty$.

The Δ_2-condition for an Orlicz function is given as follows:

$$\mathsf{M}(2x) \leq \mathrm{K}\mathsf{M}(x) \quad \forall\, x \in [0, \infty) \tag{1.1}$$

Where K is some positive real constant.

This condition is equivalent to

$$\mathsf{M}(\mathrm{L}x) \leq \mathrm{KL}.\mathsf{M}(x) \quad \forall\, x \in [0, \infty) \quad \text{and} \quad \forall\, \mathrm{L} \in [1, \infty) \tag{1.2}$$

Also an Orlicz function satisfies the inequality $\mathsf{M}(\delta x) \leq \delta \mathsf{M}(x)\ \forall \delta \in (0, 1)$.

The integral representation of Orlicz function M :

$$\mathsf{M}(x) = \int_0^x \mathrm{q(t)\,dt} \tag{1.3}$$

where, q denotes the Kernel of M vanishing at t = 0, positive valued for non-zero t and differentiable on $t \in [0, \infty)$ also the kernel has infinite limit at ∞.

Ruckle[16] defined the modulus function M as follows:

$$\mathsf{M}(x+y) \leq \mathsf{M}(x) + \mathsf{M}(y) \tag{1.4}$$

which is equivalent to the convexity of Orlicz function. Further, Maddox[17], Srivastava *et al.*[18] discussed this modulus function.

Lindenstrauss and Tzafriri[15] defined a sequence space using Orlicz function as follows:

$$l_{\mathsf{M}} = \left\{ \{x_k\} \in \omega : \sum_{k=1}^{\infty} \mathsf{M}\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\} \tag{1.5}$$

Also the norm $||.||_{\mathsf{M}}$ on the space l_{M} :

$$||\{x_k\}||_{\mathsf{M}} = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} \mathsf{M}\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\} \tag{1.6}$$

The space (1.5) with the norm (1.6) is a Banach space, known as Orlicz sequence space, where ω is the space of all sequences of real or complex numbers. Further Gungor *et al.*[3], Esi *et al.*[4], Nuray *et al.* [5], Bhardwaj *et al.* [6], Mursaleen *et al.*[7], Parasar *et al.* [8], Isik *et al.*[9], Dutta *et al.*[10], Karakaya *et al.*[11] and others used Orlicz function to construct several new sequence spaces. Later on Basu and Srivastava [19] extended the study of composite vector valued single and double sequences and their various convergence methods with the help of Orlicz function, modulus function, multiplier sequences etc.

The concept of Soft sets was first introduced by Molodtsov[12]. In his paper he presented the fundamental results of this new theory and successfully applied it to smoothness of functions. Sujoy Das and S. K. Samanta[1] introduced the definition of soft element, soft real sets and soft real numbers, also studied sequences of soft real numbers. Further Maji et al.[13] studied the soft set theory and presented useful applications also. Sujoy Das and S. K. Samanta[2] presented an idea of soft metric spaces and studied some properties.

2 Preliminaries

Definition 2.1. *[12]Let U be a universe and E be a set of parameters. Let $P(U)$ denotes the power set of U and A be a non-empty subset of E. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.*

In other words, soft set over U is a parametrized family of subsets of the universe U. For $\epsilon \in A, F(\epsilon)$ may be considered as the set of ϵ-approximate elements of the soft set (F, A).

Definition 2.2. *[1]Let X be a non-empty universal set and E be a non-empty parameter set. Then a function $\epsilon : E \to X$ is said to be a soft element of X. A soft element ϵ of X is said to belong to a soft set A of X, denoted by $\epsilon \in A$, if $\epsilon(e) \in A(e), \forall e \in E$. Thus a soft set A of X with respect to the index set E can be expressed as $A(e) = \{\epsilon(e), \epsilon \in A\}, e \in E$.*

Definition 2.3. *[1] Let $\mathbb{R}$ be the set of real numbers. $P(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and A be a set of parameters. Then a mapping $F : A \to P(\mathbb{R})$ is called a soft real set. It is denoted by (F, A).*

If (F, A) is singleton soft set i.e. $F(e)$ is singleton for all $e \in A$, then identifying (F, A) with the corresponding soft element, it will be called a soft real number.

$\mathbb{R}(\mathbb{A})^*$ denotes the set of all non-negative soft real numbers.
A non-negative soft real number $\epsilon \tilde{\geq} 0$ is defined as a mapping

$$\epsilon : E \to \mathbb{R} \quad \text{s. t.} \quad \epsilon(e) \geq 0 \ \forall \ e \in A.$$

Proposition 2.1. *[2] Any collection of soft elements of a soft set can generate a soft subset of that soft set.*

Let X be universal set and E be set of parameters, $A \subset E$ and $F : A \to P(X)$ be soft set then collection of all soft elements of soft set $(F, A) = \bar{X}$ is denoted by $SE(X)$ and collection of all soft subsets denoted by $S(X)$.

Definition 2.4. *[2] A mapping $d : SE(X) \times SE(X) \to R(A)^*$, is said to be a soft metric on the soft set $\bar{X}$ if d satisfies the following conditions:*
(i) $d(\tilde{x}, \tilde{y}) \tilde{\geq} \bar{0}$ for all $\tilde{x}, \tilde{y} \in SE(X)$
(ii) $d(\tilde{x}, \tilde{y}) = \bar{0}$ iff $\tilde{x} = \tilde{y}$
(iii) $d(\tilde{x}, \tilde{y}) = d(\tilde{y}, \tilde{x})$ for all $\tilde{x}, \tilde{y} \in SE(X)$
(iv) $d(\tilde{x}, \tilde{z}) \tilde{\leq} d(\tilde{x}, \tilde{y}) + d(\tilde{y}, \tilde{z})$ for all $\tilde{x}, \tilde{y}, \tilde{z} \in SE(X)$
The soft set $\tilde{X}$ with a soft metric d is said to be a soft metric space.

Proposition 2.2. *[2] Every crisp metric ρ on set X can be extended to soft metric on soft set $\tilde{X}$.*

Definition 2.5. *[2] Let $\tilde{\mathbb{R}}$ be the set of all soft real numbers then a mapping $f : \mathbb{N} \to \tilde{\mathbb{R}}$ is called sequence of soft real numbers.*

A sequence $\{x_n\}$ of soft real numbers is said to converge to a soft real number x if for given $\bar{\epsilon} > \bar{0}$ $\exists$ $n_0 \in \mathbb{N}$ s.t. $d(x_n(e), x(e)) < \bar{\epsilon}$ for all parameters e under consideration. in this case we write $x_n \sim x$. A sequence $\{x_n\}$ of soft real numbers is said to be bounded if there exist soft real numbers $\bar{m}, \bar{M}$ s.t. $\bar{m} \tilde{\leq} x_n(e) \tilde{\leq} \bar{M}$ for all e.

Proposition 2.3. *A convergent sequence of soft real numbers is bounded.*

Proof. Let E be the set of parameters and $\{x_n\}$ be sequence of soft real numbers and converges to x i.e. $x_n(\lambda)$ converges $x(\lambda)$ for all λ. So for given $\epsilon > 0$ $\exists$ $n_0 \in \mathbb{N}$ s.t.

$$|x_n(\lambda) - x(\lambda)| < \epsilon \qquad \forall\ n \geq n_0$$

Or

$$x(\lambda) - \epsilon < x_n(\lambda) < x(\lambda) + \epsilon \qquad \forall\ n \geq n_0$$

Let $\quad K_\lambda = \min\{x_1(\lambda), x_2(\lambda), ...x_{n_0-1}(\lambda), x(\lambda) - \epsilon\}$
and $\quad M_\lambda = \max\{x_1(\lambda), x_2(\lambda), ...x_{n_0-1}(\lambda), x(\lambda) + \epsilon\}$

Obviously, $\quad x_n(\lambda) \in [K_\lambda, M_\lambda]$ for all $n \in \mathbb{N}$
Now take $\quad K = \min_\lambda\{K_\lambda : \lambda \in E\}$ and $M = \max_\lambda\{K_\lambda : \lambda \in E\}$
Gives $\quad K \leq x_n(\lambda) \leq M \quad \forall\, n \in \mathbb{N}$ and $\forall\, \lambda \in E$
therefore, $\quad \bar{K} \tilde{\leq} x_n \tilde{\leq} \bar{M}$.
Hence, $\{x_n\}$ is bounded. □

Proposition 2.4. *[1] The set of all soft real numbers forms an abelian group with respect to addition i.e. $(\tilde{\mathbb{R}}, +)$ is a group structure, operation addition defined as follows,*

$$(\epsilon_1 + \epsilon_2)(\lambda) = \epsilon_1(\lambda) + \epsilon_2(\lambda) \quad \textit{for each fixed} \quad \lambda \in E \tag{2.1}$$

Proposition 2.5. *[1] The structure $(\tilde{\mathbb{R}}, +, \times)$ is a commutative ring with unity, operation defined as follows,*

$$(\epsilon_1 \times \epsilon_2)(\lambda) = \epsilon_1(\lambda) \times \epsilon_2(\lambda) \quad \textit{for each fixed} \quad \lambda \in E \tag{2.2}$$

Remark 2.1. *Above structure is not an integral domain,-*
take $E = \{\lambda_1, \lambda_2\}$ and $\epsilon_1, \epsilon_2 \in \tilde{\mathbb{R}}$ s.t. $\epsilon_1(\lambda_1) = \epsilon_2(\lambda_2) = 0$, $\epsilon_1(\lambda_2) = \epsilon_2(\lambda_1) = 1$ then $\epsilon_1 \times \epsilon_2 = \bar{0}$.
Therefore, zero divisors may exist.

Proposition 2.6. *The collection of all soft real valued sequences forms a vector space over the field* $\mathbb{R}$.

Definition 2.6. *Let* $S(\tilde{\mathbb{R}})$ *be the family of all soft sets defined on* $\mathbb{R}$ *with respect to the parameter set* E*. For each fixed* $e \in E$,

$$d(F(e), G(e)) = \max\{|\sup F(e) - \sup G(e)|, |\inf F(e) - \inf G(e)|\}$$

In case of soft real numbers $F(e)$ *is a singleton set.*

$$d(F(e), G(e)) = |x - y| \qquad \text{where } F(e) = \{x\},\ G(e) = \{y\} \tag{2.3}$$

d *is a metric on the set of real numbers determined by soft sets.*

Definition 2.7. *Let* $\tilde{\mathbb{R}}$ *be the collection of all soft real numbers and* E *be the finite set of parameters. Define,*

$$\bar{d}(F, G) = \sup_{e\in E} d(F(e), G(e)) = \sup_{e\in E} |f(e) - g(e)| \tag{2.4}$$

where f *and* g *are soft elements s.t.* $F(e) = \{f(e)\}$ *and* $G(e) = \{g(e)\}$

Then $\bar{d}$ *defines a soft metric on* $\tilde{\mathbb{R}}$.

(I) $\bar{d}(F, G) \geq 0$
(II) $\bar{d}(F, G) = 0$ iff $F = G$
(III) $\bar{d}(F, G) = \bar{d}(G, F)$
(IV) $\bar{d}(F, H) \leq \bar{d}(F, G) + \bar{d}(G, H)$

Proof. (I), (II) and (III) are trivial.
For IV,
Suppose $\bar{d}(F, H)$ is found at e_0 i.e. $\bar{d}(F, H) = |f(e_0) - h(e_0)|$. Then if $g(e_0) \in [f(e_0), h(e_0)]$ or $[h(e_0), f(e_0)]$,

$$\begin{aligned}|f(e_0) - h(e_0)| &= |f(e_0) - g(e_0)| + |g(e_0) - h(e_0)| \\ &\leq \sup|f(e) - g(e)| + \sup|g(e) - h(e)|.\end{aligned}$$

And if $g(e_0) \notin [f(e_0), h(e_0)]$ or $[h(e_0), f(e_0)]$

$$\begin{aligned}|f(e_0) - h(e_0)| &< |f(e_0) - g(e_0)| \text{ or } |h(e_0) - g(e_0)| \\ &\leq \sup|f(e) - g(e)| \text{ or } \sup|h(e_0) - g(e_0)| \\ &\leq \sup|f(e) - g(e)| + \sup|g(e) - h(e)|.\end{aligned}$$

Hence Proved. □

Theorem 2.1. $(\tilde{\mathbb{R}}, \bar{d})$ *is a complete soft metric space.*

Proof. Consider a cauchy sequence of soft real numbers $\langle x_n \rangle$
i.e. $x_n : E \to \mathbb{R}$ s.t. $\bar{d}(x_n, x_m) \to 0$ as $n, m \to \infty$.
Or, $\sup_{e\in E} |x_n(e) - x_m(e)| \to 0$ as $n, m \to \infty$
$\Rightarrow$ $x_n(e) - x_m(e) \to 0$ as $n, m \to \infty$ $\forall e \in E$
Therefore $\langle x_n(e) \rangle$ is a Cauchy sequence for each $e \in E$.
$\because$ $x_n(e)$ is a singleton set, we can treat it as a real number.
$\therefore$ For fixed e, $\langle x_n(e) \rangle$ can be considered as sequence of real numbers.
Since $\mathbb{R}$ is complete, there exists $x(e) \in \mathbb{R}$ s.t.

$$\begin{aligned}& x_n(e) \to x(e) \quad \text{as } n \to \infty \text{ for each fixed } e. \\ \Rightarrow\ & \sup_{e\in E} |x_n(e) - x(e)| \to 0 \quad \text{as } n \to \infty \\ \Rightarrow\ & \bar{d}(x_n, x) \to 0 \quad \text{as } n \to \infty\end{aligned}$$

Hence the proof. □

Proposition 2.7. *$\mathbb{R}$ can be embedded in $\tilde{\mathbb{R}}$, since each $r \in \mathbb{R}$ can be regarded as soft real number $\bar{r}$, defined by,-*

$$\bar{r}(e) = r \qquad \forall\ e \in E.$$

• *The additive and multiplicative identities of $\tilde{\mathbb{R}}$ are denoted by $\bar{0}$ and $\bar{1}$ respectively.*

$$\bar{0} : E \to \mathbb{R}\ s.t.\ \bar{0}(e) = 0 \quad and \quad \bar{1} : E \to \mathbb{R}\ s.t.\ \bar{1}(e) = 1 \qquad \forall\ e \in E$$

Definition 2.8. *A sequence space $\omega_{\tilde{\mathbb{R}}}$ of soft real numbers is said to be normal if,*
$\langle x_n \rangle \in \omega_{\tilde{\mathbb{R}}}$ *and* $\bar{d}(y_n, \bar{0}) \leq \bar{d}(x_n, \bar{0})\ \forall n \in \mathbb{N},\ \Rightarrow \langle y_n \rangle \in \omega_{\tilde{\mathbb{R}}}$.

Proposition 2.8. *[14] Let P_k be bounded sequence of strictly positive real numbers with $0 < p_k \leq \sup p_k = H,\ D = \max\{1, 2^{H-1}\}$ and $T = \max\{1, H\}$. Then*
1. $|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\}$
2. $|\lambda|^{p_k} \leq \max\{1, [\lambda]^H\}$.

Proposition 2.9. *The collection of all sequences of soft real numbers converging to $\bar{0}$, collection of all convergent sequences of soft real numbers and collection of all bounded sequences of soft real numbers form spaces and are denoted by $\tilde{c_o}, \tilde{c}$ and $\tilde{l}_\infty$*

3 The new class $F(\tilde{\mathbb{R}}, \mathsf{M}, p, s)$

Let F be a normal sequence space of real numbers with paranorm g_F which satisfies:
(i) g_F is a monotone paranorm.
(ii) Co-ordinatewise convergence implies convergence in paranorm g_F, which implies that for each $\{x^n\} = \{x^n_k\} \in F,\ n, k \in \mathbb{N}$,

$$x^n_k \to 0 \text{ as } n \to \infty \text{ (for each } k) \Rightarrow g_F(x^n) \to 0 \text{ as } n \to \infty.$$

Let M be an Orlicz function. We now define the new class of soft composite Orlicz sequence space as follows:

$$F(\tilde{\mathbb{R}}, \mathsf{M}, p, s) = \left\{(x_k) : x_k \in \tilde{\mathbb{R}}, k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k, \bar{0})}{\rho}\right)\right]^{p_k} \in F, \text{ for some } \rho > 0\right\} \tag{3.1}$$

where $s \geq 0$ and $\{p_k\}$ is a bounded sequence of strict positive reals with $\inf p_k > 0$.

Theorem 3.1. *$F(\tilde{\mathbb{R}}, \mathsf{M}, p, s)$ is linear space w.r.t. addition and scalar multiplication over the field of real numbers.*

Proof. We have, $\quad \bar{d}(x_k, \bar{0}) = \sup_{e \in E} |x_k(e) - \bar{0}(e)| = \sup_{e \in E} |x_k(e) - 0| = \sup_{e \in E} |x_k(e)|$
Therefore,

$$F(\tilde{\mathbb{R}}, \mathsf{M}, p, s) = \left\{(x_k) : x_k \in \tilde{\mathbb{R}}, k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e \in E} |x_k(e)|}{\rho}\right)\right]^{p_k} \in F, \text{ for some } \rho > 0\right\} \tag{3.2}$$

We have to show that $(F(\tilde{\mathbb{R}}, \mathsf{M}, p, s), +, .)$ is a linear space.

1. Closure property : Suppose $(x_k), (y_k) \in F(\tilde{\mathbb{R}}, \mathsf{M}, p, s)$. Then

$$\frac{\sup_{e\in E}|x_k(e)+y_k(e)|}{\rho} \leq \frac{\sup_{e\in E}|x_k(e)|}{\rho} + \frac{\sup_{e\in E}|y_k(e)|}{\rho}$$

$$\mathsf{M}\left(\frac{\sup_{e\in E}|x_k(e)+y_k(e)|}{\rho}\right) \leq \mathsf{M}\left(\frac{\sup_{e\in E}|x_k(e)|}{\rho} + \frac{\sup_{e\in E}|y_k(e)|}{\rho}\right)$$

$(\because$ M is increasing)

$$\leq \mathsf{M}\left(\frac{\sup_{e\in E}|x_k(e)|}{\rho}\right) + \mathsf{M}\left(\frac{\sup_{e\in E}|y_k(e)|}{\rho}\right)$$

(By the convexity of M)

$$\left[\mathsf{M}\left(\frac{\sup_{e\in E}|x_k(e)+y_k(e)|}{\rho}\right)\right]^{p_k} \leq \left[\mathsf{M}\left(\frac{\sup_{e\in E}|x_k(e)|}{\rho}\right) + \mathsf{M}\left(\frac{\sup_{e\in E}|y_k(e)|}{\rho}\right)\right]^{p_k}$$

$$\leq \left[\mathsf{M}\left(\frac{\sup_{e\in E}|x_k(e)|}{\rho}\right)\right]^{p_k} + \left[\mathsf{M}\left(\frac{\sup_{e\in E}|y_k(e)|}{\rho}\right)\right]^{p_k}$$

(By proposition 8)

$$k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|x_k(e)+y_k(e)|}{\rho}\right)\right]^{p_k} \leq k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|x_k(e)|}{\rho}\right)\right]^{p_k} + k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|y_k(e)|}{\rho}\right)\right]^{p_k}$$

Now $(x_k), (y_k) \in F(\tilde{\mathbb{R}}, \mathsf{M}, p, s) \Rightarrow$

$$k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|x_k(e)|}{\rho}\right)\right]^{p_k},\ k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|y_k(e)|}{\rho}\right)\right]^{p_k} \in F$$

F is space. Therefore,

$$k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|x_k(e)|}{\rho}\right)\right]^{p_k} + k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|y_k(e)|}{\rho}\right)\right]^{p_k} \in F$$

Also F is normal. Therefore,

$$k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|x_k(e)+y_k(e)|}{\rho}\right)\right]^{p_k} \leq k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|x_k(e)|}{\rho}\right)\right]^{p_k} + k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|y_k(e)|}{\rho}\right)\right]^{p_k}$$

Which implies that

$$k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|x_k(e)+y_k(e)|}{\rho}\right)\right]^{p_k}\in F$$

Hence, $F(\tilde{\mathbb{R}},\mathsf{M},p,s)$ is closed.

2. Associativity: $(x_k),(y_k),(z_k)\in F(\tilde{\mathbb{R}},\mathsf{M},p,s)\Rightarrow (x_k),(y_k),(z_k)\in\tilde{\omega}$ (the space of sequences of soft real numbers).
Therefore, $[(x_k)+(y_k)]+(z_k)=(x_k)+[(y_k)+(z_k)]$

3. Existence of identity: For any $(x_k)\in F(\tilde{\mathbb{R}},\mathsf{M},p,s)$

$$(\bar{0})=k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|\bar{0}_k(e)|}{\rho}\right)\right]^{p_k}\leq k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|x_k(e)|}{\rho}\right)\right]^{p_k}$$

Implies that,

$$k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|\bar{0}_k(e)|}{\rho}\right)\right]^{p_k}\in F$$

Implies that $(\bar{0})\in F(\tilde{\mathbb{R}},\mathsf{M},p,s)$
Also,

$$(\bar{0})+(x_k)=(x_k)=(\bar{0})+(x_k)$$

Therefore, $(\bar{0})$ is identity element.

4. Existence of inverse:

$$(x_k)\in F(\tilde{\mathbb{R}},\mathsf{M},p,s)\Rightarrow k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|x_k(e)|}{\rho}\right)\right]^{p_k}\in F$$

$$\Rightarrow k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|-x_k(e)|}{\rho}\right)\right]^{p_k}\in F\quad(\because |x_k(e)|=|-x_k(e)|)$$

$$\therefore (-x_k)\in F(\tilde{\mathbb{R}},\mathsf{M},p,s)\text{ such that }(x_k)+(-x_k)=(-x_k)+(x_k)=(\bar{0})$$

5. Commutativity:

$$(x_k),(y_k)\in F(\tilde{\mathbb{R}},\mathsf{M},p,s)\Rightarrow (x_k),(y_k)\in\omega$$
$$\Rightarrow (x_k)+(y_k)=(y_k)+(x_k)$$

6. Scalar multiplication:

$$(x_k)\in F(\tilde{\mathbb{R}},\mathsf{M},p,s)\Rightarrow k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|x_k(e)|}{\rho}\right)\right]^{p_k}\in F$$

$$\because\quad k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|\alpha.x_k(e)|}{|\alpha|.\rho}\right)\right]^{p_k}=k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|x_k(e)|}{\rho}\right)\right]^{p_k}$$

$$\Rightarrow (\alpha.x_k)\in F(\tilde{\mathbb{R}},\mathsf{M},p,s)\qquad\forall\,\alpha\in\mathbb{R}$$

7. Distributive law: $x=(x_k), y=(y_k)\in F(\tilde{\mathbb{R}},\mathsf{M},p,s)\Rightarrow x,\ y\in\omega$. Then for any $\alpha,\beta\in\mathbb{R}$,
(i) $\alpha(\beta x)=(\alpha\beta)x$
(ii) $\alpha(x+y)=\alpha x+\alpha y$
(iii) $(\alpha+\beta)x=\alpha x+\beta x$

8. Multiplicative identity: F is space, its multiplicative identity is sequence (1).

$$\left(k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|\bar{1}_k(e)|}{\rho_0}\right)\right]^{p_k}\right)=(k^{-s})\le(1)\qquad \text{setting}\,\rho_0\text{ s.t. }\mathsf{M}\left(\frac{1}{\rho_0}\right)=1$$

Normality of F implies $\left(k^{-s}\left[\mathsf{M}\left(\frac{\sup_{e\in E}|\bar{1}_k(e)|}{\rho_0}\right)\right]^{p_k}\right)\in F\Rightarrow(\bar{1})\in F(\tilde{\mathbb{R}},\mathsf{M},p,s)$

$$(x_k).(\bar{1})=(\bar{1}).(x_k)=(x_k)$$

Hence $F(\tilde{\mathbb{R}},\mathsf{M},p,s)$ is a linear space. □

Theorem 3.2. *The space $F(\tilde{\mathbb{R}},\mathsf{M}_1+\mathsf{M}_2,p,s)$ contains the common subspace of $F(\tilde{\mathbb{R}},\mathsf{M}_1,p,s)$ and $F(\tilde{\mathbb{R}},\mathsf{M}_2,p,s)$, where M_1 and M_2 are Orlicz functions.*

Proof. Let $(x_n)\in F(\tilde{\mathbb{R}},\mathsf{M}_1,p,s),F(\tilde{\mathbb{R}},\mathsf{M}_2,p,s)$.
Then

$$n^{-s}.\left(\mathsf{M}_1\left(\frac{\bar{d}(x_n,\bar{0})}{\rho_1}\right)\right)^{p_n},\quad n^{-s}.\left(\mathsf{M}_1\left(\frac{\bar{d}(x_n,\bar{0})}{\rho_2}\right)\right)^{p_n}\in F\qquad\text{for some }\rho_1,\rho_2>0$$

Taking $\rho=\max\rho_1,\rho_2$,

$$n^{-s}.\left(\mathsf{M}_1\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right)^{p_n},\quad n^{-s}.\left(\mathsf{M}_1\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right)^{p_n}\in F$$

Therefore,

$$n^{-s}.\left[\left(\mathsf{M}_1\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right)^{p_n}+\left(\mathsf{M}_1\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right)^{p_n}\right]\in F$$

Again taking, $D=\max\{1,2^{\sup P_n-1}\}$, we get

$$n^{-s}\left[(\mathsf{M}_1+\mathsf{M}_2)\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right]^{p_n}\le n^{-s}.D.\left[\left(\mathsf{M}_1\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right)^{p_n}+\left(\mathsf{M}_2\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right)^{p_n}\right]$$

Normality of F implies $n^{-s}\left[(\mathsf{M}_1+\mathsf{M}_2)\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right]^{p_n}\in F$.
Therefore, $(x_n)\in F(\tilde{\mathbb{R}},\mathsf{M}_1+\mathsf{M}_2,p,s)$. Hence the proof. □

Theorem 3.3. *For an Orlicz function* M

$$c_0(\tilde{\mathbb{R}},\mathsf{M},p,s)\subset c(\tilde{\mathbb{R}},\mathsf{M},p,s)\subset l_\infty(\tilde{\mathbb{R}},\mathsf{M},p,s).$$

Proof. A sequence $(x_n)\in c_0(\tilde{\mathbb{R}},\mathsf{M},p,s)\implies n^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right]^{p_n}\in c_0$
That is the sequence $n^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right]^{p_n}$ converges to zero, means $n^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right]^{p_n}$ is a convergent sequence and hence it belongs to the space c, cosequently $(x_n)\in c(\tilde{\mathbb{R}},\mathsf{M},p,s)$.
Also a sequence $(x_n)\in c(\tilde{\mathbb{R}},\mathsf{M},p,s)$ implies $\left(n^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right]^{p_n}\right)$ is convergent. Since every convergent sequence is bounded. Therefore, $\left(n^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right]^{p_n}\right)$ is bounded and this implies $(x_n)\in l_\infty(\tilde{\mathbb{R}},\mathsf{M},p,s)$.
Hence proved. □

Theorem 3.4. *If* $\mathsf{M}_1, \mathsf{M}_2$ *satisfy* Δ_2 *condition. then,*

$$F(\tilde{\mathbb{R}}, \mathsf{M}_1, p, s) \subseteq F(\tilde{\mathbb{R}}, \mathsf{M}_2 o \mathsf{M}_1, p, s).$$

Proof. Let $(x_n) \in F(\tilde{\mathbb{R}}, \mathsf{M}_1, p, s)$. For $\delta \in (0,1)$, define

$$\mathbb{N}_\delta = \left\{ n \in \mathbb{N} : \mathsf{M}_1\left(\frac{\bar{d}(x_n, \bar{0})}{\rho}\right) \leq \delta \right\} \qquad \text{for some } \delta > 0$$

When $n \in \mathbb{N}_\delta$,
By Continuity of M_2, for given $\epsilon > 0$ there exists $\delta \in (0,1)$ such that

$$\mathsf{M}_2\left(\mathsf{M}_1\left(\frac{\bar{d}(x_n, \bar{0})}{\rho}\right)\right) < \epsilon \qquad \text{whenever} \quad \mathsf{M}_1\left(\frac{\bar{d}(x_n, \bar{0})}{\rho}\right) < \delta$$

This implies that there exists $\delta \in (0,1)$ such that,

$$n^{-s}\left[\mathsf{M}_2\left(\mathsf{M}_1\left(\frac{\bar{d}(x_n, \bar{0})}{\rho}\right)\right)\right]^{p_n} < n^{-s}(\epsilon)^H \tag{3.3}$$

For $n \notin \mathbb{N}_\delta$,

$$\begin{aligned} \mathsf{M}_1\left(\frac{\bar{d}(x_n, \bar{0})}{\rho}\right) &< \frac{1}{\delta}\mathsf{M}_1\left(\frac{\bar{d}(x_n, \bar{0})}{\rho}\right) \\ &< 1 + \frac{1}{\delta}\mathsf{M}_1\left(\frac{\bar{d}(x_n, \bar{0})}{\rho}\right) \end{aligned}$$

$\mathsf{M}_2'(x) \geq 0 \implies$

$$\mathsf{M}_2\left(\mathsf{M}_1\left(\frac{\bar{d}(x_n, \bar{0})}{\rho}\right)\right) < \mathsf{M}_2\left(1 + \frac{1}{\delta}\mathsf{M}_1\left(\frac{\bar{d}(x_n, \bar{0})}{\rho}\right)\right)$$

$\mathsf{M}_2''(x) \geq 0 \implies$

$$\mathsf{M}_2\left(\mathsf{M}_1\left(\frac{\bar{d}(x_n, \bar{0})}{\rho}\right)\right) < \mathsf{M}_2(1) + \mathsf{M}_2\left(\frac{1}{\delta}\mathsf{M}_1\left(\frac{\bar{d}(x_n, \bar{0})}{\rho}\right)\right) \tag{3.4}$$

Since, M_2 satisfies Δ_2 condition. Therefore, there exist $K_1, K_2 > 0$ such that

$$\mathsf{M}_2(1.1) = \left[\frac{K_1}{2}\frac{1}{\delta}\mathsf{M}_1\left(\frac{\bar{d}(x_n, \bar{0})}{\rho}\right)\right].1.\mathsf{M}_2(1)$$

and

$$\mathsf{M}_2\left(\frac{1}{\delta}\mathsf{M}_1\left(\frac{\bar{d}(x_n, \bar{0})}{\rho}\right).1\right) \leq \frac{K_2}{2}.\left(\frac{1}{\delta}\mathsf{M}_1\left(\frac{\bar{d}(x_n, \bar{0})}{\rho}\right)\right).\mathsf{M}_2(1)$$

taking, $K = \max\{K_1, K_2\}$ then by equation (3.3), we have,

$$\mathsf{M}_2\left(\mathsf{M}_1\left(\frac{\bar{d}(x_n, \bar{0})}{\rho}\right)\right) < K\delta^{-1}\mathsf{M}_2(1)\mathsf{M}_1\left(\frac{\bar{d}(x_n, \bar{0})}{\rho}\right)$$

$$n^{-s}\left[\mathsf{M}_2\left(\mathsf{M}_1\left(\frac{\bar{d}(x_n, \bar{0})}{\rho}\right)\right)\right]^{p_n} < n^{-s}\left[K\delta^{-1}\mathsf{M}_2(1)\right]^{p_n}\left[\mathsf{M}_1\left(\frac{\bar{d}(x_n, \bar{0})}{\rho}\right)\right]^{p_n}$$

$$n^{-s}\left[\mathsf{M}_2\left(\mathsf{M}_1\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right)\right]^{p_n} < n^{-s}D_1\left[\mathsf{M}_1\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right]^{p_n} \tag{3.5}$$

Where $D_1 = \max\{1, \left[K\delta^{-1}\mathsf{M}_2(1)\right]^{p_n}\}$. From (3.3) and (3.5),

$$n^{-s}\left[\mathsf{M}_2\left(\mathsf{M}_1\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right)\right]^{p_n} < \frac{1}{2}\left[n^{-s}(\epsilon)^H + n^{-s}D_1\left[\mathsf{M}_1\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right]^{p_n}\right]$$

Since F is normal space, therefore $n^{-s}\left[\mathsf{M}_2\left(\mathsf{M}_1\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right)\right]^{p_n} \in F$.
Hence $(x_n) \in F(\tilde{\mathbb{R}}, \mathsf{M}_2 o \mathsf{M}_1, p, s)$.
Which follows the proof. □

Theorem 3.5. *$F(\tilde{\mathbb{R}}, \mathsf{M}, p, s)$ is a normal space.*

Proof. To show $F(\tilde{\mathbb{R}}, \mathsf{M}, p, s)$ is normal, we have to show that $(x_n) \in F(\tilde{\mathbb{R}}, \mathsf{M}, p, s)$, $\bar{d}(y_n,\bar{0}) \leq \bar{d}(x_n,\bar{0}) \implies (y_n) \in F(\tilde{\mathbb{R}}, \mathsf{M}, p, s)$.

Let $(x_n) \in F(\tilde{\mathbb{R}}, \mathsf{M}, p, s)$. Then, $\exists\ \rho > 0$

$$n^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right]^{p_n} \in F$$

and $\bar{d}(y_n,\bar{0}) \leq \bar{d}(x_n,\bar{0})$. Then,

$$\frac{\bar{d}(y_n,\bar{0})}{\rho} \leq \frac{\bar{d}(x_n,\bar{0})}{\rho}$$

$$\mathsf{M}\left(\frac{\bar{d}(y_n,\bar{0})}{\rho}\right) \leq \mathsf{M}\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right) \qquad (\because \mathsf{M}' \geq 0)$$

$$n^{-s}.\left[\mathsf{M}\left(\frac{\bar{d}(y_n,\bar{0})}{\rho}\right)\right]^{p_n} \leq n^{-s}.\left[\mathsf{M}\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right]^{p_n} \in F$$

Since F is normal, $\left(n^{-s}.\left[\mathsf{M}\left(\frac{\bar{d}(y_n,\bar{0})}{\rho}\right)\right]^{p_n}\right) \in F$. Therefore, $(y_n) \in F(\tilde{\mathbb{R}}, \mathsf{M}, p, s)$.
Hence $F(\tilde{\mathbb{R}}, \mathsf{M}, p, s)$ is normal space. □

Theorem 3.6. *$F(\tilde{\mathbb{R}}, \mathsf{M}, p, s)$ is symmetric space.*

Proof. Let (x_n) be squence of soft real numbers in $F(\tilde{\mathbb{R}}, \mathsf{M}, p, s)$ and (y_{m_n}) be another sequence such that for each $n \in \mathbb{N}$, $(x_n) = (y_{m_n})$. Since,

$$\left(n^{-s}.\left[\mathsf{M}\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right]^{p_n}\right) \in F$$

And

$$\left(n^{-s}.\left[\mathsf{M}\left(\frac{\bar{d}(x_n,\bar{0})}{\rho}\right)\right]^{p_n}\right) = \left(n^{-s}.\left[\mathsf{M}\left(\frac{\bar{d}(y_{m_n},\bar{0})}{\rho}\right)\right]^{p_n}\right)$$

Therefore,

$$\left(n^{-s}.\left[\mathsf{M}\left(\frac{\bar{d}(y_{m_n},\bar{0})}{\rho}\right)\right]^{p_n}\right) \in F \implies (y_{m_n}) \in F(\tilde{\mathbb{R}}, \mathsf{M}, p, s)$$

Hence $F(\tilde{\mathbb{R}}, M, p, s)$ is symmetric. □

4 New Metric On the new class defined in section 3

We define a metric on $F(\tilde{\mathbb{R}}, \mathsf{M}, p, s)$ as follows:
For $x = (x_k),\ y = (y_k) \in F(\tilde{\mathbb{R}}, \mathsf{M}, p, s)$,

$$d(x, y) = \inf\left\{\rho^{p_k/T} > 0 : \left[g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k, y_k)}{\rho}\right)\right]^{p_k}\right)\right]^{1/T} \leq 1\right\} \tag{4.1}$$

Where, $T = \max\{1, H\},\ H = \sup_k p_k < \infty$ and $\inf p_k > 0$.

Now, we show that d is a metric on $F(\tilde{\mathbb{R}}, \mathsf{M}, p, s)$.
(i) Since $\inf\{\rho^{p_k/T} > 0\} \geq 0$. Therefore, $\quad d(x, y) \geq 0$
(ii) $x = y \Rightarrow \quad x_k = y_k \Rightarrow \bar{d}(x_k, y_k) = 0$ and also,

$$\left[g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{0}{\rho}\right)\right]^{p_k}\right)\right]^{1/T} = 0 < 1 \quad \text{for all} \quad \rho > 0$$

Therefore, $0 = \inf\{\rho > 0\} = \inf\{\rho^{p_k/T} > 0\} = d(x, y)$
Further,

$$\begin{aligned} d(x, y) = 0 &\Rightarrow \inf_\rho\left\{\rho^{p_k/T} > 0 : \left[g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k, y_k)}{\rho}\right)\right]^{p_k}\right)\right]^{1/T} \leq 1\right\} = 0 \\ &\Rightarrow \rho^{p_k/T} \to 0 \quad \text{or} \quad \rho \to 0 \\ &\Rightarrow \left[g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k, y_k)}{\rho}\right)\right]^{p_k}\right)\right]^{1/T} \leq 1 \quad \text{at} \quad \rho \to 0 \end{aligned}$$

(for, $\{p_k = 1\}, s = 0$ if $x_k \neq y_k$, we get, $g_F(\xi), \xi \to \infty$ which fails to be ≤ 1.)

$$\therefore \quad x_k = y_k \quad \forall n \in \mathbb{N}$$
$$\Rightarrow \quad x = y$$

(iii)Obviously

$$d(x, y) = d(y, x) \qquad \forall\ x, y \in F(\tilde{\mathbb{R}}, \mathsf{M}, p, s)$$

(iv) Let, $x = (x_k),\ y = (y_k),\ z = (z_k)$

$$\begin{aligned} \bar{d}(x_k, z_k) &\leq \bar{d}(x_k, y_k) + \bar{d}(y_k, z_k) \\ \frac{\bar{d}(x_k, z_k)}{\rho} &\leq \frac{\bar{d}(x_k, y_k)}{\rho} + \frac{\bar{d}(y_k, z_k)}{\rho} \qquad (\because \rho > 0) \\ \mathsf{M}\left(\frac{\bar{d}(x_k, z_k)}{\rho}\right) &\leq \mathsf{M}\left(\frac{\bar{d}(x_k, y_k)}{\rho} + \frac{\bar{d}(y_k, z_k)}{\rho}\right) \\ &\leq \mathsf{M}\left(\frac{\bar{d}(x_k, y_k)}{\rho}\right) + \mathsf{M}\left(\frac{\bar{d}(y_k, z_k)}{\rho}\right) \text{ (convexity of M)} \end{aligned}$$

$$\left[\mathsf{M}\left(\frac{\bar{d}(x_k,z_k)}{\rho}\right)\right]^{p_k} \le \left[\mathsf{M}\left(\frac{\bar{d}(x_k,y_k)}{\rho}\right)+\mathsf{M}\left(\frac{\bar{d}(y_k,z_k)}{\rho}\right)\right]^{p_k}$$

$$\le \left[\mathsf{M}\left(\frac{\bar{d}(x_k,y_k)}{\rho}\right)\right]^{p_k}+\left[\mathsf{M}\left(\frac{\bar{d}(y_k,z_k)}{\rho}\right)\right]^{p_k} \qquad \text{(By proposition 8)}$$

$$k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k,z_k)}{\rho}\right)\right]^{p_k} \le k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k,y_k)}{\rho}\right)\right]^{p_k}+k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(y_k,z_k)}{\rho}\right)\right]^{p_k}$$

$$g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k,z_k)}{\rho}\right)\right]^{p_k}\right) \le g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k,y_k)}{\rho}\right)\right]^{p_k}+k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(y_k,z_k)}{\rho}\right)\right]^{p_k}\right)$$

$$(\because g_F \text{ is monotonic increasing})$$

$$\le g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k,y_k)}{\rho}\right)\right]^{p_k}\right)+g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(y_k,z_k)}{\rho}\right)\right]^{p_k}\right)$$

$$(\text{triangle prop. of } g_F)$$

$$\left[g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k,z_k)}{\rho}\right)\right]^{p_k}\right)\right]^{1/T} \le \left[g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k,y_k)}{\rho}\right)\right]^{p_k}\right)\right.$$
$$\left.+\,g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(y_k,z_k)}{\rho}\right)\right]^{p_k}\right)\right]^{1/T}$$
$$\le \left[g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k,y_k)}{\rho}\right)\right]^{p_k}\right)\right]^{1/T}$$
$$+\left[g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(y_k,z_k)}{\rho}\right)\right]^{p_k}\right)\right]^{1/T}$$

$$(\because 1/T \in [0,1])$$

$$\inf\left\{\rho^{p_k/T}>0:\left[g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k,z_k)}{\rho}\right)\right]^{p_k}\right)\right]^{1/T}\le 1\right\}$$
$$\le \inf\left\{\rho^{p_k/T}>0:\left[g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k,y_k)}{\rho}\right)\right]^{p_k}\right)\right]^{1/T}\right.$$
$$\left.+\left[g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(y_k,z_k)}{\rho}\right)\right]^{p_k}\right)\right]^{1/T}\le 1\right\}$$
$$\le \inf\left\{\rho_1^{p_k/T}>0:\left[g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k,y_k)}{\rho_1}\right)\right]^{p_k}\right)\right]^{1/T}\le 1\right\}$$
$$+\inf\left\{\rho_2^{p_k/T}>0:\left[g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(y_k,z_k)}{\rho_2}\right)\right]^{p_k}\right)\right]^{1/T}\le 1\right\}$$

where ρ_1, ρ_2 are such that $\rho_1+\rho_2=\rho$

$$d(x,z)\le d(x,y)+d(y,z)$$

Therefore, d is a metric on $F(\tilde{\mathbb{R}},\mathsf{M},p,s)$.

Theorem 4.1. *$F(\tilde{\mathbb{R}}, \mathsf{M}, p, s)$ with metric d is a complete metric space.*

Proof. Let (x^i) be a cauchy sequence in $F(\tilde{\mathbb{R}}, \mathsf{M}, p, s)$. Then,

$$d(x^i, x^j) \to 0 \qquad \text{as} \quad i, j \to \infty$$

Or,

$$\inf\left\{\rho^{p_k/T} > 0 : \left[g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k^i, x_k^j)}{\rho}\right)\right]^{p_k}\right)\right]^{1/T} \leq 1\right\} \to 0 \ (or \ = 0) \quad \text{as}\, i, j \to \infty$$

We claim that (x_k^i) is Cauchy sequence in $\tilde{\mathbb{R}}$ for each k.
Suppose, (x_k^i) is not a Cauchy sequence i.e. $\bar{d}(x_k^i, x_k^j)$ does not converge to 0 for $i, j \to \infty$.
Then

$$\left(\frac{\bar{d}(x_k^i, x_k^j)}{\rho}\right) \to \infty \qquad \text{as}\ \rho \to 0 \qquad \text{for}\ i, j \to \infty$$

which gives the failure of condition $\left[g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k^i, x_k^j)}{\rho}\right)\right]^{p_k}\right)\right]^{1/T} \leq 1$

$$\Rightarrow \quad \inf\left\{\rho^{p_k/T} > 0 : \left[g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k^i, x_k^j)}{\rho}\right)\right]^{p_k}\right)\right]^{1/T} \leq 1\right\} \neq 0$$

which is contradiction. Therefore, (x_k^i) is Cauchy sequence.

Since $\tilde{\mathbb{R}}$ is complete.

$$\exists \qquad x = (x_k), \ x_k \in \tilde{\mathbb{R}} \quad \text{s.t.} \quad x_k^i \to x_k \qquad \text{for each}\ k$$

i.e.

$$\bar{d}(x_k^i, x_k) \to 0 \implies \left[\mathsf{M}\left(\frac{\bar{d}(x_k^i, x_k^j)}{\rho}\right)\right] \to 0 \qquad (\because M \text{ is continuous})$$

for each k and $\rho > 0$ as $i \to \infty$
Or

$$\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k^i, x_k)}{\rho}\right)\right]^{p_k}\right) \to 0 \qquad \text{as}\ i \to \infty$$

$\because$ $\quad g_F$ is monotone paranorm. So, for each $\rho > 0$

$$\left(\left[g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k^i, x_k)}{\rho}\right)\right]^{p_k}\right)\right]^{1/T}\right) \to 0 \qquad \text{as}\ i \to \infty$$

$$\implies d(x^i, x) = \inf\left\{\rho^{p_k/T} > 0 : \left[g_F\left(k^{-s}\left[\mathsf{M}\left(\frac{\bar{d}(x_k^i, x_k)}{\rho}\right)\right]^{p_k}\right)\right]^{1/T} \leq 1\right\} = 0$$

Hence, $F(\tilde{\mathbb{R}}, \mathsf{M}, p, s)$ is complete metric space. □

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