An Analytical model with off diagonal impact on Solute Transport in Two-dimensional Homogeneous Porous Media with Dirichlet and Cauchy type boundary conditions

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Abstract

To stabilize the groundwater resources in good quality is of primary importance for both the human communities and the environment. Mathematical models give very good description to observe the contaminant concentration pattern in finite and semi-infinite aquifer. This paper presents a two dimensional analytical model for a semi-infinite homogeneous porous formation with temporally dependent dispersion coefficient for longitudinal, lateral as well as off diagonal directions. Initially the aquifer is not clean i.e. some space dependent initial concentration exists in the aquifer. The input concentration is considered as time dependent point-source contamination in the form of logistic sigmoid function for both Dirichlet as well as Cauchy type boundary conditions. The analytical solution is obtained,with the help of Laplace Transform Technique. The velocity distribution pattern is assumed to be transient in the form of algebraic sigmoid function.

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1 Introduction

In groundwater modeling a two-dimensional advection-dispersion equation play vital role to observe the contaminant concentration pattern for small groundwater bodies where the longitudinal as well as lateral components of the aquifer are also taken into account. Due to high number of accuracy and large efficiency, analytical solution assumed to be more reliable as compared to the numerical solution.Considering a wide range of Reynolds number in two-dimensional isotropic porous media, an analytical and experimental investigation of the longitudinal and lateral dispersion coefficient was studied by Harleman and Rumer (1963). The analytical solution for chemical transport in two-dimensional aquifer was presented assuming a constant velocity field. The solution was obtained by integrating the solution of a modified dimensional differential equation (Latinopolous et al., 1988). Assuming time dependent dispersion along uniform flow the two dimensional advection diffusion equation was solved analytically for instantaneous point injection and continuous point source. The dispersion coefficient was considered as uniform, linear, asymptotically and exponentially varying temporally dependent term (Aral and Liao, 1996). The analytical solution of the advection dispersion equation for unsteady flow from instantaneous sources and for steady flow from continuous sources in one, two and three spatial dimensions was derived for space dependent dispersion coefficient in infinite media (Hunt, 1998). Using Green's function the analytical solution of one, two and three dimensional solute transport problem with time dependent dispersion coefficient was obtained subjected to Dirichlet as well as Neumann type boundary conditions (Marinoschi, et al., 1999). Chen (2007) derived an analytical solution of two-dimensional advection-dispersion equation in cylindrical co-ordinates for non-axisymmetrical solute transport in a tracer test system using a power series technique coupled with the Laplace and finite Fourier cosine transform techniques. Here, the longitudinal and transverse dispersivities were assumed to be a linear function of solute distance. Considering dispersion coefficient as directly proportional to seepage velocity, the analytical solution of two dimensional solute transport problem was presented with the help of Laplace transform technique. The solution was obtained for both the first and third type boundary conditions considering constant longitudinal and lateral dispersion coefficient (Zhan et al., 2009). However, sometimes the dispersion coefficient and seepage velocity may vary with time. Keeping in mind this fact, a two dimensional solute transport problem in a homogeneous finite aquifer was solved by Hankel Transform Technique in which the input source concentration was taken at the far end away from the origin. Initially the aquifer was assumed to be clean and the input concentration was taken as time dependent exponentially decreasing function (Singh et al., 2010). The analytical solution of two-dimensional advection–dispersion equation subject to first- and third-type inlet boundary conditions was studied in the cylindrical co-ordinate system. The finite Hankel transform technique of second kind and the generalized integral transform technique were used to solve the problem (Chen et al., 2011). Most of the problem solved in the previous studied have taken only the longitudinal and lateral dispersion coefficient term. However, the off diagonal dispersion coefficient term also affect the contaminant concentration pattern of the aquifer. Frind and Germain (1986) examined the evolution of narrow, sharply defined contaminant plumes often observed in the field by numerical techniques. The solution was obtained by considering all the components of dispersion tensor as longitudinal, transverse as well as off diagonal. In order to find the accuracy and efficiency in the simulation of plumes the result was compared among principle direction method (PD), alternative direction Galerkin method (ADG) and Conventional finite element method. An analytical solution for the advection-dispersion equation (ADE) usually assumes that boundary and initial conditions are orthogonal to the principal axes of the dispersion tensor. However, this is not always the case in field studies or modeling scenarios. Using the method of Green's functions, a generalized analytical solution of the three-dimensional ADE in a semi-infinite porous medium was obtained. The solution was derived in an arbitrary Cartesian co-ordinate system subject to arbitrary initial condition and third type boundary condition with constant dispersion coefficients (Ellsworth and Butters, 1993). Massabo et al., (2006) solved a two dimensional advection equation with anisotropic dispersion for a homogeneous semi-infinite aquifer considering the constant dispersion coefficient. The effect of Chemical decay or adsorption like reaction inside the liquid phase was also considered. The analytical solution was obtained using Bessel function expansion for impulsive, continuous and finite pulse type pollutant release. In order to study the pollutant transport characteristics of the Han river (Korea) a two-dimensional advection dispersion model was developed using streamline-upwind Petrov-Galarkin method (SUPG). The solution was obtained with the help of finite element method considering all the components of dispersion tensor in the problem (Lee, 2007). In order to predict the depth averaged concentration of solute transport in shallow water a two-dimensional solute transport equation was solved numerically. The solution was derived for the dispersion diffusion tensor of depth-averaged mixing, whose principal direction coincides with the flow direction. In the diffusion stage, a second-order accurate central scheme is used while in the advection stage, a five-point Total Variation Diminishing (TVD) modification ismade to the standard MacCormack scheme (Liang et al., 2010).Kong et al., (2011) developed a high resolution model for solving the two-dimensional advection and anisotropic diffusion problem of solute transport in shallow water. The numerical solution was obtained with the help of finite volume method considering all the components of diffusion coefficient tensor. In these problems the dispersion coefficient tensors were assumed as constant term, however it may vary with time.

In the present study a two-dimensional advection dispersion equation having different components of dispersion tensor is considered. At the initial stage, aquifer is assumed to be not clean i.e. there are some initial concentration exists in the aquifer in the form of exponentially decreasing function of space dependent terms. At the origin the input point source concentration is taken as time-dependent in the form of logistic sigmoid function for both 1) Dirichlet and 2) Cauchy type boundary conditions. The logistic sigmoid function is horizontally asymptotic in nature i.e. it increases continuously for $t > 0$ and tends to 1 as $t \to \infty$. In the solute transport modeling context, the input point source concentration can be taken as of this form assuming that input concentration would initially increase with time and after a certain time period it would stabilize at

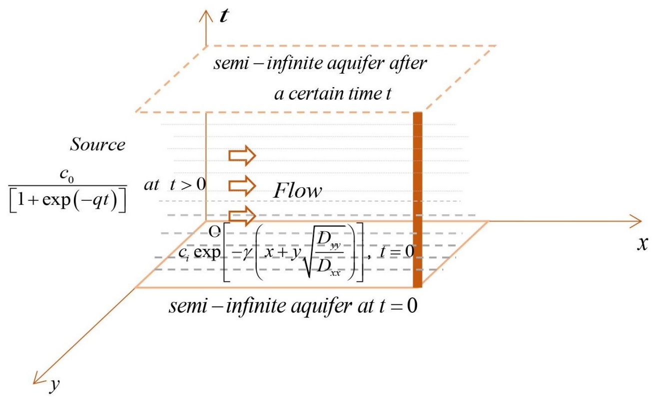

Fig. 1: Physical Modal of the Problem

an asymptotic value. The dispersion coefficient tensor is directly proportional to seepage velocity concept is usedand the analytical solution is derived for transient form of velocity distribution in the form of algebraic sigmoid expression.

2 Mathematical Formulation

2.1 Time dependent input source contaminationwith Dirichlet type boundary condition

Consider a two-dimensional homogeneous semi-infinite aquifer subjected to a time-dependent point source contamination in the form of logistic sigmoid function at the origin. The longitudinal and lateral directions at the origin are taken as x and y axes, respectively. Let $c\left[ML^{-3}\right]$is the contaminant concentration in the aquifer at any time $t\,[T]$; $u\,[LT^{-1}]$ and $v\,[LT^{-1}]$ are x and y groundwater velocity components, respectively; $D_{xx}\,[L^2T^{-1}]$,$D_{yy}\,[L^2T^{-1}]$ are the dispersion coefficients along x and y axes, respectively; $D_{xy}\left[L^2T^{-1}\right]$and $D_{yx}\left[L^2T^{-1}\right]$ are the off diagonal dispersion coefficients. Initially the aquifer has some initial background concentration, which is function of space variable, say,$c_i \exp\left[-\gamma\left(x + y\sqrt{D_{yy}/D_{xx}}\right)\right]$, where $\gamma\,[L^{-1}]$ is the decay parameter and $c_i\left[ML^{-3}\right]$ is the solute concentration.

The advection-dispersion equation representing the two-dimensional homogeneous semi-infinite aquifer can be written as

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x}\left(D_{xx}\frac{\partial c}{\partial x} + D_{xy}\frac{\partial c}{\partial y}\right) + \frac{\partial}{\partial y}\left(D_{yy}\frac{\partial c}{\partial y} + D_{yx}\frac{\partial c}{\partial x}\right) - u\frac{\partial c}{\partial x} - v\frac{\partial c}{\partial y} \tag{2.1}$$

Let u and v be expressed as

$$u = u_0 f(t) \quad and \quad v = v_0 f(t) \tag{2.2}$$

where $u_0\,[LT^{-1}]$ *and* $v_0\,[\,LT^{-1}]$are the initial values of u *and* v, respectively, and $f(t)$ is assumed to be a sinusoidally varying function or an exponentially decreasing function. The dispersion coefficient terms D_{xx} , D_{yy} *and* D_{xy} *or* D_{yx} can be expressed as (Batu, 2006)

$$\text{(2.3)}\qquad \begin{aligned} D_{xx} &= \frac{a_L u^2 + a_T v^2}{\sqrt{u^2+v^2}} = \frac{\left(a_L u_0^2 + a_T v_0^2\right) f(t)}{\sqrt{u_0^2+v_0^2}} = D_{x0} f(t), \\ D_{yy} &= \frac{a_T u^2 + a_L v^2}{\sqrt{u^2+v^2}} = \frac{\left(a_T u_0^2 + a_L v_0^2\right) f(t)}{\sqrt{u_0^2+v_0^2}} = D_{y0} f(t), \\ \text{and } D_{xy} &= \frac{(a_L - a_T) uv}{\sqrt{u^2+v^2}} = \frac{(a_L - a_T) u_0 v_0 f(t)}{\sqrt{u_0^2+v_0^2}} = D_{xy0} f(t) = D_{yx}. \end{aligned}$$

where $a_L$and $a_T$are the dispersivity$[L]$ along longitudinal and lateral directions that depends on the distribution of aquifer heterogeneities and scale of the field problem(Bedient et al., 1999); D_{x0}, D_{y0} and D_{xy0} are the initial values ofD_{xx} , D_{yy} and D_{xy} or D_{yx}, respectively.

As stated earlier, the initial contaminant concentration is a function of space say $c_i \exp\left[-\gamma\left(x + y\sqrt{D_{yy}/D_{xx}}\right)\right]$at $t = 0$ and at the origin the time dependent point source concentration is taken in the form of logistic sigmoid function as$\frac{c_0}{[1+\exp(-qt)]}$. Here $q\left[T^{-1}\right]$is contaminant decay rate coefficient and $c_0\left[ML^{-3}\right]$is solute concentration. Hence, the initial and boundary conditions are expressed as

$$\text{(2.4)}\qquad c(x, y, t) = c_i \exp\left[-\gamma\left(x + y\sqrt{\frac{D_{yy}}{D_{xx}}}\right)\right], \quad x > 0, y > 0,\ t = 0$$

Using Eq. (2.3), Eq. (2.4) can be expressed as

$$\text{(2.5)}\qquad c(x, y, t) = c_i \exp\left[-\gamma\left(x + y\sqrt{\frac{D_{y0}}{D_{x0}}}\right)\right], \quad x > 0, y > 0,\ t = 0$$

$$\text{(2.6)}\qquad c(x, y, t) = \frac{c_0}{[1 + \exp(-qt)]} \quad x = 0,\ y = 0,\ t > 0$$

and

$$\text{(2.7)}\qquad \frac{\partial c}{\partial x} = 0, \qquad \frac{\partial c}{\partial y} = 0, \qquad x \to \infty,\ y \to \infty,\ \ t > 0$$

Using Eqs. (2.2) and (2.3), Eq. (2.1) becomes

$$\text{(2.8)}\qquad \frac{1}{f(t)}\frac{\partial c}{\partial t} = D_{x0}\frac{\partial^2 c}{\partial x^2} + D_{y0}\frac{\partial^2 c}{\partial y^2} + 2D_{xy0}\frac{\partial^2 c}{\partial x \partial y} - u_0\frac{\partial c}{\partial x} - v_0\frac{\partial c}{\partial y}$$

Introducing a new time variable T^* by the following transformation (Crank, 1975):

$$\text{(2.9)}\qquad T^* = \int_0^t f(t)\, dt$$

The term $f(t)$ represents time dependent form of velocity expression as an algebraic sigmoid function and it is expressed as

$$\text{(2.10)}\qquad f(t) = \frac{mt}{\sqrt{1 + (mt)^2}} \Rightarrow T^* = \frac{\left(\sqrt{1 + (mt)^2} - 1\right)}{m}$$

with the help of Eq. (2.9), Eq. (2.8) subject to Eqs. (2.5)-(2.7) reduce to

$$\frac{\partial c}{\partial T^*} = D_{x0}\frac{\partial^2 c}{\partial x^2} + D_{y0}\frac{\partial^2 c}{\partial y^2} + 2D_{xy0}\frac{\partial^2 c}{\partial x \partial y} - u_0\frac{\partial c}{\partial x} - v_0\frac{\partial c}{\partial y} \tag{2.11}$$

$$c\left(x, y, T^*\right) = c_i \exp\left[-\gamma\left(x + y\sqrt{\frac{D_{y0}}{D_{x0}}}\right)\right], \quad x > 0, y > 0, \ T^* = 0 \tag{2.12}$$

$$c\left(x, y, T^*\right) = \frac{c_0}{\left[1 + \exp(-qT^*)\right]}, \quad x = 0, y = 0, \ T^* > 0 \tag{2.13}$$

$$\frac{\partial c}{\partial x} = 0, \qquad \frac{\partial c}{\partial y} = 0, \qquad x \to \infty, \ y \to \infty, \ \ T^* > 0 \tag{2.14}$$

Let us define a new space variable as

$$\xi = x + y\sqrt{\frac{D_{y0}}{D_{x0}}} \quad \text{or} \quad x + y\sqrt{\frac{v_0}{u_0}} \tag{2.15}$$

with the aid of Eq. (2.15), Eqs. (2.11)-(2.14) can be written as

$$\frac{\partial c}{\partial T^*} = D_1\frac{\partial^2 c}{\partial \xi^2} - U_1\frac{\partial c}{\partial \xi} \tag{2.16}$$

where, $D_1 = D_{x0} + \frac{D_{y0}^2}{D_{x0}} + 2D_{xy0}\sqrt{\frac{D_{y0}}{D_{x0}}}$ *and* $U_1 = u_0 + v_0\sqrt{\frac{v_0}{u_0}}$.

$$c\left(\xi, T^*\right) = c_i \exp\left(-\gamma\xi\right), \quad \xi > 0, \quad T^* = 0 \tag{2.17}$$

$$c\left(\xi, T^*\right) = \frac{c_0}{2}\left[1 + \frac{qT^*}{2}\right], \quad \xi = 0, \ \ T^* > 0 \tag{2.18}$$

and,

$$\frac{\partial c}{\partial \xi} = 0, \quad \xi \to \infty, \ \ T^* > 0 \tag{2.19}$$

In order to reduce the convective term from Eq. (2.16), using the following transformation

$$c\left(\xi, T^*\right) = K\left(\xi, T^*\right)\exp\left(\frac{U_1}{2D_1}\xi - \frac{U_1^2}{4D_1}T^*\right) \tag{2.20}$$

Eqs. (2.16)- (2.19) become

$$\frac{1}{D_1}\frac{\partial K}{\partial T^*} = \frac{\partial^2 K}{\partial \xi^2} \tag{2.21}$$

subject to

$$K(\xi, T^*) = c_i \exp\left[-\left(\gamma + \frac{U_1}{2D_1}\right)\xi\right], \quad \xi > 0, \quad T^* = 0 \tag{2.22}$$

$$K(\xi, T^*) = \frac{c_0}{2}\left[1 + \frac{qT^*}{2}\right]\exp\left(\frac{U_1^2}{4D_1}T^*\right), \quad \xi = 0, \quad T^* > 0 \tag{2.23}$$

$$\frac{\partial K}{\partial \xi} = -\frac{U_1}{2D_1}K, \quad \xi \to \infty, \quad T^* > 0 \tag{2.24}$$

Applying Laplace transform to Eqs. (2.21) -(2.22) it give

$$\bar{K}(\xi, p) = C_1 \exp\left(\sqrt{\frac{p}{D_1}}\xi\right) + C_2 \exp\left(-\sqrt{\frac{p}{D_1}}\xi\right) + \frac{c_i}{p - \left(\gamma + \frac{U_1}{2D_1}\right)^2 D_1}\exp\left(-\gamma - \frac{U_1}{2D_1}\right)\xi \tag{2.25}$$

where, C_1 *and* C_2 are constants.
To applying Laplace transform on Eqs. (2.23) and (2.24), then using in Eq. (2.25), the values of C_1 *and* C_2 can be obtained as

$$C_1 = 0 \text{ and } C_2 = \frac{c_0}{2}\left[\frac{1}{\left(p - \frac{U_1^2}{4D_1}\right)} + \frac{q}{2}\frac{1}{\left(p - \frac{U_1^2}{4D_1}\right)^2}\right] - \frac{c_i}{p - \left(\gamma + \frac{U_1}{2D_1}\right)^2 D_1} \tag{2.26}$$

Computing values of c_1 *and* c_2 from Eq. (2.26), in Eq. (2.25), it can be written as

$$\begin{aligned}\overline{K}(\xi, p) = &\left[\frac{c_0}{2}\left\{\frac{1}{\left(p - \frac{U_1^2}{4D_1}\right)} + \frac{q}{2}\frac{1}{\left(p - \frac{U_1^2}{4D_1}\right)^2}\right\} - \frac{c_i}{p - \left(\gamma + \frac{U_1}{2D_1}\right)^2 D_1}\right]\exp\left(-\sqrt{\frac{p}{D_1}}\xi\right) \\ &+ \frac{c_i}{p - \left(\gamma + \frac{U_1}{2D_1}\right)^2 D_1}\exp\left(-\gamma - \frac{U_1}{2D_1}\right)\xi\end{aligned} \tag{2.27}$$

Now, taking the inverse Laplace transform of Eq. (2.27) and substituting the values of $K(\xi, T)$ in Eq. (2.20), the required solution can be written as

$$c(\xi, T^*) = \frac{c_0}{2}c_1(\xi, T^*) + \frac{qc_0}{4}c_2(\xi, T^*) - c_i c_3(\xi, T^*) + c_i c_4(\xi, T^*) \tag{2.28}$$

where,

$$c_1(\xi, T^*) = \frac{1}{2}erfc\left(\frac{\xi}{2\sqrt{D_1T^*}} - \frac{U_1T^*}{2\sqrt{D_1T^*}}\right) + \frac{1}{2}\exp\left(\frac{U_1}{D_1}\xi\right)erfc\left(\frac{\xi}{2\sqrt{D_1T^*}} + \frac{U_1T^*}{2\sqrt{D_1T^*}}\right)$$

$$\begin{aligned}c_2(\xi, T^*) = &\frac{1}{2U_1}(U_1T^* - \xi)\,erfc\left(\frac{\xi}{2\sqrt{D_1T^*}} - \frac{U_1T^*}{2\sqrt{D_1T^*}}\right) \\ &+ \frac{1}{2U_1}(U_1T^* + \xi)\exp\left(\frac{U_1}{D_1}\xi\right)erfc\left(\frac{\xi}{2\sqrt{D_1T^*}} + \frac{U_1T^*}{2\sqrt{D_1T^*}}\right)\end{aligned}$$

$$\begin{aligned}c_3(\xi, T^*) = &\frac{1}{2}\exp\left(\gamma^2 D_1T^* + \gamma U_1T^* - \gamma\xi\right)erfc\left(\frac{\xi}{2\sqrt{D_1T^*}} - \left(\gamma + \frac{U_1}{2D_1}\right)\sqrt{D_1T^*}\right) \\ &+ \frac{1}{2}\exp\left(\gamma^2 D_1T^* + \gamma U_1T^* + \gamma\xi + \frac{U_1}{2D_1}\xi\right)erfc\left(\frac{\xi}{2\sqrt{D_1T^*}} + \left(\gamma + \frac{U_1}{2D_1}\right)\sqrt{D_1T^*}\right)\end{aligned}$$

$$c_4(\xi, T^*) = \exp\left(\gamma^2 D_1 T^* + \gamma U_1 T^* - \gamma\xi\right)$$

2.2 Time dependent input source contamination in the form of Cauchy type boundary condition

In case of Cauchy type boundary condition, Eq. (2.6) will be replaced by

$$-D_{xx}\frac{\partial c}{\partial x} + uc = \frac{uc_0}{[1+\exp(-qt)]}, \quad x = 0, \quad t > 0 \tag{2.29}$$

$$-D_{yy}\frac{\partial c}{\partial x} + vc = \frac{vc_0}{[1+\exp(-qt)]}, \quad y = 0, \quad t > 0 \tag{2.30}$$

Using Eqs. (2.2) and (2.3), and then using the new time variable T^* defined in (2.9), (2.29) and (2.30) become

$$-D_{x0}\frac{\partial c}{\partial x} + u_0 c = \frac{u_0 c_0}{[1+\exp(-qT^*)]}, \quad x = 0, \quad T^* > 0 \tag{2.31}$$

$$-D_{y0}\frac{\partial c}{\partial y} + v_0 c = \frac{v_0 c_0}{[1+\exp(-qT^*)]}, \quad y = 0, \quad T^* > 0 \tag{2.32}$$

Applying Eq. (2.15) in (2.31) and (2.32) and then adding each other it gives

$$-D_2\frac{\partial c}{\partial \xi} + U_2 c = \frac{U_2 c_0}{2}\left[1 + \frac{qT^*}{2}\right], \quad \xi = 0, \quad T^* > 0 \tag{2.33}$$

where,$D_2 = D_{x0} + D_{y0}\sqrt{\frac{D_{y0}}{D_{x0}}}$ *and* $U_2 = u_0 + v_0$

Using the transformation given in Eq. (2.20), Eq. (2.33) can be written as

$$-D_2\frac{\partial K}{\partial \xi} + \left(U_2 - \frac{U_1 D_2}{2D_1}\right)K = \frac{U_2 c_0}{2}\left[1 + \frac{qT^*}{2}\right]\exp\left(\frac{U_1^2}{4D_1}T^*\right), \quad \xi = 0, \quad T^* > 0 \tag{2.34}$$

Applying Laplace transform technique on Eq. (2.34), and using it in Eq. (2.25), the values of C_1 *and* C_2 can be obtained as

$$C_1 = 0$$

$$\begin{aligned} C_2 = {} & \frac{U_2 c_0}{2\left(p - \frac{U_1^2}{4D_1}\right)\left(D_2\sqrt{\frac{p}{D_1}} + U_2 - \frac{U_1 D_2}{2D_1}\right)} + \frac{qU_2 c_0}{4\left(p - \frac{U_1^2}{4D_1}\right)^2\left(D_2\sqrt{\frac{p}{D_1}} + U_2 - \frac{U_1 D_2}{2D_1}\right)} \\ & - \frac{\gamma c_i D_2}{\left[p - \left(\gamma + \frac{U_1}{2D_1}\right)^2 D_1\right]\left(D_2\sqrt{\frac{p}{D_1}} + U_2 - \frac{U_1 D_2}{2D_1}\right)} \end{aligned} \tag{2.35}$$

Puting the values of C_1 *and* C_2 from Eq. (2.35) in Eq. (2.25) it gives

$$\begin{aligned} \overline{K}(\xi, p) = {} & \left[\begin{array}{l} \frac{U_2 c_0}{2\left(p - \frac{U_1^2}{4D_1}\right)\left(D_2\sqrt{\frac{p}{D_1}} + U_2 - \frac{U_1 D_2}{2D_1}\right)} \\ + \frac{qU_2 c_0}{4\left(p - \frac{U_1^2}{4D_1}\right)^2\left(D_2\sqrt{\frac{p}{D_1}} + U_2 - \frac{U_1 D_2}{2D_1}\right)} \\ - \frac{\gamma c_i D_2}{\left[p - \left(\gamma + \frac{U_1}{2D_1}\right)^2 D_1\right]\left(D_2\sqrt{\frac{p}{D_1}} + U_2 - \frac{U_1 D_2}{2D_1}\right)} \end{array}\right]\exp\left(-\sqrt{\frac{p}{D_1}}\xi\right) \\ & + \frac{c_i}{\left[p - \left(\gamma + \frac{U_1}{2D_1}\right)^2 D_1\right]}\exp\left(-\gamma - \frac{U_1}{2D_1}\right)\xi \end{aligned} \tag{2.36}$$

Taking the inverse Laplace transform of Eq. (2.36) and substituting the values of $K(\xi, T)$ in Eq. (2.20), the required solution can be written as

$$c(\xi, T^*) = \frac{U_2 c_0}{2} c_{11}(\xi, T^*) + \frac{qU_2 c_0}{4} c_{22}(\xi, T^*) - \gamma c_i c_{33}(\xi, T^*) + c_i c_{44}(\xi, T^*) \tag{2.37}$$

where,

$$c_{11}(\xi, T^*) = \frac{\sqrt{D_1}}{D_2} \left[\begin{array}{l} \frac{1}{2(\alpha+\beta)} erfc\left(\frac{\xi}{2\sqrt{D_1 T^*}} - \frac{U_1 T^*}{2\sqrt{D_1 T^*}} \right) \\ -\frac{1}{2(\alpha-\beta)} \exp\left(\frac{U_1}{D_1}\xi \right) erfc\left(\frac{\xi}{2\sqrt{D_1 T^*}} + \frac{U_1 T^*}{2\sqrt{D_1 T^*}} \right) \\ +\frac{\beta}{\alpha^2-\beta^2} \exp\left(\frac{U_2^2 D_1}{D_2^2} T^* - \frac{U_1 U_2}{D_2} T^* + \frac{U_2}{D_2}\xi \right) \\ erfc\left(\frac{\xi}{2\sqrt{D_1 T^*}} + \frac{U_2\sqrt{D_1 T^*}}{D_2} - \frac{U_1 T^*}{2\sqrt{D_1 T^*}} \right) \end{array} \right]$$

$$c_{22}(\xi, T^*) = \frac{\sqrt{D_1}}{D_2} \left[\begin{array}{l} \frac{1-\left(2\alpha T^* - \xi/\sqrt{D_1}\right)(\alpha+\beta)}{4\alpha(\alpha+\beta)^2} erfc\left(\frac{\xi}{2\sqrt{D_1 T^*}} - \frac{U_1 T^*}{2\sqrt{D_1 T^*}} \right) \\ +\frac{\left(2\alpha T^* + \xi/\sqrt{D_1}\right)(\alpha-\beta)-1}{4\alpha(\alpha-\beta)^2} \exp\left(\frac{U_1}{D_1}\xi \right) erfc\left(\frac{\xi}{2\sqrt{D_1 T^*}} + \frac{U_1 T^*}{2\sqrt{D_1 T^*}} \right) \\ -\sqrt{\frac{T^*}{\pi}} \frac{1}{\alpha^2-\beta^2} \exp\left(\frac{U_1}{2D_1}\xi - \frac{\xi^2}{4D_1 T^*} - \frac{U_1^2}{4D_1} T^* \right) + \frac{\beta}{\alpha^2-\beta^2} \\ \exp\left(\frac{U_2^2 D_1}{D_2^2} T^* - \frac{U_1 U_2}{D_2} T^* + \frac{U_2}{D_2}\xi \right) \\ erfc\left(\frac{\xi}{2\sqrt{D_1 T^*}} + \frac{U_2\sqrt{D_1 T^*}}{D_2} - \frac{U_1 T^*}{2\sqrt{D_1 T^*}} \right) \end{array} \right]$$

$$c_{33}(\xi, T^*) = \sqrt{D_1} \left[\begin{array}{l} \frac{1}{2(\delta+\beta)} \exp\left(\gamma^2 D_1 T^* + \frac{\gamma U_1}{\sqrt{D_1}} T^* - \gamma\xi \right) \\ erfc\left(\frac{\xi}{2\sqrt{D_1 T^*}} - \gamma\sqrt{D_1 T^*} - \frac{U_1 T^*}{2\sqrt{D_1 T^*}} \right) \\ -\frac{1}{2(\delta-\beta)} \exp\left(\gamma^2 D_1 T^* + \frac{\gamma U_1}{\sqrt{D_1}} T^* + \gamma\xi + \frac{U_1}{D_1}\xi \right) \\ erfc\left(\begin{array}{l} \frac{\xi}{2\sqrt{D_1 T^*}} \\ +\gamma\sqrt{D_1 T^*} + \frac{U_1 T^*}{2\sqrt{D_1 T^*}} \end{array} \right) \\ +\frac{\beta}{\alpha^2-\beta^2} \exp\left(\frac{U_2^2 D_1}{D_2^2} T^* - \frac{U_1 U_2}{D_2} T^* + \frac{U_2}{D_2}\xi \right) \\ erfc\left(\begin{array}{l} \frac{\xi}{2\sqrt{D_1 T^*}} \\ +\frac{U_2\sqrt{D_1 T^*}}{D_2} - \frac{U_1 T^*}{2\sqrt{D_1 T^*}} \end{array} \right) \end{array} \right]$$

$$c_{44}(\xi, T^*) = \exp\left(\gamma^2 D_1 T^* + \gamma U_1 T^* - \gamma\xi \right)$$

and $\alpha = \frac{U_1}{2\sqrt{D_1}}, \beta = \frac{U_2\sqrt{D_1}}{D_2} - \frac{U_1}{2\sqrt{D_1}}, \delta = (\gamma + \frac{U_1}{2D_1})\sqrt{D_1}$

3 Numerical Results and Discussion

To discuss the numerical results of the problem, the time-dependent algebraic sigmoid form of velocity expression defineding Eq. (1.10) are taken into consideration. The values $mt = 3k + 2mt = 3k + 2kk$ mt of is chosen where is the whole number. Here, we have considered the value of k as $10 \le k \le 12$ corresponds mt as 32,35 and 38. These values ofmt yield t =1940,2122 and 2304 days at a regular interval of approximately 182 days. If the value t = 1940 days represent some date in

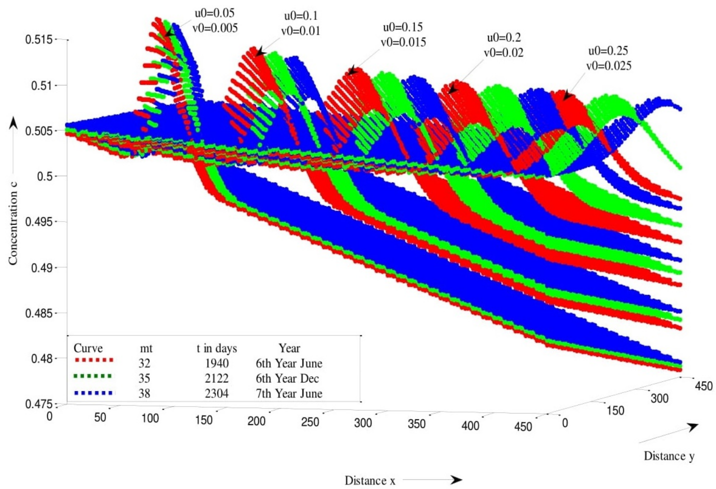

Fig. 2: Time dependent contaminant concentration pattern along groundwater flow for Dirichlet type boundary condition

the month of 6^{th} year June when groundwater velocity is minimum, then $t = 2122$days represent some date in 6^{th} year December when groundwater velocity is maximum, and similarly $t = 2304$ days represents some date in the 7^{th} year June.Therefore, these value of mt shows the concentration pattern for the period of 6^{th} year June to 7^{th} year June. The analytical solution is obtained for both Dirichlet and Cauchy type boundary condition separately and the contamination pattern of solute is depicted graphically in the Fig. 2 and Fig. 3 respectively.

The two-dimensional analytical solution given in Eq. (2.28) for Dirichlet type boundary condition is computed for the input values $a_L = 1\ km$,a_T=0.1 km,$c_i = 0.5$,c_0=1,$x = 450km$, $y = 450km$,$q = 0.00001(/day)$,$\gamma = 0.0001(/km)$is shown graphically in the Fig.2.The groundwater velocity along lateral and longitudinal directions has been taken in the range of $0.05 \leq u_0 \leq 0.25$ and $0.005 \leq v_0 \leq 0.025$ with a difference of 0.05 and 0.005 respectively. It is observed that for $u_0 = 0.05$ and $v_0 = 0.005$, the contaminant concentration value decreases slowly near the origin upto one-fourth length of the aquifer, then it starts increasing and attains its maximum value and finally it goes on decreasing and reaches upto harmless concentration. In case of $u_0 = 0.1$ and $v_0 = 0.01$ the contaminant concentration decreases gradually upto one-third length of the aquifer and then it follows the same pattern as of the previous one. Similarly, for $u_0 = 0.25$ and $v_0 = 0.025$ the contaminant concentration decreases with distance upto three-fourth length of the aquifer and then it starts increases to attain maximum peak value and then it decreases. Here, it can be observed that on increasing the values of u_0 and v_0, the concentration values at the peak start decreasing and moving towards the other end of the aquifer.

The solution obtained for Cauchy type boundary condition given in Eq. (2.37) is computed for the same set of values except $q = 0.0001\,(/day)$ and it is shown in fig.3. It shows that initially the contaminant concentration decreases with time and after travelling some distance it shows the reverse pattern. The contaminant concentration pattern have been observed for the range of$0.05 \leq u_0 \leq 0.25$ and $0.005 \leq v_0 \leq 0.025$. In this case it is observed that contaminant concentration is stagnant in nature near the origin, after some distance itdecreases gradually and then it reaches upto harmless concentration. For $u_0 = 0.05$ and $v_0 = 0.005$ the contaminant concentration shows stagnant nature upto one-fifth length of the aquifer, then it goes down upto minimum value. Similarly, for $u_0 = 0.15$ and $v_0 = 0.015$ the concentration value is stagnant upto one-half length of the aquifer and then it shows decreasing pattern. It is observed that on

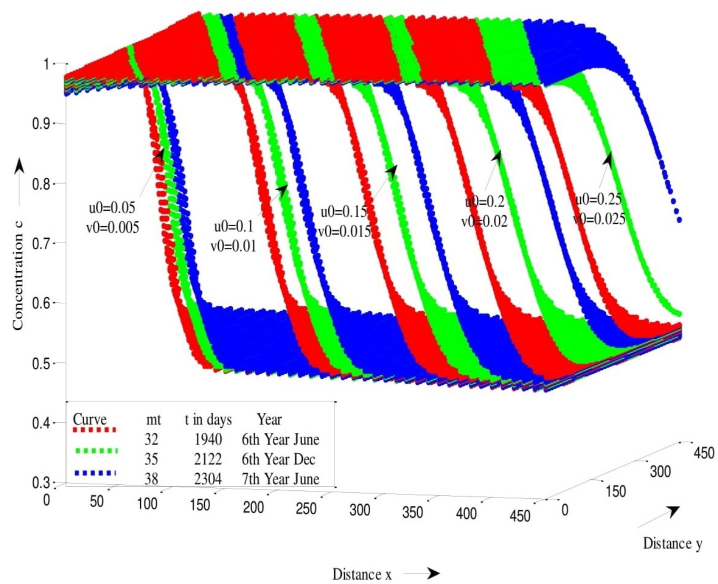

Fig. 3: Time dependent contaminant concentration pattern along groundwater flow for Cauchy type boundary condition

increasing the values of u_0 and v_0, the stagnant pattern for the contaminant concentration cover the larger distance near the origin and lesser distance towards the other end of the aquifer.

4 Conclusion

On the basis of the observations shown in Fig.2 and Fig. 3 following conclusions are made
1. The contaminant concentration of the aquifer decreases with time in both Dirichlet and Cauchy type boundary condition.
2. For Dirichlet type boundary conditions, the contaminant concentration starts increases initially with distance, goes to the maximum concentration value and then decreases continuously upto harmless concentration.
3. The concentration pattern decreases more rapidly for Cauchy type boundary condition as compared to Dirichlet type boundary condition.
4. On increasing the groundwater velocity in both longitudinal and lateral directions, the contaminant concentration stagnant pattern increases and shifted towards the other end of the aquifer.

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