A construction of admissible frame scaling sets on reducing subspaces of $L^2(\mathbb{R})$

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Abstract

In this paper, we construct admissible frame scaling sets for reducing subspace of $L^2(\mathbb{R})$ and also frame wavelet sets corresponding to this frame scaling set. Some examples of admissible frame scaling set on reducing subspace $H^2(\mathbb{R})$ is constructed with the help of frame scaling set on $L^2(\mathbb{R})$.

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1 Introduction

Let H be a Hilbert space. A family of elements $\{a_j : j \in \tau\}$ in H is called a *frame* for H if there exist constants c_1 and c_2, $0 < c_1 \leq c_2 < \infty$ such that for each $f \in H$, we have

$$c_1\|f\|^2 \leqslant \sum_{i\in\tau} |< f, a_j >|^2 \leqslant c_2\|f\|^2.$$

If supremum and infimum of all such numbers c_1 and c_2 are C_1 and C_2 respectively, then C_1 is called the *lower frame* bound and C_2 is called the *upper frame* bound of the frame $\{a_j : j \in \tau\}$. A frame is said to be the *tight frame* when $C_1 = C_2$ and the *normalized tight frame*(NT Frame) when $C_1 = C_2 = 1$ [3]. Any orthonormal basis in a Hilbert space is a normalized tight frame but all normalized tight frames are not necessarily an orthonormal basis[11].

Let $D : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, defined by $Df(\cdot) = \sqrt{2}f(2\cdot)$ and $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, defined by $T_k f(\cdot) = f(\cdot - k), k \in \mathbb{Z}$ then D and T_k are called *dilation* and *translation* operators respectively. Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then its Fourier transform is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-ix\xi}dx$$

for a.e $\xi \in \mathbb{R}$. Let $\mathbb{T} := [-\pi, \pi]$. Hilbert space of 2π periodic functions is denoted by $L^2(\mathbb{T})$ and inner product on $L^2(\mathbb{T})$ is defined by

$$< f, g >= \int_{\mathbb{T}} f(t)\overline{g(t)}dt,$$

where $f, g \in L^2(\mathbb{T})$. For a function f, the support of f is defined by

$$supp(f) = \{x \in \mathbb{R} : f(x) \neq 0\}.$$

Define, symbol $L^2_E := \{f \in L^2(\mathbb{R}) : supp(\hat{f}) \subset E\}$. With the condition $E \subset 2E$, L^2_E is a closed subspace of $L^2(\mathbb{R})$ called *reducing subspace* of $L^2(\mathbb{R})$. Most important example of reducing subspace of $L^2(\mathbb{R})$ is $H^2(\mathbb{R}) := \{f \in L^2(\mathbb{R}) : supp(\hat{f}) \subset [0, \infty)\}$, also known as Hardy space. For $\psi \in L^2_E$, let

$$\Psi_{m,n}(x) = \{D^m T^n \psi : m, n \in \mathbb{Z}\}.$$

If $\Psi_{m,n}$ is a frame for L^2_E, then the function ψ is called a frame wavelet for L^2_E [2]. Let $\psi \in L^2(\mathbb{R})$ defined by $\hat{\psi} = \chi_F$, where F is a measurable set of finite measure. If ψ so defined is a frame wavelet for $L^2(\mathbb{R})$, then the set F is called a *frame wavelet set*. Similarly F is defined as *tight (normalized) frame wavelet set* if ψ is a tight (normalized) frame wavelet.

Multiresolution analysis (MRA) was introduced by Mallat and Meyer and it is the most general method for the construction of orthonormal wavelets. A method to construct a class of wavelet sets is also discussed in [6]. The classical theory of MRA has been extended to frame multiresolution analysis (FMRA). FMRA theory was introduced by Benedetto and Li [1] and generalized in different spaces like $L^2(\mathbb{R}^d)$ and reducing subspaces in [4], [5], [7]. Some important frame wavelets can we constucted with this process of FMRA, we can see in [7], [8] that construction of smooth wavelets has been done with the help of wavelets arising from FMRA process. We can also construct FMRA on $L^2(\mathbb{R}^d)$ with the help of FMRA on $L^2(\mathbb{R})$. In this paper, section 2 constitute basic preliminaries and notations. Section 3 is devoted to a construction of admissible frame scaling sets for reducing subspaces of $L^2(\mathbb{R})$ and corresponding frame wavelet sets. We have also constructed frame scaling set for Hardy Space with the help of frame scaling set for $L^2(\mathbb{R})$. In section 4 we have constructed a class of examples of admissible frame scaling sets for $L^2(\mathbb{R})$ and for reducing subspace $H^2(\mathbb{R})$.

2 Notations and Preliminaries

Let $S \subset \mathbb{R}$ be a Lebesgue measurable set. For any $l \in \mathbb{Z}$, define

$$\tau(S) = \cup_{l\in\mathbb{Z}} (S \cap [2\pi l, 2\pi(l+1)) - 2\pi l)$$

and by $(S)_{\mathbb{T}}$ the set

$$(S)_{\mathbb{T}} := \{x \in \mathbb{T} : x = x' + k\}$$

for some $x' \in S$ and $k \in \mathbb{Z}$. For $\phi \in L^2(\mathbb{R})$, we denote by Φ the function

$$\Phi(\cdot) = \sum_{k\in\mathbb{Z}} |\hat{\phi}(\cdot - 2\pi k)|^2 \tag{2.1}$$

on $\mathbb{R}$, and by N the set

$$N := \{x \in \mathbb{T} : \Phi(x) = 0\}.$$

We use following notations

$$\Gamma := \{\xi \in \mathbb{T} : \Phi(2\xi) = 0, \Phi(\xi) > 0, \Phi(\xi + \pi) > 0\} \tag{2.2}$$

and $\Delta = \Gamma \cap [0, \pi]$. Also

$$T_1 = \{\xi \in \mathbb{T} : \Phi(\xi) > 0, \Phi(\xi + \pi) > 0\} \tag{2.3}$$

and

$$T_2 = \{\xi \in \mathbb{T} : \Phi(\xi) > 0, \Phi(\xi + \pi) = 0\}. \tag{2.4}$$

Definition 2.1. *[9] Let L^2_E be a reducing subspace of $L^2(\mathbb{R})$ and V_j be the sequence of closed*

subspaces of L^2_E, which satisfies following properties:
(i) $V_j \subset V_{j+1} \quad for \quad j \in \mathbb{Z}$,
(ii)$\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2_E$ and $\cap_{j\in\mathbb{Z}} V_j = \{0\}$,
(iii) $V_j = D^j V_0$ for $j \in \mathbb{Z}$,
(iv) $f \in V_0 \implies T_k f \in V_0$ for $k \in \mathbb{Z}$,
(v) there exists $\phi \in L^2_E$ such that $\{T_k\phi : k \in \mathbb{Z}\}$ is a frame for V_0.
Then the sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ is called FMRA for L^2_E.

The sequence $\{V_j\}_{j\in\mathbb{Z}}$ is called MRA for L^2_E if condition(v) is replaced by the condition that the set $\{T_k\phi : k \in \mathbb{Z}\}$ forms Riesz basis for L^2_E. In such case ϕ is said to be frame scaling function. In addition if $\{T_k\phi : k \in \mathbb{Z}\}$ is a tight frame (Normalized tight frame) for V_0, then $\{V_j\}_{j\in\mathbb{Z}}$ is tight(normalized) FMRA for L^2_E. For a measurable set $S \subset \mathbb{R}$ if the scaling function ϕ is given by $\hat{\phi} = \chi_S$, then we say that S is a frame scaling set.

Let us consider W_j, the orthogonal complement of V_j in V_{j+1} for each $j \in \mathbb{Z}$ such that

$$L^2_E = \oplus_{j\in\mathbb{Z}} W_j$$

If $\psi \in W_0$ such that $\{D^j T_k \psi : j, k \in \mathbb{Z}\}$ is a frame for L^2_E, then the function ψ is called frame wavelet associated with the FMRA for L^2_E [9].

Definition 2.2. *[2] Let $f \in L^2_E$, f is said to be refinable if there exists a 2π-periodic function $m_f \in L^2(\mathbb{T})$ such that*

$$\hat{f}(2\cdot) = m_f(\cdot)\hat{f}(\cdot)$$

on $\mathbb{R}$.

Theorem 2.1. *[9] Let $\phi \in L^2_E$ and V_j be the sequence of closed subspaces defined by $V_j := \overline{span}\{D^j T_k \phi : k \in \mathbb{Z}\}$ for $j \in \mathbb{Z}$. Then V_j is an FMRA for L^2_E if and only if*
(1) $A \le \Phi(\cdot) \le B$ on $\mathbb{T} \setminus N$,
(2) there exists $m_\phi \in L^2(\mathbb{T})$ such that $\hat{\phi}(2\cdot) = m_\phi(\cdot)\hat{\phi}(\cdot)$,
(3) $\cup_{j\in\mathbb{Z}} 2^j supp(\hat{\phi}) = E$.

If ϕ generates an MRA for $L^2(\mathbb{R})$, then there always exist a wavelet ψ corresponding to the scaling function ϕ, but in case of FMRA it is not necessary that there always exist a frame wavelet corresponding to FMRA generated by ϕ. In [2] a characterization for FMRA in $L^2(\mathbb{R})$ to admit a frame wavelet in $L^2(\mathbb{R})$ is given. This result is extended for reducing subspace of $L^2(\mathbb{R})$ in [9].

Theorem 2.2. *[9] Let $\phi \in L^2_E$ generates an FMRA. Then there exists a function $\psi \in L^2_E$ such that $\{D^j T_k \psi : j, k \in \mathbb{Z}\}$ forms a frame for L^2_E iff $|\Delta| = 0$.*

Theorem 2.3. *[9] Let ϕ generates FMRA for L^2_E with $|\Delta| = 0$. Define $m_\psi \in L^2(\mathbb{T})$ by*

$$m_\psi(\cdot) = \begin{cases} e^{-2\pi i} T_\pi(\overline{m}_\phi \Phi)(\cdot) & on \quad T_1 \\ 1 & on \quad T_2 \cap \{\xi \in \mathbb{T} : m_\phi(\xi) = 0\} \\ 0 & otherwise. \end{cases}$$

on $\mathbb{T}$, define $\psi \in W_0$ via its Fourier transform $\hat{\psi}(\cdot) = m_\psi(\frac{\cdot}{2})\hat{\phi}(\frac{\cdot}{2})$. Then $\{D^j T_k \psi : j, k \in \mathbb{Z}\}$ forms a frame for L^2_E.

Definition 2.3. *[10] An FMRA $\{V_j\}_{j\in\mathbb{Z}}$ is said to be admissible if there exist frame wavelet for $\{V_j\}_{j\in\mathbb{Z}}$. Let ϕ be an admissible scaling function given by $\hat{\phi} = \chi_S$, S is said to be an admissible frame scaling set if there exist frame wavelet set corresponding to S.*

In [11], a characterization of frame scaling sets is given as follows

Theorem 2.4. *[11] Let S be a bounded closed subset of $\mathbb{R}$. Then there exist a frame scaling function ϕ with supp $\hat{\phi} = S$ iff*
(1)$S \subset 2S$ (2) $\cup_{m\in\mathbb{Z}} 2^m S = \mathbb{R}$ (3)$(S - \frac{1}{2}S) \cap (\frac{1}{2}S + 2\pi k) = \emptyset$ (4) $\tau(S) \subset [0, 2\pi)$.

In [12], we have constructed a class of three interval frame scaling sets as follows

Theorem 2.5. *[12] Four points α, β, γ, and δ in $\mathbb{R}$ with $0 < \alpha < \beta < \gamma < 2\pi < \delta < 2\pi + \alpha$ provide three kinds of frame scaling set described as follows:*
(1) $S = [\gamma - 2\pi, \delta - 2\pi) \cup [\alpha + 2\pi, \beta + 2\pi) \cup [\beta, \gamma)$,
where (i) $2\gamma \geq \beta + 2\pi$ (ii) $2\beta \leq \alpha + 2\pi$ (iii) $2\delta - 4\pi \geq \gamma$.
(2) $S = [\alpha - 2\pi, \beta - 2\pi) \cup [\gamma - 2\pi, \delta - 2\pi) \cup [\beta, \gamma)$,
where (i) $2\gamma - 4\pi \leq \alpha - 2\pi$ (ii) $2\delta - 4\pi \geq \gamma$.
(3) $S = [\beta - 4\pi, \gamma - 4\pi) \cup [\alpha - 2\pi, \beta - 2\pi) \cup [\gamma - 2\pi, \delta - 2\pi)$,
where (i) $2\gamma - 2\pi \leq \alpha$ (ii) $2\beta \geq \gamma$ (iii) $2\alpha \leq \beta$.

3 Admissible frame scaling sets for reducing subspace

In [12] we have constructed a class of frame scaling sets S, where S is bounded measurable subset of $\mathbb{R}$. Function ϕ defined by $\hat{\phi} = \chi_S$, generates FMRA together with function $m_\phi \in L^2(\mathbb{T})$ defined by $m_\phi = \chi_{\frac{S}{2}}$. It is not necessary that there always exists frame wavelet corresponding to an FMRA generated by ϕ, we can see this by following example.
Example 3.1 Let us consider a function $\phi \in L^2(\mathbb{R})$ defined by $\hat{\phi} = \chi_S$, where

$$S = \left[\frac{-\pi}{4}, \frac{7\pi}{8}\right) \cup \left[\frac{3\pi}{2}, \frac{7\pi}{4}\right) \cup \left[3\pi, \frac{7\pi}{2}\right).$$

From equation(2.2), we have $\Gamma = \left[\frac{\pi}{8}, \frac{7\pi}{32}\right)$ and $\Delta = \Gamma \cap [0, \pi] = \left[\frac{\pi}{8}, \frac{7\pi}{32}\right)$
Therefore we have $|\Delta| \neq 0$. Thus by theorem(2.2) we can say that there does not exist any frame wavelet corresponding to FMRA generated by $\phi \in L^2(\mathbb{R})$. Next we prove a theorem which admits a frame wavelet corresponding to an FMRA.

Theorem 3.1. *Let $S \subset E$ be a measurable set. A function $\phi \in L^2_E$ is defined by $\hat{\phi} = \chi_S$, where S satisfies following properties*
(1)$S \subset 2S$ (2) $\cup_{j\in\mathbb{Z}} 2^j S = E$ (3)$S \subset [-\pi/2, \pi/2)$.
Then S is an admissible frame scaling set and its corresponding function ϕ generates an admissible FMRA for L^2_E.

Proof. Function $\phi \in L^2_E$ is defined by $\hat{\phi} = \chi_S$. For ϕ to generate FMRA it has to satisfy all the conditions of theorem(2.1).

$$\sum_{k\in\mathbb{Z}} |\hat{\phi}(\cdot - 2k\pi)|^2 = \sum_{k\in\mathbb{Z}} \chi_S(\cdot - 2k\pi) = \sum_{k\in\mathbb{Z}} \chi_{S+2k\pi}(\cdot).$$

As $S \subset \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$, we have $(S + 2k\pi) \cap (S + 2j\pi) = \emptyset$ when $j \neq k$. Therefore $\sum_{k\in\mathbb{Z}} |\hat{\phi}(\cdot - 2k\pi)|^2 = 1$ on $\mathbb{T} \setminus N$. Let us define m_ϕ by

$$m_\phi(\cdot) = \begin{cases} 1 & on \quad \frac{S}{2} \\ 0 & on \quad \mathbb{T} \setminus \frac{S}{2}. \end{cases}$$

Then $m_\phi \in L^2(\mathbb{T})$. By extending m_ϕ periodically it becomes 2π-periodic function on $\mathbb{R}$. It is easy to check that the refinable condition $\hat{\phi}(2\cdot) = m_\phi(\cdot)\hat{\phi}(\cdot)$ is satisfied for this m_ϕ. As $S \subset 2S$ and S contains that neighbourhood of origin which is contained in E, hence $\cup_{j\in\mathbb{Z}} 2^j S = E$. Thus by theorem(2.1), ϕ generates an FMRA $\{V_j\}_{j\in\mathbb{Z}}$ for L^2_E.

Now we show that there exist a frame wavelet corresponding to FMRA.

$$\Phi(\xi) = \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi - 2k\pi)|^2 = \sum_{k\in\mathbb{Z}} \chi_S(\xi - 2k\pi) = \chi_{\cup_{k\in\mathbb{Z}}(S+2k\pi)}(\xi)$$

.

$$supp(\Phi(\xi)) = \left[\frac{-\pi}{2}, \frac{\pi}{2}\right) \quad when \quad \xi \in \mathbb{T}.$$

Similarly

$$\Phi(2\xi) = \chi_{\cup_{k\in\mathbb{Z}}(\frac{1}{2}S+k\pi)}(\xi)$$

$$supp(\Phi(2\xi)) = \left[-\pi, \frac{-3\pi}{4}\right) \cup \left[\frac{-\pi}{4}, \frac{\pi}{4}\right) \cup \left[\frac{3\pi}{4}, \pi\right) \quad when \quad \xi \in \mathbb{T}.$$

And

$$\Phi(\xi + \pi) = \chi_{\cup_{k\in\mathbb{Z}}(S+\pi+2k\pi)}(\xi) = \chi_{\cup_{k\in\mathbb{Z}}(S+(2k+1)\pi)}(\xi).$$

For $k \neq 0, -1$, $supp(\Phi(\xi+\pi)) \notin \mathbb{T}$, for $k = 0$, $supp(\Phi(\xi+\pi)) = \left[\frac{\pi}{2}, \pi\right)$ and for $k = -1$, $supp(\Phi(\xi + \pi)) = \left[-\pi, \frac{-\pi}{2}\right)$. So for this, set Γ defined in equation(2.2) is an empty set, which implies that $|\Gamma| = 0$ and also $|\Delta| = 0$. Thus by theorem (2.2) we can say that there must exist a frame wavelet corresponding to FMRA generated by ϕ, i.e. S is an admissible frame scaling set. □

Theorem 3.2. *Let $\phi \in L^2_E$ be a function defined by $\hat{\phi} = \chi_S$, where S satisfies all the conditions of theorem(3.1). Then frame wavelet set corresponding to admissible frame scaling set S is of the form $2S \setminus S$.*

Proof. Since $\hat{\phi} = \chi_S$, where S satisfies conditions of theorem(3.1), then ϕ generates admissible FMRA i.e. it admits frame wavelet. Now to find $m_\psi \in L^2(\mathbb{T})$ we use theorem(2.3). Since $T_1 = S \cap ((S+\pi) \cup (S-\pi)) = \emptyset$, and

$T_2 = S \cap \{([0,\pi] \setminus (S+\pi)) \cup ([-\pi, 0] \setminus (S-\pi))\} = S$, we define $m_\psi \in L^2(\mathbb{T})$ by

$$m_\psi(\cdot) = \begin{cases} 1 & on \quad S \setminus \frac{S}{2} \\ 0 & on \quad \mathbb{T} \setminus (S \setminus \frac{S}{2}). \end{cases}$$

Fourier transform of $\psi \in L^2_E$ is given by

$$\begin{aligned} \hat{\psi}(\cdot) &= m_\psi\left(\frac{\cdot}{2}\right)\hat{\phi}\left(\frac{\cdot}{2}\right) \\ &= \chi_{S\setminus\frac{1}{2}S}\left(\frac{\cdot}{2}\right)\chi_S\left(\frac{\cdot}{2}\right) \\ &= \chi_{2S\setminus S}(\cdot)\chi_{2S}(\cdot) \\ &= \chi_{(2S\setminus S)\cap 2S}(\cdot) \\ &= \chi_{2S\setminus S}(\cdot). \end{aligned}$$

Thus, frame wavelet set corresponding to admissible frame scaling set S is given by $2S \setminus S$. □

Example 3.2. Let $S = \left[\frac{-3\pi}{8}, \frac{\pi}{8}\right)$ and ϕ be a function defined by $\hat{\phi} = \chi_S$. We can easily check that

$S \subset 2S$ and $\cup_{j\in\mathbb{Z}} 2^j S = \mathbb{R}$. Thus by theorem 3.1, S is an admissible frame scaling set.

Example 3.3. Define a function $\hat{\phi} = \chi_{[-a,a)}$ for some $a \in [0, \frac{\pi}{2})$, then by using theorem(3.1) we can check that ϕ generates FMRA for $L^2(\mathbb{R})$ and admits a frame wavelet ψ defined by $\hat{\psi} = \chi_{[-2a,-a)\cup[a,2a)}$.

Let us consider a function $f \in L^2(\mathbb{R})$ defined by $\hat{f} = \chi_S$, which generates an FMRA for $L^2(\mathbb{R})$, where S is a bounded frame scaling set of $\mathbb{R}$. Write $S = S^- \cup S^+$ where S^- and S^+ are intervals lying on negative and positive real axis respectively. Define a function ϕ via its Fourier transform by $\hat{\phi} = \chi_{S^+}$, then we show that ϕ generates FMRA for $H^2(\mathbb{R})$. Since f generates FMRA for $L^2(\mathbb{R})$ by theorem(2.1) condition(1) holds i.e

$$C_1 \le \sum_{k\in\mathbb{Z}} |\hat{f}(\cdot + 2k\pi)|^2 \le C_2 \qquad on \quad \mathbb{T} \setminus N.$$

Now note that $(S^+)_{\mathbb{T}} \subset (S)_{\mathbb{T}}$, therefore ϕ also satisfies condition(1) of theorem(2.1). Thus $\hat{\phi}(2\cdot) = \chi_{S^+}(2\cdot) = \hat{f}(2\cdot)\chi_{S^+}(2\cdot) = \hat{f}(2\cdot)\chi_{\frac{S^+}{2}}(\cdot)$ (as $\hat{f}(\cdot) = \chi_S(\cdot)$). Since f generates FMRA for $L^2(\mathbb{R})$, there exists $m_f \in L^2(\mathbb{T})$ such that

$$\hat{f}(2\cdot) = m_f(\cdot) f(\cdot)$$

on $\mathbb{R}$. Therefore

$$\begin{aligned}
\hat{\phi}(2\cdot) &= \hat{f}(2\cdot)\chi_{\frac{S^+}{2}}(\cdot) \\
&= \hat{f}(2\cdot)\chi_{(S^+\cap\frac{S^+}{2})}(\cdot) \\
&= m_f(\cdot) f(\cdot)\chi_{S^+}(\cdot)\chi_{\frac{S^+}{2}}(\cdot) \\
&= m_f(\cdot)\chi_S(\cdot)\chi_{S^+}(\cdot)\chi_{\frac{S^+}{2}}(\cdot) \\
&= m_f(\cdot)\chi_{S^+}(\cdot)\chi_{\frac{S^+}{2}}(\cdot) \\
&= m_f(\cdot)\chi_{\frac{S^+}{2}}(\cdot)\hat{\phi}(\cdot).
\end{aligned}$$

By taking $m_\phi(\cdot) = m_f(\cdot)\chi_{\frac{S^+}{2}}(\cdot)$, we can easily check that $m_\phi \in L^2(\mathbb{T})$. So ϕ satisfies condition(2) of theorem(2.1) i.e $\hat{\phi}(2\cdot) = m_\phi(\cdot)\hat{\phi}(\cdot)$. Now to show the condition(3) of theorem(2.1), we note that $\cup_{j\in\mathbb{Z}} 2^j S = \mathbb{R}$ and $S \subset 2S$. Since $\hat{\phi}(2\cdot) = m_\phi(\cdot)\hat{\phi}(\cdot)$, therefore $S^+ \subset 2S^+$ and S^+ contains origin, thus $\cup_{j\in\mathbb{Z}} 2^j S^+ = [0, \infty)$. Therefore ϕ satisfies all the condition of theorem(2.1) and hence generates FMRA for $H^2(\mathbb{R})$. Thus we have the following theorem.

Theorem 3.3. *Let f be a function in $L^2(\mathbb{R})$, defined by $\hat{f} = \chi_S$, where S is bounded measurable subset of $\mathbb{R}$ and f generates FMRA for $L^2(\mathbb{R})$. Define a function ϕ via its Fourier transform by $\hat{\phi}(\cdot) = \chi_{S^+}(\cdot)$, then ϕ generates FMRA for $H^2(\mathbb{R})$.*

Like $L^2(\mathbb{R})$, if ϕ defined by $\hat{\phi} = \chi_{S^+}$ generates FMRA for $H^2(\mathbb{R})$ then it is not necessary that there always exist frame wavelet corresponding to this FMRA. We can see this by following example.

Example 3.4. From example(3.1) we define a function $\hat{\phi} = \chi_{S^+}$, where

$$S^+ = \left[0, \frac{7\pi}{8}\right) \cup \left[\frac{3\pi}{2}, \frac{7\pi}{4}\right) \cup \left[3\pi, \frac{7\pi}{2}\right).$$

Then ϕ generates FMRA for $H^2(\mathbb{R})$. Here $\Gamma = \left[\frac{\pi}{8}, \frac{7\pi}{32}\right)$ and $|\Delta| > 0$. From theorem(2.2) there does not exist frame wavelet corresponding to FMRA generated by ϕ for $H^2(\mathbb{R})$.

Remark 3.1. *If S^+ defined in theorem(3.3) satisfies all the condition of theorem(3.1), then ϕ generates admissible FMRA for $H^2(\mathbb{R})$, i.e there exist frame wavelet for $H^2(\mathbb{R})$.*

4 Examples of admissible frame scaling sets on $L^2(\mathbb{R})$ and $H^2(\mathbb{R})$

In this section, we construct a class of admissible frame scaling sets on $L^2(\mathbb{R})$ and $H^2(\mathbb{R})$, and their corresponding frame wavelet sets with the help of frame scaling sets constructed in theorem(2.5). We have already seen in example(3.1) that not all FMRA generated by $\hat{\phi} = \chi_S$ admits frame wavelet. We find subset of the set S given in theorem(2.5), which provides admissible FMRA.

Consider case(1) of theorem(2.5). Let

$$S_1 = \frac{1}{2^3}S = \left[\frac{\gamma}{8} - \frac{\pi}{4}, \frac{\delta}{4} - \frac{\pi}{4}\right) \cup \left[\frac{\beta}{8}, \frac{\gamma}{8}\right) \cup \left[\frac{\alpha}{8} + \frac{\pi}{4}, \frac{\beta}{8} + \frac{\pi}{4}\right). \tag{4.1}$$

Since $\frac{\gamma}{8} - \frac{\pi}{4} < 0$ and $\gamma > 0$, we have

$$\frac{\beta}{8} + \frac{\pi}{4} < \frac{\gamma}{8} + \frac{\pi}{4} < \frac{\pi}{2}$$

and $\frac{\gamma}{8} - \frac{\pi}{4} > \frac{-\pi}{2}$. Therefore $S_1 \subset \left[\frac{-\pi}{2}, \frac{\pi}{2}\right)$. Hence $S_1 \subset 2S_1$ is satisfied because $S \subset 2S$ and $\mathbb{R} = \cup_{j\in\mathbb{Z}} 2^j S_1$ because S_1 contains neighborhood of origin. Thus all the conditions of theorem(3.1) are satisfied. If $\phi \in L^2(\mathbb{R})$ be a function defined by $\hat{\phi} = \chi_{S_1}$, then ϕ generates admissible FMRA for $L^2(\mathbb{R})$ and corresponding frame wavelet ψ is given by $\hat{\psi} = \chi_F$, where $F = 2S_1 \setminus S_1$ is frame wavelet set, i.e

$$F = \left[\frac{\gamma}{4} - \frac{\pi}{2}, \frac{\gamma}{8} - \frac{\pi}{4}\right) \cup \left[\frac{\delta}{8} - \frac{\pi}{4}, \frac{\beta}{8}\right) \cup \left[\frac{\gamma}{8}, \frac{\delta}{4} - \frac{\pi}{2}\right) \\ \cup \left[\frac{\beta}{4}, \frac{\alpha}{8} + \frac{\pi}{4}\right) \cup \left[\frac{\beta}{8} + \frac{\pi}{4}, \frac{\gamma}{4}\right) \cup \left[\frac{\alpha}{4} + \frac{\pi}{2}, \frac{\beta}{4} + \frac{\pi}{2}\right).$$

Now from theorem(3.3) and remark(3.1) function ϕ defined by $\hat{\phi} = \chi_{S_1^+}$, will generate admissible FMRA for $H^2(\mathbb{R})$ and there exists corresponding wavelet ψ defined by $\hat{\psi} = \chi_{2S_1^+ \setminus S_1^+}$, where S_1^+ is interval of S_1 lying on positive real axis.

Let us take $\alpha = \pi, \beta = \frac{3\pi}{2}, \gamma = \frac{7\pi}{4}$ and $\delta = \frac{23\pi}{8}$ in case(1) of theorem(2.5). By equation(4.1) we get

$$S_1 = \left[\frac{-\pi}{32}, \frac{7\pi}{64}\right) \cup \left[\frac{3\pi}{16}, \frac{7\pi}{32}\right) \cup \left[\frac{3\pi}{8}, \frac{7\pi}{16}\right).$$

The function ϕ defined by $\hat{\phi} = \chi_{S_1}$ generates admissible FMRA for $L^2(\mathbb{R})$ together with the function $m_\phi \in L^2(\mathbb{T})$ defined by $m_\phi(\cdot) = \chi_{\frac{S_1}{2}}(\cdot)$ such that $\hat{\phi}(2\cdot) = m_\phi(\cdot)\hat{\phi}(\cdot)$. Frame wavelet ψ corresponding to FMRA is defined by $\hat{\psi} = \chi_F$, where

$$F = \left[\frac{-\pi}{16}, \frac{-\pi}{32}\right) \cup \left[\frac{7\pi}{64}, \frac{3\pi}{16}\right) \cup \left[\frac{3\pi}{4}, \frac{7\pi}{8}\right).$$

For the case of $H^2(\mathbb{R})$, define ϕ by $\hat{\phi} = \chi_{S_1^+}$, where

$$S_1^+ = \left[0, \frac{7\pi}{64}\right) \cup \left[\frac{3\pi}{16}, \frac{7\pi}{32}\right) \cup \left[\frac{3\pi}{8}, \frac{7\pi}{16}\right).$$

S_1^+ satisfies all the condition of theorem(3.1). Therefore ϕ generates FMRA for $H^2(\mathbb{R})$ together with the function $m_\phi \in L^2(\mathbb{T})$ defined by $m_\phi(\cdot) = \chi_{\frac{S_1^+}{2}}(\cdot)$ such that $\hat{\phi}(2\cdot) = m_\phi(\cdot)\hat{\phi}(\cdot)$ and the corresponding frame wavelet is given by $\hat{\psi} = \chi_F$, where $F = \left[\frac{7\pi}{64}, \frac{3\pi}{16}\right) \cup \left[\frac{3\pi}{4}, \frac{7\pi}{8}\right)$.

Similarly in case(2) of theorem(2.5), we take

$$S_2 = \frac{1}{2^3}S = \left[\frac{\alpha}{8} - \frac{\pi}{4}, \frac{\beta}{4} - \frac{\pi}{4}\right) \cup \left[\frac{\gamma}{4} - \frac{\pi}{4}, \frac{\delta}{8} - \frac{\pi}{4}\right) \cup \left[\frac{\beta}{8}, \frac{\gamma}{8}\right). \tag{4.2}$$

If $\frac{\alpha}{8} - \frac{\pi}{4} \leq \frac{-\pi}{2}$ then $\frac{\alpha}{8} \leq \frac{-\pi}{4}$ i.e $\alpha < 0$ which is not possible. Thus $\frac{\alpha}{8} - \frac{\pi}{4} > \frac{-\pi}{2}$. From condition $2\delta - 4\pi \geq \gamma$ i.e $\frac{\gamma}{2} \leq \delta - 2\pi < \alpha$ we have $\frac{\gamma}{2} < \frac{\pi}{2}$. Therefore $S_2 \subset \left[\frac{-\pi}{2}, \frac{\pi}{2}\right)$. It can be checked that other conditions of theorem(3.1)are satisfied for S_2. Thus the function ϕ defined by $\hat{\phi} = \chi_{S_2}$ generates admissible FMRA for $L^2(\mathbb{R})$. Function ϕ whose Fourier transform is supported in the interval lying on positive real axis of S_2 will generate admissible FMRA for $H^2(\mathbb{R})$.

Also in case(3) of theorem(2.5),we take

$$S_3 = \frac{1}{2^3}S = \left[\frac{\beta}{8} - \frac{\pi}{2}, \frac{\gamma}{8} - \frac{\pi}{2}\right) \cup \left[\frac{\alpha}{8} - \frac{\pi}{4}, \frac{\beta}{8} - \frac{\pi}{4}\right) \cup \left[\frac{\gamma}{8} - \frac{\pi}{4}, \frac{\delta}{8} - \frac{\pi}{4}\right). \tag{4.3}$$

As $\beta > 0$, we have $\frac{\beta}{8} - \frac{\pi}{2} > \frac{-\pi}{2}$. Also $\frac{\beta}{8} - \frac{\pi}{2} < 0$ i.e $\frac{\beta}{8} < \frac{\pi}{2}$. Since $\delta - 2\pi < \alpha$, we have $\frac{\delta}{8} - \frac{\pi}{4} < \frac{\alpha}{8} < \frac{\beta}{8} < \frac{\pi}{2}$. Therefore $S_3 \subset \left[\frac{-\pi}{2}, \frac{\pi}{2}\right)$. It can be checked that other conditions of theorem(3.1) are satisfied for S_3. Thus function ϕ defined by $\hat{\phi} = \chi_{S_3}$ generates admissible FMRA for $L^2(\mathbb{R})$, and S_3^+ generates admissible FMRA for $H^2(\mathbb{R})$.

In case(2) and case(3) corresponding frame wavelet for S_2 and S_3 given in equation(4.2) and (4.3)can be constructed by using theorem(3.2)as illustrated in example(4.1) for the case of S_1.

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