Nano Generalized Fuzzy Semi Open Sets and Nano Fuzzy β -Open Sets in Nano Fuzzy Topological Spaces

Purva Rajwade [1], **Rachna Navalakhe** [2] **and Vaibhav Jain** [3]

[1] *Department of Applied Mathematics, Institute of Engineering & Technology, DAVV, Indore (M.P.), India*
rajwadepurva@gmail.com

[2]*Department of Applied Mathematics & Computational Science, Shri G.S. Institute of Technology & Science, Indore (M.P.), India*
sgsits.rachna@gmail.com

[3] *Department of Computer Science, Institute of Engineering & Technology, DAVV, Indore (M.P.), India*
vjain@ietdavv.edu.in

Abstract

Nano fuzzy topology draws a close link between fuzzy topological structures and Nano topological structures. In this paper, a new class of Nano fuzzy sets named Nano generalized fuzzy semi-open sets and Nano generalized fuzzy semi-closed sets are defined with the help of Nano generalized fuzzy semi closure and Nano generalized fuzzy semi interior. Further, Nano fuzzy β-open sets in Nano fuzzy topological spaces are defined and analyzed.
Subject Classification:[2020] Primary 54A40 ; Secondary 03E72.

Keywords: Nano generalized fuzzy semi-open/closed sets, Nano generalized fuzzy semi closure, Nano generalized fuzzy semi interior, Nano fuzzy β- open sets and Nano fuzzy β- closed sets.

1 Introduction

The concept of generalized semi closed sets was introduced by S. P. Arya et.al [16] which is the particular case of generalized closed sets in topology [11]. Later when Nano topology [8, 9] and various Nano sets [6, 1, 2, 5, 13, 3, 12], their closure and interior [17] were defined, A. Ezhilarasi et.al [1, 2] has introduced Nano generalized semi cosed sets in 2017. Abd El Monsef et al. [10] introduced the notion of B-open set in topology, and the equivalent notion of semi-pre-open set was given by Andrijevic in [4], and further investigated by Ganster and Andrijevic [16]. A. Revathy et.al [3] has given the theory of Nano β- open sets. Nano topology was introduced by Lellis Thivagar [9]. Also, he has given the introduction to Nano semi-open sets, Nano pre-open sets and Nano α-open sets [9]. After Zadeh L.[7] has given the concept of fuzzy sets and fuzzy theory was came to existence, research works have been done by defining topology and it's properties in fuzzy environment. This has given us the motivation to expand the theory of Nano Fuzzy Topological Spaces [15] which was defined in terms of upper approximation, lower approximation and boundary region of any fuzzy subset of universe. Earlier Nano fuzzy pre-open sets, Nano fuzzy semi open sets, Nano fuzzy regular open sets and Nano fuzzy α - open sets [14] were introduced and defined. In this paper, we have defined Nano fuzzy generalized semi-closed sets with the help of Nano fuzzy semi-closure and Nano fuzzy semi-interior. Also, we have defined Nano β-closed/open sets and studied their properties and relation with other types of Nano fuzzy sets. This work will be helpful in expansion of the theory of Nano fuzzy topological spaces.

2 Preliminaries

Definition 2.1. *[12] Let $(U, \tau_R(X))$ be a Nano topological space and $A \subseteq U$. Then A is said to be Nano semi open if $A \subseteq NCl(NInt(A))$.*

Definition 2.2. *[2] The Nano semi-closure of a set A of a Nano topological space $(U, \tau_R(X)))$ is the intersection of all Nano semi-closed sets that contain A and denoted by $NsCl(A)$. Similarly, the union of all Nano semi-open subsets of A is called the Nano semi-interior of A and it is denoted by $NsInt(A)$.*

Definition 2.3. *[2] For a subset A of $(U, \tau_R(X))$, if $NsCl(A) \subseteq V, A \subseteq V$ and V is Nano open in $(U, \tau_R(X))$, then A is called Nano generalized semi closed sets, briefly we can say Ngs-closed set.*

Definition 2.4. *[2] A subset A is called Ngs-open in $(U, \tau_R(X))$ if A^c is Ngs-closed.*

Definition 2.5. *[2] The Nano generalized semi closure briefly Ngs-closure of A is defined as the intersection of all Nano generalized semi-closed sets containing A. It is the smallest Ngs-closed set containing A. If A is Ngs- closed, then $NgsCl(A) = A$. Further $NgsCl(A) \subseteq A$.*

Definition 2.6. *[2] The union of all Nano generalized semi open subsets of A is called the Nano generalized semi interior of A. It is the largest Ngs-open set and if A is Ngs-open then $NgsInt(A) = A$. In general, $NgsInt(A) \subseteq A$.*

Definition 2.7. *[3] If $(U, \tau_R(X))$ is Nano topological space with respect to X and $X \subseteq U$. R is an equivalence relation on U, U/R denotes the family of equivalence classes of U by R. A subset P of a Nano topological space $(U, \tau_R(X))$ is called Nano β-open in U if $P \subseteq NCl(NInt(NCl(P)))$. The set of all Nano β-open sets of U is denoted by $N\beta O(U, X)$.*

Remark 2.1. *[14][Properties of Fuzzy Approximation Space] Let R be an arbitrary relation from X to Y. The lower and upper approximation operators of a fuzzy set $\underline{R}$ and $\overline{R}$ respectively satisfy the following properties: for all $\alpha, \beta \in F(X)$, where $F(X)$ is the set of all fuzzy subsets of X:*

FL1 $\underline{R}(\alpha) = (\overline{R}(\alpha^C))^C$

FL2 $\underline{R}(\alpha \wedge \beta) = \underline{R}(\alpha) \wedge \underline{R}(\beta)$

FL3 $\alpha \leq \beta \Longrightarrow \underline{R}(\alpha) \leq \underline{R}(\beta)$

FL4 $\underline{R}(\alpha \vee \beta) = \underline{R}(\alpha) \vee \underline{R}(\beta)$

FU1 $\overline{R}(\alpha) = (\underline{R}(\alpha^C))^C$

FU2 $\overline{R}(\alpha \vee \beta) = \overline{R}(\alpha) \vee \overline{R}(\beta)$

FU3 $\alpha \leq \beta \Longrightarrow \overline{R}(\alpha) \leq \overline{R}(\beta)$

FU4 $\overline{R}(\alpha \wedge \beta) = \overline{R}(\alpha) \wedge \overline{R}(\beta)$

Definition 2.8. *[15] Let X be a non-empty finite set, R be an equivalence relation on X, $\lambda \leq X$ be a fuzzy subset and $\tau_{(R)}(\lambda) = \{1_\lambda, 0_\lambda, \underline{R}(\lambda), \overline{R}(\lambda), Bd(\lambda)\}$. Then by property (2.7), $\tau_{(R)}(\lambda)$ satisfies the following axioms:*

- *i* $0_\lambda, 1_\lambda \in \tau_{(R)}(\lambda)$ *where $0_\lambda : \lambda \longrightarrow I$ denotes the null fuzzy set and $1_\lambda : \lambda \longrightarrow I$ denotes the whole fuzzy set.*
- *ii Arbitrary union of members of $\tau_{(R)}(\lambda))$ is a member of $\tau_{(R)}(\lambda)$.*
- *iii Finite intersection of members of $\tau_{(R)}(\lambda)$ is a member of $\tau_{(R)}(\lambda)$.*

That is, $\tau_{(R)}(\lambda)$ is a topology on X called the Nano fuzzy topology on X with respect to λ. We call $(X, \tau_{(R)}(\lambda))$ as the Nano fuzzy topological space (NFTS). The elements of the Nano fuzzy topological space that is $\tau_{(R)}(\lambda)$, are called Nano fuzzy open sets and elements of $\tau_{(R)}(\lambda)^C$ are called Nano fuzzy closed sets.

Definition 2.9. *[15] Let $(X, \tau_{(R)}(\lambda))$ be a Nano fuzzy topological space with respect to λ where $\lambda \leq X$ and if $\mu \leq X$ then the Nano fuzzy interior of μ is defined as union of all Nano fuzzy open subsets of μ and it is denoted by $NfInt(\mu)$. That is, it is the largest Nano fuzzy open subset contained in μ. Similarly, the Nano fuzzy closure of μ is defined as the intersection of all Nano fuzzy closed sets containing μ . It is denoted by $NfCl(\mu)$ and it is the smallest Nano fuzzy closed set containing μ .*

Definition 2.10. *[14] Let $(X, \tau_{(R)}(\lambda))$ be a Nano fuzzy topological space and $\mu \leq X$. Then μ is said to be*

i *Nano fuzzy semi open if $\mu \leq NfCl(NfInt(\mu))$*

ii *Nano fuzzy pre-open if $\mu \leq NfInt(NfCl(\mu))$*

iii *Nano fuzzy α-open if $\mu \leq NfInt(NfCl(NfInt(\mu)))$*

$NFSO(X, \lambda)$, $NFPO(X, \lambda)$ and $\tau^{\alpha}_{(R)}(\lambda)$ respectively denotes the families of all Nano fuzzy semi open, Nano fuzzy pre open and Nano fuzzy α-open subsets of X.

Definition 2.11. *[15] If $(X, \tau_{(R)}(\lambda))$ be a Nano fuzzy topological space and $\mu \leq X$, then μ is said to be Nano fuzzy semi-closed (resp. Nano fuzzy pre closed, Nano fuzzy α-closed), if it's compliment is Nano fuzzy semi open (Nano fuzzy pre-open, Nano fuzzy α-open).*

3 Nano Generalized Fuzzy Semi-open sets

Throughout this paper $(X, \tau_{(R)}(\lambda))$ represents Nano fuzzy topological space with respect to λ where $\lambda \leq X$ and $\mu \leq X$ (fuzzy subsets of X) and R is an equivalence relation on X where X/R denotes the family of equivalence classes of X by R.

Definition 3.1. *The Nano fuzzy semi-closure briefly Nfs-closure of $\mu \leq X$ is defined as the intersection of all Nano fuzzy semi-closed sets containing μ. It is the smallest Nfs-closed set containing μ. If μ is Nfs- closed, then $NfsCl(\mu) = \mu$. Further $NfsCl(\mu) \leq \mu$.*

Definition 3.2. *The union of all Nano fuzzy semi-open subsets of μ is called the Nano fuzzy semi-interior of μ briefly Nfs-interior. It is the largest Nfs-open set and if μ is Nfs-open then $NfsInt(\mu) = \mu$. In general, $NgsInt(\mu) \leq \mu$.*

Definition 3.3. *For a fuzzy subset α of Nano fuzzy topological space $(X, \tau_{(R)}(\lambda))$, if $NfsCl(\alpha) \leq \beta, \alpha \leq \beta$ and β is Nano fuzzy semi-open in $(X, \tau_{(R)}(\lambda))$, then α is called Nano fuzzy generalized semi-closed set (briefly $Nfgs$-closed).*

Definition 3.4. *A fuzzy subset μ is called $Nfgs$-open in $(X, \tau_{(R)}(\lambda))$ if μ^C is $Nfgs$-closed.*

Let $X = \{a, b, c\}$ and $X/R = \{\{a, b\}, \{c\}\}$. Let a fuzzy subset $\lambda = \{a_{0.6}, b_{0.2}, c_{0.5}\}$ of X. Then $\underline{R}(\lambda) = \{a_{0.2}, b_{0.2}, c_{0.5}\}$, $\overline{R}(\lambda) = \{a_{0.6}, b_{0.6}, c_{0.5}\}$ and $Bd(\lambda) = \{a_{0.4}, b_{0.4}, c_0\}$. So the open sets are $\{0_\lambda, 1_\lambda, \underline{R}(\lambda), \overline{R}(\lambda), Bd(\lambda)\}$ and closed sets will be $\{0_\lambda, 1_\lambda, \underline{R}(\lambda)^c, \overline{R}(\lambda)^c, Bd(\lambda)^c\}$. Now let a fuzzy subset $\alpha = \{a_{0.6}, b_{0.6}, c_{0.5}\}$ of X. For α, $NfCl(NfInt(\alpha)) = \{a_{0.6}, b_{0.6}, c_{0.5}\}$. Set of all Nano fuzzy semi-open sets will be $NFSO(\alpha) = \{0_\lambda, \{a_{0.2}, b_{0.2}, c_{0.5}\}, \{a_{0.6}, b_{0.6}, c_{0.5}\}, \{a_{0.4}, b_{0.4}, c_0\}\}$ and set of all Nano fuzzy semi closed sets will be $\{1_\lambda, \{a_{0.8}, b_{0.8}, c_{0.5}\}, \{a_{0.4}, b_{0.4}, c_{0.5}\}, \{a_{0.6}, b_{0.6}, c_1\}\}$. Now let fuzzy subset $\mu = \{a_{0.4}, b_{0.3}, c_{0.3}\}$ so Nano fuzzy semi-closure for $\alpha, NfsCl(\alpha) = \{a_{0.4}, b_{0.4}, c_{0.5}\} \leq \{a_{0.6}, b_{0.6}, c_{0.5}\} = \nu$ which is a Nano fuzzy semi-open set. Hence, μ is Nano fuzzy generalized semi-open set.

Definition 3.5. *The intersection of all $Nfgs$-closed sets containing μ is defined as the Nano fuzzy generalized semi-closure of μ, denoted by $NfgsCl(\mu)$ which is the smallest $Nfgs$-closed set containing μ. Also, $NfgsCl(\mu) \leq \mu$ and thus $NfgsCl(\mu) = \mu$ if μ is Nfgs-closed set.*

Definition 3.6. *The union of all $Nfgs$-open subsets of μ is named as the Nano fuzzy generalized semi-interior of μ, denoted by $NfgsInt(\mu)$. $NfgsInt(\mu)$ is the largest $Nfgs$-open subset of μ. Also, $NfgsInt(\mu) \leq \mu$. If μ is $Nfgs$-open, then $NfgsInt(\mu) = \mu$.*

4 Nano Fuzzy β-open sets

In this section, we will define new family of Nano fuzzy open sets, which will be termed as Nano fuzzy β-open sets and we will discuss some of the properties of Nano fuzzy β-open sets.

Definition 4.1. *A subset μ of a Nano fuzzy topological space $(X, \tau_{(R)}\lambda)$ is called Nano fuzzy β-open in X if $\mu \leq NfCl(NfInt(NfCl(\mu)))$. The set of all Nano fuzzy β-open sets of X is denoted by $Nf\beta O(X, \lambda)$.*

Theorem 4.1. *If $\overline{R}(\lambda) = 1_\lambda$ in a Nano fuzzy topological space then $Nf\beta O(X, \lambda)$ is $F(X)$, where $F(X)$ is the set of all fuzzy subsets of X.*

Proof. Let $\overline{R}(\lambda) = 1_\lambda$

i If $\underline{R}(\lambda) = 0_\lambda$ then $Bd(\lambda) = 1_\lambda$. Then $\tau_R(\lambda) = \{1_\lambda, 0_\lambda\}$. Thus for any fuzzy set $\mu, \overline{N}fCl(NfInt(NfCl(\mu))) = 1_\lambda$. Hence $\mu \leq NfCl(NfInt(NfCl(\mu)))$. Therefore, μ is Nano fuzzy β-open in X. Hence $Nf\beta O(X, \lambda)$ is$F(X)$.

ii If $\underline{R}(\lambda) \neq 0_\lambda$. Each element of X is in either $\underline{R}(\lambda)$ or in $Bd(\lambda)$, since $Bd(\lambda) = \overline{R}(\lambda) - \underline{R}(\lambda)$. If $\mu \leq \underline{R}(\lambda)$ then $NfCl(NfInt(NfCl(\mu))) = \underline{R}(\lambda)$ and if $\mu \leq Bd(\lambda)$ then $NfCl(NfInt(NfCl(\mu))) = Bd(\lambda)$. Since $\underline{R}(\lambda)$ and $Bd(\lambda)$ are compliments of each other, and $\overline{R}(\lambda) = 1_\lambda$, it is possible that μ intersects both $\underline{R}(\lambda)$ and $Bd(\lambda)$ then $NfCl(\mu) = 1_\lambda$, also $NfCl(NfInt(NfCl(\mu))) = 1_\lambda$. Hence, $\mu \leq NfCl(NfInt(NfCl(\mu)))$. Therefore, μ is Nano fuzzy β-open in X. Hence $Nf\beta O(X, \lambda)$ is $F(X)$.

□

Theorem 4.2. *If μ is Nano fuzzy open in $(X, \tau_{(R)}(\lambda))$, then it is Nano fuzzy β-open in X.*

Proof. Since μ is Nano fuzzy open in X, $NfInt(\mu) = \mu$ and $\mu \leq NfCl(\mu)$.
Then $\mu \leq NfCl(NfInt(\mu)) \leq NfCl(NfInt(NfCl(\mu)))$. Therefore, μ is Nano fuzzy β-open in X. □

Theorem 4.3. *Every Nano fuzzy semi-open set is Nano fuzzy β open in X.*

Proof. If μ is Nano fuzzy semi-open, then $\mu \leq NfCl(NfInt(\mu)) \leq NfCl(NfInt(NfCl(\mu)))$. Hence, μ is Nano fuzzy β-open. □

Theorem 4.4. *Every Nano fuzzy pre-open set is Nano fuzzy β-open in $(X, \tau_{(R)}(\lambda))$.*

Proof. If μ is Nano fuzzy pre-open, then $\mu \leq NfInt(NfCl(\mu)) \leq NfCl(NfInt(NfCl(\mu)))$. Hence, μ is Nano fuzzy β-open. □

Theorem 4.5. *Every Nano fuzzy α-open set is Nano fuzzy β-open in $(X, \tau_{(R)}(\lambda))$.*

Proof. If μ is Nano fuzzy open then $NfInt(\mu) \leq \mu, NfCl(NfInt(\mu)) \leq NfCl(\mu)$. Therefore, $NfInt(NfCl(NfInt(\mu))) \leq NfInt(NfCl(\mu)) \leq NfCl(NfInt(NfCl(\mu)))$. Hence, μ is Nano fuzzy β-open. □

Remark 4.1. *The converse of the above theorems* (4.3), (4.4), (4.5) *and* (4.6) *need not to be true always and it can be seen from the following example.*

Let $X = \{a, b, c, d\}$, $X/R = \{\{a, b\}, \{c\}, \{d\}\}$. Let a fuzzy subset of X is $\lambda = \{a_{0.2}, b_{0.3}, c_{0.7}, d_{1.0}\}$. Then $\underline{R}(\lambda) = \{a_{0.2}, b_{0.2}, c_{0.7}, d_{1.0}\}$, $\overline{R}(\lambda) = \{a_{0.3}, b_{0.3}, c_{0.7}, d_{1.0}\}$ and $Bd(\lambda) = \{a_{0.1}, b_{0.1}, c_0, d_0\}$. Therefore, Nano fuzzy topology will be $\tau_{(R)}(\lambda) = \{0_\lambda, 1_\lambda, \{a_{0.2}, b_{0.2}, c_{0.7}, d_{1.0}\}, \{a_{0.3}, b_{0.3}, c_{0.7}, d_{1.0}\}, \{a_{0.1}, b_{0.1}, c_0, d_0\}\}$. Let $\mu = \{a_{0.4}, b_{0.2}, c_{0.1}, d_{0.6}\} \leq X$. Closed sets with respect to topology are $\tau_{(R)}(X)^C = \{0_\lambda, 1_\lambda, \{a_{0.8}, b_{0.8}, c_{0.3}, d_0\}, \{a_{0.7}, b_{0.7}, c_{0.3}, d_0\}, \{a_{0.9}, b_{0.9}, c_{1.0}, d_{1.0}\}\}$. By definitions, we can observe that μ is Nano fuzzy β-open in X but neither Nano fuzzy open, nor Nano fuzzy semi-open nor Nano fuzzy pre-open nor Nano fuzzy α-open set in X.

Proposition 4.1. *If γ is Nano fuzzy open and μ is Nano fuzzy β-open then $\gamma \wedge \mu$ is Nano fuzzy β-open.*

Proof. Since, $(\gamma \wedge \mu) \leq \gamma \wedge NfCl(NfInt(NfCl(\mu))$
$\leq NfCl(\gamma \wedge NfInt(NfCl(\mu))) \leq NfCl(NfInt(NfCl(\gamma \wedge \mu)))$. Therefore, $\gamma \wedge \mu$ is Nano fuzzy β-open. □

Acknowledgement: We are grateful for the insightful comments offered by the anonymous peer reviewers of this manuscript. The generosity and expertise of one and all have improved this study in innumerable ways. We would like to thank editorial board of the journal for guidance.

References

[1] A. Ezhilarasi and K. Bhuvaneswari, On Nano Semi-generalized and Nano Generalized-semi Closed Sets in Nano Topological Spaces, International Journal of Mathematics and Computer Applications Research, (2014), 117-124.

[2] A. Ezhilarasi and K. Bhuvaneswari, Properties of Nano generalized-semi closed sets in Nano topological space, International Journal of Pure and Applied Mathematics, 116 (12) (2017), 167-175.

[3] A. Revathy and G. Ilango, On Nano β-open sets, International Journal of Engineering, Contemporary Mathematics and Sciences, 1 (2) (2015), 1-6.

[4] D. Andrijevic, Semi-pre-open sets, Mat. Vesnik , 38 (1986), 24-32.

[5] D. Andrijevic and M. Ganster, On some questions concerning semi-preopen sets, J. Inst. Mathematics and Computer Science (Math. Ser.), 1 (2) (1988), 65-75.

[6] K. Bhuvaneswari and K. Mythili Gnanapriya, Nano Generalized Closed Sets in Nano Topological Spaces, International Journal of Scientific and Research Publications, (2014), 1-3.

[7] L.A. Zadeh, Fuzzy sets, Information and Control, 8 (1965), 338 - 353.

[8] L. Thivagar and C. Richard, On Nano Continuity, Mathematical Theory on Modeling, 3 (7) (2013), 32-37.

[9] L. Thivagar and C. Richard., On Nano Forms of Weakly Open Set, International Journal of Mathematics and Statistics Invention, 1 (1)(2013).

[10] M. E. Abd EL-Monsef, S. N. EL-Deep and R. A. Mahmoud, B-Open Sets and B-continuous mappings, Bull. Fac. Sci., Assiut Univ., 12 (1983), 77-90.31-37.

[11] N. Levine, Generalized Closed Sets in Topology, Rend. Cric. Mat. Palermo, 19 (1970), 89-96.

[12] N. Thanga and K. Bhuvaneswari, On Nano generalized α-closed Sets & Nano α-generalized Closed Sets in Nano Topological Spaces, International Journal of Engineering Trends and Technology, 13 (6) (2014), 257-260.

[13] Q. H. Imran, On Nano Generalized Semi Generalized Closed Sets, Iraqi Journal of Science, 57 (2C) (2016), 1521-1527.

[14] R. Navalakhe and P.Rajwade, On Nano fuzzy forms of weakly fuzzy open sets, (Preprint).

[15] R. Navalakhe and P. Rajwade, On Nano Fuzzy Topological Spaces, International Review of Fuzzy Mathematics, 14 (2) (2019), 127-136.

[16] S. P. Arya and R. Gupta, On Strongly Continuous mappings, Kyungpook Math. J., 14 (1974), 131-143.

[17] Y.Y. Yao, A Comparative Study of Fuzzy Sets and Rough Sets, Information Sciences, 109 (1-4) (1998), 227-242.