2-Norm On Fuzzy Linear spaces over Fuzzy Fields

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Abstract

In this paper we have studied the concept of 2-norm on fuzzy linear space over fuzzy field. We have also discussed some fundamental properties of 2-norm.
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1 Introduction

The concept of fuzzy set was introduced by L.A. Zadeh [14] in 1965 and fuzzy topology was introduced by C.L. Cheng [2] in1968. Thereafter many researchers introduced the notions of fuzzy norm in different points of view. In 1984 Katsaras [6] defined a fuzzy norm on a linear space and thereafter Wu and Fang [11] introduced a fuzzy normed space. R. Biswas [1] in 1991 defined fuzzy norm and fuzzy inner product of elements on a linear space. In 1992, Felbin [4] introduced fuzzy norm on a linear space by assigning a fuzzy real number to each element of the linear space. Another important approach of fuzzy norm on a linear space was introduced in 1994, by Cheng and Mordeson [3], on a parallel line as the corresponding fuzzy metric is of Kramosil and Michelek [8] type. Thereafter Krishna and Sarma [7], Xiao and Zhu [13] discussed fuzzy norms on linear spaces at different points of aspects. All these researchers have done their work in the area of crisp linear space. Gu Wenxiang and Lu Tu [12] were the first to introduce the concept of fuzzy fields and fuzzy linear spaces over fuzzy fields. In 2011, C.P. Santhosh and T.V. Ramakrishnan [10] introduced norm on fuzzy linear space over fuzzy field. In 2018, Noori F. AL-Mayahi and Suadad M. Abbas [9] defined fuzzy normed algebra over fuzzy field. A satisfactory theory of 2-norm on a linear space has been introduced and developed by Gähler [5]. In this paper we introduce the idea of 2-norm on fuzzy linear spaces over fuzzy fields and have also discussed some properties of it.

2 Preliminaries

In this section, some definitions and preliminary results are given which will be needed in proof of our main results.

Definition 2.1. *[12] Let X be a field and F a fuzzy set in X with the following conditions*

(i) $F(x+y) \geq \min\{F(x), F(y)\}, \ x, y \in X,$

(ii) $F(-x) \geq F(x), \ \ x \in X,$

(iii) $F(xy) \geq \min\{F(x), F(y)\}, \quad x, y \in X,$

(iv) $F\left(x^{-1}\right) \geq F(x), \quad x(\neq 0) \in X.$

Then we call F a fuzzy field in X and denote it by (F, X). Also, (F, X) is called a fuzzy field of X.

Theorem 2.1. *[12] If (F, X) is a fuzzy field of X, then*

(i) $F(0) \geq F(x), \ x \in X.$

(ii) $F(1) \geq F(x), \quad x(\neq 0) \in X.$

(iii) $F(0) \geq F(1).$

Proof. We have

(i) $F(x - x) \geq \min\{F(x), F(-x)\}$
$F(0) \geq F(x).$

(ii) $F(xx^{-1}) \geq \min\{F(x), F\left(x^{-1}\right)\}$
$F(1) \geq F(x), \quad x(\neq 0) \in X.$

(iii) $F(1 - 1) \geq \min\{F(1), F(-1)\}$
$F(0) \geq F(1).$

□

Theorem 2.2. *[12] Let X and Y be field and f a homomorphism of X into Y suppose that (F, X) is a fuzzy field of X and (G, Y) is a fuzzy field of Y. Then*

(i) $(f(F), Y)$ *is a fuzzy field of Y.*

(ii) $(f^{-1}(G), X)$ *is a fuzzy field of X.*

Definition 2.2. *[12] Let X be a field and (F, X) be a fuzzy field of X. Let Y be a linear space over X and V a fuzzy set of Y. Suppose the following condition hold:*

(i) $V(x + y) \geq \min\{V(x), V(y)\}, \ x, y \in Y,$

(ii) $V(-x) \geq V(x), \ x \in Y,$

(iii) $V(\lambda x) \geq \min\{F(\lambda), V(x)\}, \ \lambda \in X, x \in Y,$

(iv) $F(1) \geq V(0).$

Then (V, Y) is called a fuzzy linear space over (F, X).

Theorem 2.3. *[12] If (V, Y) is a fuzzy linear space over (F, X), then*

(i) $F(0) \geq V(0).$

(ii) $V(0) \geq V(x), \quad x \in Y.$

(iii) $F(0) \geq V(x), \quad x \in Y.$

Proof.

(i) $F(0) = F(1 - 1) \geq F(1) \geq V(0)$
$F(0) \geq V(0).$

(ii) $V(x - x) \geq min\{V(x), \ V(-x)\} = V(x)$
$V(0) \geq V(x), \quad x \in Y.$

(iii) $F(0) \geq V(x), \quad x \in Y.$ by (i) and (ii)

□

Theorem 2.4. *[12] Let (F,X) be a fuzzy field of X and Y a linear space over X . Let V be a fuzzy set of Y. Then (V,Y) is a fuzzy linear space over (F,X) if and only if*

(i) $V(\lambda x + \mu y) \geq \min\{F(\lambda), F(\mu), V(x), V(y)\}, \lambda, \mu \in X$ *and* $x, y \in Y$.

(ii) $F(1) \geq V(x), x \in Y$.

Proof. (i) $V(\lambda x + \mu y) \geq \min\{V(\lambda x), V(\mu y)\}$

$$\geq \min\{F(\lambda) \wedge V(x), F(\mu) \wedge V(y)\}$$
$$= \min\{F(\lambda), V(x), F(\mu), V(y)\}$$

Also, as $F(1) \geq V(0)$, (by definition 2.2)
And as $V(0) \geq V(x)$
$F(1) \geq V(x)$, for all $x \in Y$.

Conversely if $V(\lambda x + \mu y) \geq \min\{F(\lambda), V(x), F(\mu), V(y)\}$
Now consider $V(x+y) \geq \min\{F(1), V(x), V(y)\}$
$\geq \min\{V(x), V(y)\}$ Thus (i) is satisfied.
$V(\lambda x) = V(0.x + \lambda x)$
$= \min\{F(0), V(x), F(\lambda)\}$
$= \min\{V(x), F(\lambda)\}$ (by theorem 2.3)

(ii) By definition 2.2 and theorem 2.3(iii).

□

Theorem 2.5. *[12] Let Y and Z be linear space over the field X and f a linear transformation of Y into Z. Let (F,X) be a fuzzy field of X and (W,Z) be a fuzzy linear space over (F,X) . Then ($f^{-1}(W)$, Y) is a fuzzy linear space over (F,X).*

Theorem 2.6. *[12] Let Y and Z be linear spaces over the field X and f a linear transformation of Y into Z. Let (F,X) be a fuzzy field of X and (V,Y) be a fuzzy linear space over (F,X). Then $(f(V),Z)$ is a fuzzy linear space over (F,X).*

Proof. Let $u, v \in Z$ and $\lambda, \mu \in X$
As $u, v \in Z$ so let r $\in f^{-1}(u)$ and s $\in f^{-1}(v)$ then

$$f(\lambda r + \mu s) = \lambda f(r) + \mu f(s) = \lambda u + \mu v$$

Now consider

$$\begin{aligned} f(V)(\lambda u + \mu v) &= \sup_{w \in f^{-1}(\lambda u + \mu v)} V(w) \\ &\geq \sup_{r \in f^{-1}(u), s \in f^{-1}(v)} V(\lambda r + \mu s) \\ &\geq \sup_{r \in f^{-1}(u), s \in f^{-1}(v)} \min\{V(\lambda r), V(\mu s)\} \\ &\geq \sup_{r \in f^{-1}(u), s \in f^{-1}(v)} \min\{F(\lambda), V(r), F(\mu), V(s)\} \\ &= \min\{F(\lambda), \sup_{r \in f^{-1}(u)} V(r), F(\mu), \sup_{s \in f^{-1}(v)} V(s)\} \\ &= \min\{F(\lambda), f(V)(u), F(\mu), f(V)(v)\} \end{aligned}$$

Also, for any $x \in Z$ $F(1) \geq f(V)(x)$
Thus $(f(V), Z)$ is a fuzzy linear space over (F,X). □

Theorem 2.7. *[10] Let (F, X) be a fuzzy field of x and Let $(V_1, Y_1), (V_2, Y_2), \ldots, (V_n, Y_n)$ be fuzzy linear spaces over (F, X). Then $(V_1 \times V_2 \times \ldots \times V_n,\ Y_1 \times Y_2 \times \ldots \times Y_n)$ is a fuzzy linear space over (F, X).*

Definition 2.3. *[10] Let (F, K) be a fuzzy field of K (K denotes either R or C), X be a linear space over K and (V, X) be a fuzzy linear space over (F, K). A norm on (V, X) is a function $\|.\| : X \to [0, \infty)$ such that*

(i) $F(\|x\|) \geq V(x)$ *for all* $x \in X$,

(ii) $\|x\| \geq 0$ *for all* $x \in X$ *and* $\|x\| = 0$ *if and only if* $x = 0$,

(iii) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$,

(iv) $\|kx\| = |k| \|x\|$ *for all* $k \in K$ *and for all* $x \in X$.

Then $(V, X, \|.\|)$ *is called a normed fuzzy linear space (NFLS).*

3 2-Norm on Fuzzy Linear Spaces

Here K denotes either R (set of real numbers) or C (set of complex numbers).

Definition 3.1. *Let (F, K) be a fuzzy field of K , X be a linear space over K and (V, X) be a fuzzy linear space over (F, K). A 2-norm on (V, X) is function $\|.,.\| : X \times X \to [0, \infty)$ such that*

(i) $F(\|x, y\|) \geq \min\{V(x), V(y)\}$,

(ii) $\|x, y\| = 0 \Leftrightarrow$ *x and y are linearly dependent,*

(iii) $\|x, y\| = \|y, x\|$,

(iv) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$,

(v) $\|kx, y\| \leq |k| \|x, y\|$ *for all* $k \in K$ *and for all* $x, y \in X$.

Then $(V, X, \|.,.\|)$ *is called a 2-normed fuzzy linear space.*

Let (F, R) be a fuzzy field of R the 2-norm faction is a 2-norm on (F, R).

Proof. As here $X = R$ and $V = F$ so we have only to show.
$F(\|x, y\|) \geq \min\{F(x), F(y)\}$
If x and y are linearly dependent then $F(\|x, y\|) = F(0) \geq F(x)$ for all $x \in R$
If x and y are linearly independent then obviously $F(\|x, y\|) \geq \min\{F(x), F(y)\}$. □

(F, R) be a fuzzy field of R the 2-norm $\|.,.\|$ on R^n defined by $\|x, y\| = \left\|x_1, y_1\right\| + \ldots + \|x_n, y_n\|$ is 2-norm on the fuzzy linear space $(F \times F \times \ldots \times F, R^n)$.

Proof. Let $x = (x_1, x_2, x_3, \ldots, x_n) \in R^n,\ y = (y_1, y_2, y_3, \ldots, y_n) \in R^n$

$$\begin{aligned} F(\|x, y\|) &= F\{\|x_1, y_1\| + \ldots + \|x_n, y_n\|\} \\ &\geq \min\{F\|x_1, y_1\|, \ldots, F\|x_n, y_n\|\} \\ &\geq \min[\min\{F(x_1), F(y_1)\}, \ldots, \min\{F(x_n), F(y_n)\}] \\ &= \min[\min\{F(x_1), \ldots, F(x_n)\}, \min\{F(y_1), \ldots, F(y_n)\}] \\ &= \min\{F \times F \times \ldots \times F(x_1, x_2, \ldots, x_n),\ F \times F \times \ldots \times F(y_1, y_2, \ldots, y_n)\} \\ &= \min\{F^n(x),\ F^n(y)\}. \end{aligned}$$

□

Let (F, R) be a fuzzy field of R the 2-norm $\|.,.\|$ on R^n defined by $\|x, y\| = min\{\left\|x_1, y_1\right\|, \ldots, \|x_n, y_n\|\}$ is 2-norm on the fuzzy linear space $(F \times F \times \ldots \times F,\ R^n)$.

Proof. Let $x = (x_1, x_2, x_3, \ldots, x_n) \in R^n,\ y = (y_1, y_2, y_3, \ldots, y_n) \in\ R^n$

$$\begin{aligned} F\left(\|x, y\|\right) &= F[\min\{\|x_1, y_1\|, \ldots, \|x_n, y_n\|\}] \\ &= F\{\left\|x_{i,} y_i\right\|\} \\ &\geq [\min\{F(x_i),\ F(y_i)\} \\ &= \min[\min\{F(x_1), \ldots, F(x_n)\}, \min\{F(y_1), \ldots, F(y_n)\}] \\ &= \min\{\ F \times F \times F \times \ldots \times F(x),\quad F \times F \times F \times \ldots \times F(y)\}. \end{aligned}$$

□

Theorem 3.1. *Let $(V, X,\ \|.,.\|)$ be a 2- normed linear space over (F, K). Then $F(\|x, y\|)^2 \geq F(\|x, y\|)$ for all $x, y \in X$.*

Proof. For all $k_1, k_2 \in K$
$F(k_1, k_2) \geq \min\{F(k_1),\ F(k_2)\}$
So, $F(k^2) \geq \min\{F(k), F(k)\}$
$F\left(k^2\right) \geq F(k)$, for all $k \in K$
Hence $F\left(\|x, y\|^2\right) \geq F(\|x, y\|)$, for all $x, y \in X$. □

Theorem 3.2. *Let $(V_j, X_j, \|.,.\|_j)$ be 2-normed fuzzy linear spaces over fuzzy field (F, K) for $j = 1, 2, 3, \ldots, n$. The 2-norms $\|.,.\|$ defined by $\|x, y\| =\ \|x_1, y_1\|_1 + \ldots + \|x_n, y_n\|_n$ and $\|.,.\|_\infty$ defined by $\|x, y\|_\infty = \max\{\|x_1, y_1\|_1, \ldots, \|x_n, y_n\|_n\}$ are 2-norms on the fuzzy linear space $(V_1 \times V_2 \times \ldots \times V_n,\ X_1 \times X_2 \times \ldots \times X_n)$.*

Proof. (i) Let $x = (x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \ldots \times X_n,\quad y = (y_1, y_2, \ldots, y_n) \in X_1 \times X_2 \ldots \times X_n$.

$$\begin{aligned} F(\|x, y\| &= F(\|x_1, y_1\|_1 + \ldots + \|x_n, y_n\|_n) \\ &\geq \min\{F(\|x_1, y_1\|_1, \ldots, F(\|x_n, y_n\|_n\} \\ &\geq \min\{V_1(x_1), V_1(y_1),\ V_2(x_2), V_2(y_2), \ldots, V_n(x_n), V_n(y_n)\} \\ &= \min\{(V_1(x_1), \ldots, V_n(x_n),\ \ (V_1(y_1), \ldots, V_n(y_n)\} \\ &= \min\{(V_1 \times V_2 \times \ldots \times V_n)(x), (V_1 \times V_2 \times \ldots \times V_n)(y)\} \\ &= (V_1 \times V_2 \times \ldots \times V_n)(x, y). \end{aligned}$$

Therefore $\|.,.\|$ is a 2-norm on $(V_1 \times V_2 \times \ldots \times V_n,\ X_1 \times X_2 \times \ldots \times X_n)$.
(ii) $F(\|x, y\|_\infty) = F[\max\{\|(x_1, y_1)\|_1, \ldots, \left\|(x_n, y_{n)}\right\|_n\}$

$$\begin{aligned} &= F\{\|(x_r, y_r)\|_r\ \geq\ V_r(x_r, y_r) \\ &\geq \min\{V_1(x_1, y_1), \ldots, V_n(x_n, y_n)\} \\ &=\ V_1 \times V_2 \times \ldots \times V_n(x,\ y). \end{aligned}$$

Therefore $\|.,.\|_\infty$ is a 2-norm on $(V_1 \times V_2 \times \ldots \times V_n,\ X_1 \times X_2 \times \ldots \times X_n)$. □

Theorem 3.3. *[10] Let (V, X) be a fuzzy linear space over a fuzzy field (F, K), Y be a linear space over K and $T : X \to Y$ be an injective linear transformation. $T(V)(T(x)) = V(x)$ for all $x \in X$.*

Theorem 3.4. *Let (V, X) be a fuzzy linear space over fuzzy field (F, K), Y be a linear space over K and T be an isomorphism of X onto Y. (V, X) is a 2-normed fuzzy linear space over (F, K) if and only if $(T(V), Y)$ is a 2-normed fuzzy linear space over (F, K).*

Proof. Let $\|.,.\|_X$ be a 2-norm on (V, X). Let $x_i \in X$ so $T(x_i) \in Y$, $(i = 1, 2)$. Now take $T(x_i) = y_i$

Now consider the 2-norm $\|.,.\|_Y$ on Y as $\|y_1, y_2\|_Y = \|x_1, x_2\|_X$. Then

$$
\begin{aligned}
F(\|y_1, y_2\|_Y) &= F(\|x_1, x_2\|_X) \\
&\geq \min\{V(x_1), \ V(x_2)\} \\
&= \min\{T(V)T(x_1), \ \ T(V)T(x_2)\} \\
&= \min\{T(V)y_1, \ \ T(V)y_2\}
\end{aligned}
$$

Thus $\|.,.\|_Y$ is a 2-norm on (T(V),Y).

Conversely, Assume that $\|.,.\|_Y$ is a 2-norm on $(T(V), Y)$. Consider the 2-norm $\|.,.\|_X$ on X as $\|x_1, x_2\|_X = \|Tx_1, Tx_2\|_Y$

Then

$$
\begin{aligned}
F(\|x_1, x_2\|_X) &= (\|Tx_1, Tx_2\|_Y) \\
&\geq \min\{T(V)Tx_1, T(V)Tx_2\} \\
&= \min\{V(x_1), \ V(x_2)\}
\end{aligned}
$$

Thus $\|.,.\|_X$ is a 2-norm on (V, X). □

Theorem 3.5. *Let X be a linear space over K, (W, Y) a fuzzy linear space over a fuzzy field (F, K) and $T : X \to Y$ be an injective linear transformation. If (W, Y) is a 2-normed fuzzy linear space over (F, K) . Then $(T^{-1}(W), X)$ is a 2-normed fuzzy linear space over (F, K).*

Proof. Let $\|.,.\|_Y$ be a 2-norm on (W, Y). Consider the 2-norm $\|.,.\|_X$ on X as

$$
\begin{aligned}
\|x_1, x_2\|_X &= \|Tx_1, Tx_2\|_Y \text{ then} \\
F(\|x_1, x_2\|_X) &= F(\|Tx_1, Tx_2\|_Y) \\
&\geq \min\{WT(x_1), \ WT(x_2)\} \\
&= \min\{T^{-1}W(x_1), \ T^{-1}W(x_2)\}
\end{aligned}
$$

Hence $\|.,.\|_X$ is a 2-norm on $\left(T^{-1}(W), X\right)$. □

Theorem 3.6. *Let (V, X) be a 2-normed fuzzy linear space over a fuzzy field (F, K) and $T : X \to X$ be an injective linear transformation. Then $(T^{-1}(V), X)$ is a 2- normed fuzzy linear space over (F, K).*

Proof. Obvious by theorem 3.5. □

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